

THE HORMONES

PHYSIOLOGY, CHEMISTRY AND APPLICATIONS

VOLUME III

THE HORMONES

Physiology, Chemistry and Applications

Edited by

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VOLUME III



1955

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Preface to Volume III

Seven years have passed since the appearance of Volume I of *The Hormones*. At that time it was suggested that the subject might have completed its first unfolding—virtually all the major hormones especially in the mammals and man having been discovered bioassayed and at least partially identified chemically. The subsequent period it was felt would be one first of consolidation, then of increasing emphasis upon physiological problems and the inner mechanism of the action of hormones. This prognostication, it now appears was true only in part. It was true for the hormones of the thyroid anterior pituitary and gonads which have always occupied so central a position in research and application. To a lesser extent it was true in other areas. But as regards the hormones of the invertebrates and insects and in the realms of the posterior pituitary and the nervous systems the appearance of new facts and ideas has opened new horizons. The concept of nerve as secreting and transmitting hormones as well as the long awaited isolation of the pure posterior pituitary hormones themselves are both notable. Even in the plants the discovery that auxin plays a role in flowering brings what was thought to be a growth hormone into relationship with the reproduction process. Not less to be regarded as new departures are the discoveries concerning the pituitary adrenal axis and the widespread medical applications just touched on in Volume II which have followed.

The expected emphasis upon mechanism of action on the contrary has not yet developed to the degree expected and the major discoveries in this direction are still probably for the future although there have been interesting and suggestive developments. In the case of the animal hormones particularly intensive investigations of their metabolic fate have been considered necessary preludes to studies of their specific roles in biochemical reactions. In several instances it has been indicated that a metabolite of the secretory product rather than the secreted substance itself may be responsible for the presumed specific effects. Furthermore with the broadening of our knowledge of hormone interaction the need for distinction between primary and secondary effects of a given hormone has become more evident. For example the adrenocorticomimetic effects of estrogens are cited in the text the effects are presumably exerted through stimulating the secretion of ACTH from the anterior pituitary. Another complication

which has emerged with clarity is that mammals differ more widely in their physiology than was formerly thought. For this reason, any attempt to study mechanisms of action as problems in general physiology must recognize species differences in response to hormones, and the past septennium has unearthed additional remarkable hormonal responses occurring in one species and completely absent in another.

For all these reasons the present volume is not merely a supplement to Volumes I and II—that is, it is not merely a chronicle of recent experiments, extending in detail what has already been laid down in outline. Instead some parts of the volume supplant their predecessors and certainly the majority of the chapters at least modify or recast the picture which had been presented. While part of its information is certainly supplementary an important part must be regarded as revision or perhaps as reassessment.

The partition of subject matter between the authors and the planned content of the individual chapters has been somewhat revised from that of the previous volumes to allow of more unification and changed emphasis. Some of the authors are those who already reviewed their subjects in Volumes I and II, but the regrouping of the subject matter as well as death, retirement, or preoccupation with other affairs has necessitated a number of changes. In any event the chapters reflect individual viewpoints as much as ever. It is hoped that the elements both of variety and of uniformity will combine to make the book a useful tool in the difficult task of integrating modern biology.

G. Pincus

A. V. THIMANN

June 1950

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CHAPTER I

Plant Growth Hormones

BY KENNETH A. THIMANN AND A. CARL HOFHOLD

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I Introduction

The progress of any field of science is highly irregular: some parts may make a major breakthrough whereas other parts—apparently no more complex—seem blocked by some difficulty of procedure or of concepts so that ideas and experiments proceed in spirals without advancing. The study of plant hormones has shown the expected irregularity of advance during the six years since the appearance of Volume I of *The Hormones*. Some aspects like that of the inhibition of the growth of buds have not undergone any really fundamental change whereas the role of auxin in flowering and the mechanism of auxin action in growth have developed with great rapidity. Recent evidence for an *in vitro* action of auxin and

But C is also equal to AD

Therefore each of the straight lines AE , C is equal to AD ,
so that AE is also equal to C

[C N 1]

Therefore given the two straight lines AB , C , from AB the greater AE has been cut off equal to C the less

(Being) what it was required to do

PROPOSITION 4

If two triangles have the two sides equal to two sides respectively and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend

Let ABC , DEF be two triangles having the two sides AB , AC equal to the two sides DE , DF respectively, namely AB to DE and AC to DF , and the angle BAC equal to the angle EDF

I say that the base BC is also equal to the base EF the triangle ABC will be equal to the triangle DEF , and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend that is, the angle ABC to the angle DEF and the angle ACB to the angle DFE

For, if the triangle ABC be applied to the triangle DEF

and if the point A be placed on the point D

and the straight line AB on DE

then the point B will also coincide with E because AB is equal to DE

Again, AB coinciding with DE

the straight line AC will also coincide with DF because the angle BAC is equal to the angle EDF

hence the point C will also coincide with the point F , because AC is again equal to DF

But B also coincided with E

hence the base BC will coincide with the base EF

[For if when B coincides with E and C with F the base BC does not coincide with the base EF two straight lines will enclose a space which is impossible

Therefore the base BC will coincide with EF] and will be equal to it [C N 4]
Thus the whole triangle ABC will coincide with the whole triangle DEF , and will be equal to it

And the remaining angles will also coincide with the remaining angles and will be equal to them

the angle ABC to the angle DEF
and the angle ACB to the angle DFE

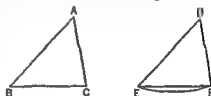
Therefore etc

(Being) what it was required to prove

PROPOSITION 5

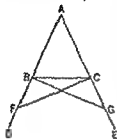
In isosceles triangles the angles at the base are equal to one another and if the equal straight lines be produced further the angles under the base will be equal to one another

Let ABC be an isosceles triangle having the side AB equal to the side AC ,



and let the straight lines BD , CE be produced further in a straight line with AB , AC [Post 2]

I say that the angle ABC is equal to the angle ACB and the angle CBD to the angle BCE



Let a point F be taken at random on BD , from AE the greater let AG be cut off equal to AF the less [I 3]

and let the straight lines FC , GB be joined [Post 1]

Then since AF is equal to AG and AB to AC , the two sides FA , AC are equal to the two sides GA , AB respectively,

and they contain a common angle, the angle FAG

Therefore the base FC is equal to the base GB

and the triangle AFC is equal to the triangle AGB

and the remaining angles will be equal to the remaining angles respectively namely those which the equal sides subtend

that is the angle ACF to the angle ABG

and the angle AFC to the angle AGB [I 4]

And, since the whole AF is equal to the whole AG

and in these AB is equal to AC

the remainder BF is equal to the remainder CG

But FC was also proved equal to GB

therefore the two sides BF , FC are equal to the two sides CG , GB respectively, and the angle BFC is equal to the angle CGB

while the base BC is common to them

therefore the triangle BFC is also equal to the triangle CGB and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend

therefore the angle FBC is equal to the angle GCB ,

and the angle BCF to the angle CBG

Accordingly since the whole angle ABG was proved equal to the angle ACF

and in these the angle CBG is equal to the angle BCF

the remaining angle ABC is equal to the remaining angle ACB

and they are at the base of the triangle ABC

But the angle FBC was also proved equal to the angle GCB

and they are under the base

Therefore etc

Q E D

PROPOSITION 6

If in a triangle two angles be equal to one another the sides which subtend the equal angles will also be equal to one another

Let ABC be a triangle having the angle ABC equal to the angle ACB

I say that the side AB is also equal to the side AC

For if AB is unequal to AC one of them is greater

Let AB be greater and from AB the greater let DB be cut off equal to AC the less

let DC be joined

Then since DB is equal to AC

and BC is common,



the two sides DB BC are equal to the two sides AC , CB respectively,
 and the angle DBC is equal to the angle ACB ,
 therefore the base DC is equal to the base AB ,
 and the triangle DBC will be equal to the triangle ACB
 the less to the greater
 which is absurd

Therefore AB is not unequal to AC ,
 it is therefore equal to it

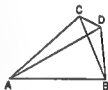
Therefore etc

Q E D

PROPOSITION 7

Given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there cannot be constructed on the same straight line (from its extremities) and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it

For, if possible given two straight lines AC , CB constructed on the straight line AB and meeting at the point C , let two other straight lines AD DB be constructed on the same straight line AB , on the same side of it meeting in another point D and equal to the former two respectively namely each to that which has the same extremity with it so that CA is equal to DA which has the same extremity A with it and CB to DB which has the same extremity B with it, and let CD be joined



Then since AC is equal to AD ,
 the angle ACD is also equal to the angle ADC , [1 5]
 therefore the angle ADC is greater than the angle DCB
 therefore the angle CDB is much greater than the angle DCB

Again since CB is equal to DB
 the angle CDB is also equal to the angle DCB

But it was also proved much greater than it
 which is impossible

Therefore etc

Q E D

PROPOSITION 8

If two triangles have the two sides equal to two sides respectively and have also the base equal to the base they will also have the angles equal which are contained by the equal straight lines

Let ABC , DEF be two triangles having the two sides AB , AC equal to the two sides DE DF respectively namely AB to DE , and AC to DF and let them have the base BC equal to the base EF

I say that the angle BAC is also equal to the angle EDF



For if the triangle ABC be applied to the triangle DEF and if the point B be placed on the point E and the straight line BC on EF ,
 the point C will also coincide with F ,

because BC is equal to EF

Then BC coinciding with EF ,

BA AC will also coincide with ED DF ,

for if the base BC coincides with the base EF and the sides BA , AC do not coincide with ED DF but fall beside them as EG GF

then given two straight lines constructed on a straight line (from its extremities) and meeting in a point there will have been constructed on the same straight line (from its extremities) and on the same side of it two other straight lines meeting in another point and equal to the former two respectively namely each to that which has the same extremity with it

But they cannot be so constructed

[1 7]

Therefore it is not possible that if the base BC be applied to the base EF , the sides BA , AC should not coincide with ED DF

they will therefore coincide,

so that the angle BAC will also coincide with the angle EDF and will be equal to it

If therefore etc

Q E D

PROPOSITION 9

To bisect a given rectilineal angle



Let the angle BAC be the given rectilineal angle

Thus it is required to bisect it

Let a point D be taken at random on AB

let AE be cut off from AC equal to AD

[1 3]

let DE be joined and on DE let the equilateral triangle DEF be constructed

let AF be joined

I say that the angle BAC has been bisected by the straight line AF

For, since AD is equal to AE

and AF is common

the two sides DA AF are equal to the two sides EA AF respectively

And the base DF is equal to the base EF

therefore the angle DAF is equal to the angle EAF

[1 8]

Therefore the given rectilineal angle BAC has been bisected by the straight line AF

Q E F

PROPOSITION 10

To bisect a given finite straight line

Let AB be the given finite straight line

Thus it is required to bisect the finite straight line AB

Let the equilateral triangle ABC be constructed on it

[1 1]

and let the angle ACB be bisected by the straight line CD

[1 9]

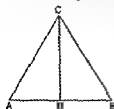
I say that the straight line AB has been bisected at the point D

For since AC is equal to CB

and CD is common

the two sides AC CD are equal to the two sides BC , CD respectively

and the angle ACD is equal to the angle BCD



therefore the base AD is equal to the base BD [I 4]

Therefore the given finite straight line AB has been bisected at D \square $E F$

PROPOSITION 11

To draw a straight line at right angles to a given straight line from a given point on it

Let AB be the given straight line and C the given point on it

Thus it is required to draw from the point C a straight line at right angles to the straight line AB

Let a point D be taken at random on AC ,

let CE be made equal to CD , [I 3]

on DE let the equilateral triangle FDE be constructed, [I 1]

and let FC be joined

I say that the straight line FC has been drawn at right angles to the given straight line AB from C the given point on it

For, since DC is equal to CE ,

and CF is common,

the two sides DC CF are equal to the two sides EC CF respectively,

and the base DF is equal to the base FE ,

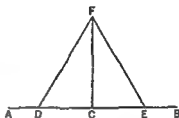
therefore the angle DCF is equal to the angle ECF [I 8]

and they are adjacent angles

But when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right [Def 10]

therefore each of the angles DCF , FCE is right

Therefore the straight line CF has been drawn at right angles to the given straight line AB from the given point C on it \square $E F$



PROPOSITION 12

To a given infinite straight line, from a given point which is not on it to draw a perpendicular straight line

Let AB be the given infinite straight line and C the given point which is not on it

thus it is required to draw to the given infinite straight line AB , from the given point C which is not on it, a perpendicular straight line

For let a point D be taken at random on the other side of the straight line AB and with centre C and distance CD let the circle EFG be described [Post 3]

let the straight line EG be bisected at H

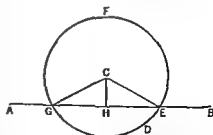
[I 10]

and let the straight lines CG , CH CE be joined [Post 1]

I say that CH has been drawn perpendicular to the given infinite straight line AB from the given point C which is not on it

For since GH is equal to HE

and HC is common



the two sides GH , HC are equal to the two sides EH , HC respectively
 and the base CG is equal to the base CE
 therefore the angle CHG is equal to the angle EHC [I 8]
 And they are adjacent angles

But when a straight line set up on a straight line makes the adjacent angles equal to one another each of the equal angles is right and the straight line standing on the other is called a perpendicular to that on which it stands [Def 10]

Therefore CH has been drawn perpendicular to the given infinite straight line AB from the given point C which is not on it Q E F

PROPOSITION 13

If a straight line set up on a straight line make angles it will make either two right angles or angles equal to two right angles

For let any straight line AB set up on the straight line CD make the angles CBA , ABD

I say that the angles CBA , ABD are either two right angles or equal to two right angles

Now if the angle CBA is equal to the angle ABD they are two right angles [Def 10]

But if not let BE be drawn from the point B at right angles to CD [I 11]

therefore the angles CBE , EBD are two right angles

Then since the angle CBE is equal to the two angles CBA , ABE , let the angle EBD be added to each therefore the angles CBE , EBD are equal to the three angles CBA , ABE , EBD [C N 2]

Again since the angle DBA is equal to the two angles DBE , EBA

let the angle ABC be added to each

therefore the angles DBA , ABC are equal to the three angles DBE , EBA , ABC [C N 2]

But the angles CBE , EBD were also proved equal to the same three angles and things which are equal to the same thing are also equal to one another [C N 1]

therefore the angles CBE , EBD are also equal to the angles DBA , ABC

But the angles CBE , EBD are two right angles

therefore the angles DBA , ABC are also equal to two right angles

Therefore etc Q E D

PROPOSITION 14

If with any straight line and at a point on it two straight lines not lying on the same side make the adjacent angles equal to two right angles the two straight lines will be in a straight line with one another

For with any straight line AB and at the point B on it let the two straight lines BC , BD not lying on the same side make the adjacent angles ABC , ABD equal to two right angles

I say that BD is in a straight line with BC

For if BD is not in a straight line with BC , let BE be in a straight line with CB

Then since the straight line AB stands on the straight line CBE ,
the angles ABC , ABE are equal to two right angles [I 13]

But the angles ABC , ABD are also equal to two right angles

therefore the angles CBA , ABE are equal to the angles CBA , ABD [Post 4 and CN 1]

Let the angle CBA be subtracted from each,
therefore the remaining angle ABE is equal to the remaining angle ABD [CN 3]

the less to the greater which is impossible

Therefore BE is not in a straight line with CB

Similarly we can prove that neither any other straight line except BD

Therefore CB is in a straight line with BD

Therefore etc

Q E D

PROPOSITION 15

If two straight lines cut one another, they make the vertical angles equal to one another

For let the straight lines AB , CD cut one another at the point E ,

I say that the angle AEC is equal to the angle DEB ,
and the angle CEB to the angle AED

For since the straight line AE stands on the straight line CD , making the angles CEA , AED ,

the angles CEA , AED are equal to two right angles [I 13]

Again since the straight line DE stands on the straight line AB , making the angles AED , DEB

the angles AED , DEB are equal to two right angles [I 13]

But the angles CEA , AED were also proved equal to two right angles,
therefore the angles CEA , AED are equal to the angles AED , DEB

[Post 4 and CN 1]

Let the angle AED be subtracted from each

therefore the remaining angle CEA is equal to the remaining angle DEB [CN 3]

Similarly it can be proved that the angles CEB , DEA are also equal

Therefore etc

Q E D

[PORISM From this it is manifest that if two straight lines cut one another they will make the angles at the point of section equal to four right angles]

PROPOSITION 16

In any triangle if one of the sides be produced the exterior angle is greater than either of the interior and opposite angles

Let ABC be a triangle and let one side of it BC be produced to D ,

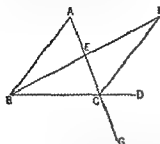
I say that the exterior angle ACD is greater than either of the interior and opposite angles CBA , BAC

Let AC be bisected at E [I 10] and let BE be joined and produced in a straight line to F

let EF be made equal to BE [I 3]

let FC be joined [Post 1] and let AC be drawn through to G [Post 2]

Then since AE is equal to EC and BE to EF ,
 the two sides AE EB are equal to the two sides CE EF respectively
 and the angle AEB is equal to the angle FEC
 for they are vertical angles [1 15]



Therefore the base AB is equal to the base FC
 and the triangle ABE is equal to the triangle CFE
 and the remaining angles are equal to the remain-
 ing angles respectively namely, those which the
 equal sides subtend [1 4]

therefore the angle BAE is equal to the angle ECF
 But the angle ECD is greater than the angle
 ECF [C 5]

therefore the angle ACD is greater than the angle BAE

Similarly also if BC be bisected the angle BCG that is the angle ACD
 [1 15] can be proved greater than the angle ABC as well

Therefore etc

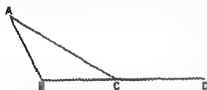
Q E D

PROPOSITION 17

In any triangle two angles taken together in any manner are less than two right angles

Let ABC be a triangle

I say that two angles of the triangle ABC taken together in any manner are
 less than two right angles



For let BC be produced to D [Post 2]

Then since the angle ACD is an exterior
 angle of the triangle ABC

it is greater than the interior and opposite
 angle ABC [1 16]

Let the angle ACB be added to each

therefore the angles ACD ACB are greater than the angles ABC , BCA

But the angles ACD ACB are equal to two right angles [1 13]

Therefore the angles ABC BCA are less than two right angles

Similarly we can prove that the angles BAC ACB are also less than two
 right angles and so are the angles CAB ABC as well

Therefore etc

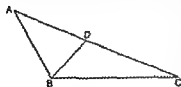
Q E D

PROPOSITION 18

In any triangle the greater side subtends the greater angle

For let ABC be a triangle having the side AC greater than AB

I say that the angle ABC is also greater than the angle BCA



For since AC is greater than AB let AD
 be made equal to AB [1 3] and let BD be
 joined

Then since the angle ADB is an exterior
 angle of the triangle BCD

it is greater than the interior and opposite
 angle DCB [1 16]

But the angle ADB is equal to the angle ABD

since the side AB is equal to AD

therefore the angle ABD is also greater than the angle ACB
 therefore the angle ABC is much greater than the angle ACB

Therefore etc

Q E D

PROPOSITION 19

In any triangle the greater angle is subtended by the greater side

Let ABC be a triangle having the angle ABC greater than the angle BCA ,

I say that the side AC is also greater than the side AB

For, if not, AC is either equal to AB or less

Now AC is not equal to AB

for then the angle ABC would also have been equal to the angle ACB [I 5]

but it is not,

therefore AC is not equal to AB

Neither is AC less than AB ,

for then the angle ABC would also have been less than the angle ACB [I 18]

but it is not

therefore AC is not less than AB

And it was proved that it is not equal either

Therefore AC is greater than AB

Therefore etc

Q E D



PROPOSITION 20

In any triangle two sides taken together in any manner are greater than the remaining one

For let ABC be a triangle

I say that in the triangle ABC two sides taken together in any manner are greater than the remaining one namely

BA AC greater than BC

AB , BC greater than AC ,

BC CA greater than AB

For let BA be drawn through to the point D , let DA be made equal to CA , and let DC be joined

Then since DA is equal to CA

the angle ADC is also equal to the angle ACD [I 5]

therefore the angle BCD is greater than the angle ADC

[C N 5]

And since DCB is a triangle having the angle BCD greater than the angle BDC

and the greater angle is subtended by the greater side [I 19]

therefore DB is greater than BC

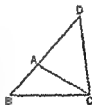
But DA is equal to AC

therefore BA AC are greater than BC

Similarly we can prove that AB BC are also greater than CA and BC CA than AB

Therefore etc

Q E D



PROPOSITION 21

If on one of the sides of a triangle from its extremities there be constructed two

straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle

On BC , one of the sides of the triangle ABC from its extremities B C , let the two straight lines BD DC be constructed meeting within the triangle

I say that BD DC are less than the remaining two sides of the triangle BA , AC , but contain an angle BDC greater than the angle BAC

For let BD be drawn through to E

Then, since in any triangle two sides are greater than the remaining one [I 20]

therefore in the triangle ABE , the two sides AB , AE are greater than BE

Let EC be added to each

therefore BA AC are greater than BE EC

Again since in the triangle CED

the two sides CE ED are greater than CD

let DB be added to each

therefore CE , EB are greater than CD DB

But BA AC were proved greater than BE EC

therefore BA AC are much greater than BD DC

Again since in any triangle the exterior angle is greater than the interior and opposite angle [I 16]

therefore in the triangle CDE ,

the exterior angle BDC is greater than the angle CED

For the same reason moreover in the triangle ABE also

the exterior angle CEB is greater than the angle BAC

But the angle BDC was proved greater than the angle CEB

therefore the angle BDC is much greater than the angle BAC

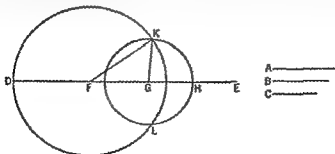
Therefore etc

Q E D

PROPOSITION 22

Out of three straight lines which are equal to three given straight lines to construct a triangle thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one [I 20]

Let the three given straight lines be A B C and of these let two taken together in any manner be greater than the remaining one



namely

A B greater than C

and

A C greater than B

B C greater than A

thus it is required to construct a triangle out of straight lines equal to A, B, C

Let there be set out a straight line DE , terminated at D but of infinite length in the direction of E

and let DF be made equal to A FG equal to B , and GH equal to C [1 3]

With centre F and distance FD let the circle DKL be described

again with centre G and distance GH let the circle LKH be described

and let KF, KG be joined

I say that the triangle KFG has been constructed out of three straight lines equal to A, B, C

For since the point F is the centre of the circle DKL

FD is equal to FK

But FD is equal to A

therefore KF is also equal to A

Again since the point G is the centre of the circle LKH ,

GH is equal to GK

But GH is equal to C

therefore KG is also equal to C

And FG is also equal to B ,

therefore the three straight lines KF, FG, GK are equal to the three straight lines A, B, C

Therefore out of the three straight lines KF, FG, GK which are equal to the three given straight lines A, B, C , the triangle KFG has been constructed

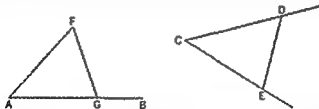
Q E F

PROPOSITION 23

On a given straight line and at a point on it to construct a rectilineal angle equal to a given rectilineal angle

Let AB be the given straight line A the point on it and the angle DCE the given rectilineal angle

thus it is required to construct on the given straight line AB and at the point A on it, a rectilineal angle equal to the given rectilineal angle DCE



On the straight lines CD, CE respectively let the points D, E be taken at random,

let DE be joined

and out of three straight lines which are equal to the three straight lines CD, DE, CE let the triangle AFG be constructed in such a way that CD is equal to AF , CE to AG and further DE to FG [1 22]

Then, since the two sides DC, CE are equal to the two sides FA, AG respectively,

and the base DE is equal to the base FG

the angle DCF is equal to the angle FAG

[1 8]

Therefore on the given straight line AB and at the point A on it the rectilinear angle FAG has been constructed equal to the given rectilinear angle DCE

Q E F

PROPOSITION 24

If two triangles have the two sides equal to two sides respectively but have the one of the angles contained by the equal straight lines greater than the other they will also have the base greater than the base

Let ABC , DEF be two triangles having the two sides AB , AC equal to the two sides DE , DF respectively namely AB to DE , and AC to DF , and let the angle at A be greater than the angle at D

I say that the base BC is also greater than the base EF



For since the angle BAC is greater than the angle EDF let there be constructed on the straight line DE , and at the point D on it, the angle EDG equal to the angle BAC , [I 23] let DG be made equal to either of the two straight lines AC , DF , and let EG FG be joined

Then since AB is equal to DE and AC to DG

the two sides BA AC are equal to the two sides ED DG , respectively

and the angle BAC is equal to the angle EDG

therefore the base BC = equal to the base EG

[I 4]

Again since DF is equal to DG

the angle DGF is also equal to the angle DFG ,

[I 5]

therefore the angle DFG is greater than the angle EGF

Therefore the angle EFG is much greater than the angle EGF

And since EFG is a triangle having the angle EFG greater than the angle EGF

and the greater angle = subtended by the greater side

[I 19]

the side EG is also greater than EF

But EG is equal to BC

Therefore BC = also greater than EF

Therefore etc

Q E D

PROPOSITION 25

If two triangles have the two sides equal to two sides respectively but have the base greater than the base they will also have the one of the angles contained by the equal straight lines greater than the other

Let ABC DEF be two triangles having the two sides AB AC equal to the two sides DE DF respectively namely AB to DE and AC to DF and let the base BC be greater than the base EF

I say that the angle BAC is also greater than the angle EDF

For if not it is either equal to it or less

Now the angle BAC is not equal to the angle EDF for then the base BC would also have been equal to the base EF

[I 4]

but it is not,

therefore the angle BAC is not equal to the angle EDF

Neither again is the angle BAC less than the angle EDF for then the base BC would also have been less than the base EF , [1 24]



but it is not therefore the angle BAC is not less than the angle EDF
But it was proved that it is not equal either,

therefore the angle BAC is greater than the angle EDF

Therefore etc

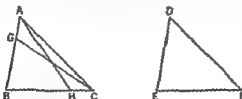
Q E D

PROPOSITION 26

If two triangles have the two angles equal to two angles respectively and one side equal to one side namely either the side adjoining the equal angles or that subtending one of the equal angles they will also have the remaining sides equal to the remaining sides and the remaining angle to the remaining angle

Let ABC DEF be two triangles having the two angles ABC BCA equal to the two angles DEF EFD respectively namely the angle ABC to the angle DEF and the angle BCA to the angle EFD , and let them also have one side equal to one side first that adjoining the equal angles namely BC to EF

I say that they will also have the remaining sides equal to the remaining sides respectively namely AB to DE and AC to DF and the remaining angle to the remaining angle namely the angle BAC to the angle EDF



For if AB is unequal to DE one of them is greater

Let AB be greater and let BG be made equal to DE and let GC be joined
Then since BG is equal to DE and BC to EF

the two sides GB BC are equal to the two sides DE EF respectively,

and the angle GBC is equal to the angle DEF

therefore the base GC is equal to the base DF

and the triangle GBC is equal to the triangle DEF

and the remaining angles will be equal to the remaining angles namely those which the equal sides subtend [1 4]

therefore the angle GCB is equal to the angle DFE

But the angle DFE is by hypothesis equal to the angle BCA

therefore the angle BCG is equal to the angle BCA

the less to the greater which is impossible

Therefore AB is not unequal to DE

and is therefore equal to it

But BC is also equal to EF

therefore the two sides AB , BC are equal to the two sides DE , EF respectively,

and the angle ABC is equal to the angle DEF ,

therefore the base AC is equal to the base DF ,

and the remaining angle BAC is equal to the remaining angle EDF [I 4]

Again, let sides subtending equal angles be equal as AB to DE ,

I say again that the remaining sides will be equal to the remaining sides namely AC to DF and BC to EF , and further the remaining angle BAC is equal to the remaining angle EDF

For, if BC is unequal to EF , one of them is greater

Let BC be greater if possible and let BH be made equal to EF , let AH be joined

Then since BH is equal to EF , and AB to DE

the two sides AB , BH are equal to the two sides DE , EF respectively, and they contain equal angles

therefore the base AH is equal to the base DF

and the triangle ABH is equal to the triangle DEF

and the remaining angles will be equal to the remaining angles namely those which the equal sides subtend [I 4]

therefore the angle BHA is equal to the angle EFD

But the angle EPD is equal to the angle BCA

therefore, in the triangle AHC the exterior angle BHA is equal to the interior and opposite angle BCA

which is impossible

[I 16]

Therefore BC is not unequal to EF

and is therefore equal to it

But AB is also equal to DE

therefore the two sides AB , BC are equal to the two sides DE , EF respectively, and they contain equal angles,

therefore the base AC is equal to the base DF

the triangle ABC equal to the triangle DEF

and the remaining angle BAC equal to the remaining angle EDF [I 4]

Therefore etc

Q E D

PROPOSITION 27

If a straight line falling on two straight lines make the alternate angles equal to one another the straight lines will be parallel to one another

For let the straight line EF falling on the two straight lines AB , CD make the alternate angles AEF , EFD equal to one another

I say that AB is parallel to CD

For if not AB , CD when produced will meet either in the direction of B , D or towards A , C

Let them be produced and meet, in the direction of B , D at G

Then in the triangle GEF

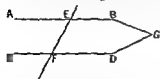
the exterior angle AEF is equal to the interior and opposite angle EFG

which is impossible

[I 16]

Therefore AB , CD when produced will not meet in the direction of B , D

Similarly it can be proved that neither will they meet towards A , C



But straight lines which do not meet in either direction are parallel

[Def 23]

therefore AB is parallel to CD

Therefore etc

Q E D

PROPOSITION 28

If a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side or the interior angles on the same side equal to two right angles the straight lines will be parallel to one another

For let the straight line EF falling on the two straight lines AB CD make the exterior angle EGB equal to the interior and opposite angle CHD or the interior angles on the same side namely BGH , GHD , equal to two right angles, I say that AB is parallel to CD

For, since the angle EGB is equal to the angle GHD

while the angle EGB is equal to the angle AGH

[I 15]

the angle AGH is also equal to the angle GHD

and they are alternate

therefore AB is parallel to CD [I 27]

Again since the angles BGH GHD are equal to two right angles and the angles AGH , BGH are also equal to two right angles [I 13]

the angles AGH BGH are equal to the angles BGH GHD

Let the angle BGH be subtracted from each

therefore the remaining angle AGH is equal to the remaining angle GHD , and they are alternate

therefore AB is parallel to CD

[I 27]

Therefore etc

Q E D

PROPOSITION 29

A straight line falling on parallel straight lines makes the alternate angles equal to one another the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles

For let the straight line EF fall on the parallel straight lines AB CD

I say that it makes the alternate angles AGH GHD equal, the exterior angle EGB equal to the interior and opposite angle GHD and the interior angles on the same side, namely BGH GHD equal to two right angles

For if the angle AGH is unequal to the angle GHD one of them is greater

Let the angle AGH be greater

Let the angle BGH be added to each

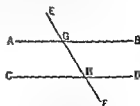
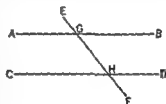
therefore the angles AGH BGH are greater than the angles BGH , GHD

But the angles AGH , BGH are equal to two right angles, [I 13]

therefore the angles BGH GHD are less than two right angles

But straight lines produced indefinitely from angles less than two right angles meet [Post 5]

therefore AB CD if produced indefinitely will meet,



but they do not meet, because they are by hypothesis parallel
Therefore the angle AGH is not unequal to the angle GHD ,
and is therefore equal to it

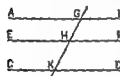
Again, the angle AGH is equal to the angle EGB , [I 15]
therefore the angle EGB is also equal to the angle GHD [C.N. 1]

Let the angle BGH be added to each,
therefore the angles EGB, BGH are equal to the angles BGH, GHD [C.N. 2]
But the angles EGB, BGH are equal to two right angles [I 13]
therefore the angles BGH, GHD are also equal to two right angles
Therefore etc Q.E.D.

PROPOSITION 30

Straight lines parallel to the same straight line are also parallel to one another

Let each of the straight lines AB, CD be parallel to EF . I say that AB is also parallel to CD .



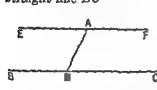
For let the straight line GA fall upon them
Then, since the straight line GA has fallen on the parallel straight lines AB, EF ,
the angle AGK is equal to the angle GHE [I 27]
Again, since the straight line GA has fallen on the parallel straight lines EF, CD
the angle GHE is equal to the angle GKD [I 29]
But the angle AGK was also proved equal to the angle GHE
therefore the angle AGK is also equal to the angle GKD [C.N. 1]
and they are alternate

Therefore AB is parallel to CD Q.E.D.

PROPOSITION 31

Through a given point to draw a straight line parallel to a given straight line

Let A be the given point and BC the given straight line
thus it is required to draw through the point A a straight line parallel to the straight line BC



Let a point D be taken at random on BC
and let AD be joined, on the straight line DA ,
and at the point A on it let the angle DAE be
constructed equal to the angle ADC [I 23] and
let the straight line AF be produced in a straight
line with EA

Then, since the straight line AD falling on the two straight lines BC, EF has
made the alternate angles EAD, ADC equal to one another
therefore EAF is parallel to BC [I 27]

Therefore through the given point A the straight line EAF has been drawn
parallel to the given straight line BC Q.E.D.

PROPOSITION 32

In any triangle, if one of the sides be produced the exterior angle is equal to the two interior and opposite angles and the three interior angles of the triangle are equal to two right angles

Let ABC be a triangle, and let one side of it BC be produced to D

I say that the exterior angle ACD is equal to the two interior and opposite angles CAB , ABC , and the three interior angles of the triangle ABC , BCA , CAB are equal to two right angles

For let CE be drawn through the point C parallel to the straight line AB [I 31]

Then since $AB \parallel CE$,
and AC has fallen upon them
the alternate angles BAC , ACE are equal to one another [I 29]

Again, since $AB \parallel CE$
and the straight line BD has fallen upon them
the exterior angle ECD is equal to the interior and opposite angle ABC [I 29]

But the angle ACE was also proved equal to the angle ABC ,
therefore the whole angle ACD is equal to the two interior and opposite angles BAC , ABC

Let the angle ACB be added to each
therefore the angles ACD , ACB are equal to the three angles ABC , BCA , CAB

But the angles ACD , ACB are equal to two right angles [I 13]
therefore the angles ABC , BCA , CAB are also equal to two right angles

Therefore etc

Q E D

PROPOSITION 33

The straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are themselves also equal and parallel

Let AB , CD be equal and parallel and let the straight lines AC , BD join them (at the extremities which are) in the same directions (respectively),

I say that AC , BD are also equal and parallel

Let BC be joined

Then since $AB \parallel CD$ and BC has fallen upon them
the alternate angles ABC , BCD are equal to one another [I 29]

And since AB is equal to CD

and BC is common

the two sides AB , BC are equal to the two sides DC , CB

and the angle ABC is equal to the angle BCD

therefore the base AC is equal to the base BD

and the triangle ABC is equal to the triangle DCB ,

and the remaining angles will be equal to the remaining angles respectively
namely those which the equal sides subtend [I 4]

therefore the angle ACB is equal to the angle CBD

And since the straight line BC falling on the two straight lines AC , BD has made the alternate angles equal to one another

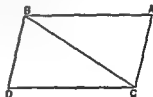
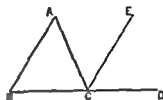
AC is parallel to BD

[I 27]

And it was also proved equal to it

Therefore etc

Q E D



PROPOSITION 34

In parallelogrammic areas the opposite sides and angles are equal to one another, and the diameter bisects the areas

Let $ACDB$ be a parallelogrammic area, and BC its diameter

I say that the opposite sides and angles of the parallelogram $ACDB$ are equal to one another and the diameter BC bisects it

For since AB is parallel to CD ,
and the straight line BC has fallen upon them
the alternate angles ABC BCD are equal to one another [I 29]

Again, since AC is parallel to BD and BC has fallen upon them,

the alternate angles ACB CBD are equal to one another [I 29]

Therefore ABC DCB are two triangles having the two angles ABC , BCA equal to the two angles DCB , CBD respectively, and one side equal to one side namely that adjoining the equal angles and common to both of them, BC , therefore they will also have the remaining sides equal to the remaining sides respectively, and the remaining angle to the remaining angle [I 26]

therefore the side AB is equal to CD ,

and AC to BD ,

and further the angle BAC is equal to the angle CDB

And, since the angle ABC is equal to the angle BCD

and the angle CBD to the angle ACB

the whole angle ABD is equal to the whole angle ACD [C.N 2]

And the angle BAC was also proved equal to the angle CDB

Therefore in parallelogrammic areas the opposite sides and angles are equal to one another

I say, next, that the diameter also bisects the areas

For, since AB is equal to CD ,

and BC is common,

the two sides AB BC are equal to the two sides DC , CB respectively,

and the angle ABC is equal to the angle BCD ,

therefore the base AC is also equal to DB

and the triangle ABC is equal to the triangle DCB [I 4]

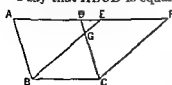
Therefore the diameter BC bisects the parallelogram $ACDB$ ■ E D

PROPOSITION 35

Parallelograms which are on the same base and in the same parallels are equal to one another

Let $ABCD$ $EBCF$ be parallelograms on the same base BC and in the same parallels AF BC ,

I say that $ABCD$ is equal to the parallelogram $EBCF$



For since $ABCD$ is a parallelogram

AD is equal to BC

[I 34]

For the same reason also

EF is equal to BC

so that AD is also equal to EF

[C.N 1]

and DE is common,

therefore the whole AF is equal to the whole DF [C 2]

But AB is also equal to DC , [1 34]

therefore the two sides EA, AB are equal to the two sides FD, DC respectively,

and the angle FDC is equal to the angle EAB ,
the exterior to the interior, [1 29]

therefore the base FB is equal to the base FC ,

and the triangle FAB will be equal to the triangle FDC [1 4]

Let DGE be subtracted from each

therefore the trapezium $ABGD$ which remains is equal to the trapezium $EGCF$
which remains [C 3]

Let the triangle GBC be added to each

therefore the whole parallelogram $ABCD$ is equal to the whole parallelogram
 $FBCF$ [C 3]

Therefore etc

Q E D

PROPOSITION 36

Parallelograms which are on equal bases and in the same parallels are equal to one another

Let $ABCD, EFGH$ be parallelograms which are on equal bases BC, FG and in the same parallels AH, BG

I say that the parallelogram $ABCD$ is equal to $EFGH$

For let BE, CH be joined

Then since BC is equal to FG ,
while

FG is equal to EH

BC is also equal to FH [C 1]

But they are also parallel

And EB, HC join them

but straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are equal and parallel [1 33]

Therefore $EBCH$ is a parallelogram [1 34]

And it is equal to $ABCD$

for it has the same base BC with it and is in the same parallels BC, AH with it [1 35]

For the same reason also $EFGH$ is equal to the same $EBCH$ [1 35]

so that the parallelogram $ABCD$ is also equal to $EFGH$ [C 1]

Therefore etc

Q E D

PROPOSITION 37

Triangles which are on the same base and in the same parallels are equal to one another

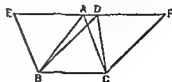
Let ABC, DBC be triangles on the same base BC and in the same parallels AD, BC

I say that the triangle ABC is equal to the triangle DBC

Let AD be produced in both directions to E, F

through B let BE be drawn parallel to CA

[1 31]



and through C let CF be drawn parallel to BD

[1 31]

Then each of the figures $EBCA$, $DBCF$ is a parallelogram, and they are equal

for they are on the same base BC and in the same parallels BC EF [1 35]

Moreover the triangle ABC is half of the parallelogram $EBCA$, for the diameter AB bisects it

[1 34]

And the triangle DBC is half of the parallelogram $DBCF$ for the diameter DC bisects it

[1 34]

[But the halves of equal things are equal to one another]

Therefore the triangle ABC is equal to the triangle DBC

Therefore etc

Q E D

PROPOSITION 38

Triangles which are on equal bases and in the same parallels are equal to one another

Let ABC DEF be triangles on equal bases BC EF and in the same parallels BC AD ,

I say that the triangle ABC is equal to the triangle DEF

For let AD be produced in both directions to G H

through B let BG be drawn parallel to CA ,

[1 31]

and through F let FH be drawn parallel to DE

Then each of the figures $GBCA$, $DEFH$ is a parallelogram

and $GBCA$ is equal to $DEFH$,

for they are on equal bases BC , EF and in the same parallels BF GH

[1 36]

Moreover the triangle ABC is half of the parallelogram $GBCA$ for the diameter AB bisects it

[1 34]

And the triangle FED is half of the parallelogram $DEFH$ for the diameter DF bisects it

[1 34]

[But the halves of equal things are equal to one another]

Therefore the triangle ABC is equal to the triangle DEF

Therefore etc

Q E D

PROPOSITION 39

Equal triangles which are on the same base and on the same side are also in the same parallels

Let ABC DBC be equal triangles which are on the same base BC and on the same side of it

[I say that they are also in the same parallels]

And [For] let AD be joined I say that AD is parallel to BC

For if not let AE be drawn through the point A parallel to the straight line BC

[1 31]

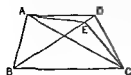
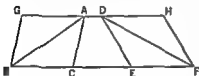
and let EC be joined

Therefore the triangle ABC is equal to the triangle EBC ,

for it is on the same base BC with it and in the same parallels

[1 37]

But ABC is equal to DBC



therefore DBC is also equal to FBC [CA 1]

the greater to the less which is impossible

Therefore AF is not parallel to BC

Similarly we can prove that neither is any other straight line except AD ,
therefore AD is parallel to BC

Therefore etc

Q E D

PROPOSITION 10

Equal triangles which are on equal bases and on the same side are also in the same parallels

Let ABC, CDE be equal triangles on equal bases BC, CE and on the same side

I say that they are also in the same parallels

I or let AD be joined

I say that AD is parallel to BF

I or if not let AF be drawn through A parallel to BF

[I 31] and let FE be joined

Therefore the triangle ABC is equal to the triangle FCE ,

for they are on equal bases BC, CE and in the same parallels BE, AF [I 35]

But the triangle ABC is equal to the triangle DCE ,

therefore the triangle DCE is also equal to the triangle FCE , [CA 1]

the greater to the less which is impossible

Therefore AF is not parallel to BE

Similarly we can prove that neither is any other straight line except AD ,
therefore AD is parallel to BE

Therefore etc

Q E D

PROPOSITION 41

If a parallelogram have the same base with a triangle and be in the same parallels the parallelogram is double of the triangle

For let the parallelogram $ABCD$ have the same base BC with the triangle EBC and let it be in the same parallels BC, AE

I say that the parallelogram $ABCD$ is double of the triangle EBC

For let AC be joined

Then the triangle ABC is equal to the triangle EBC ,
for it is on the same base BC with it and in the same parallels BC, AE [I 37]

But the parallelogram $ABCD$ is double of the triangle ABC

for the diameter AC bisects it [I 34]

so that the parallelogram $ABCD$ is also double of the triangle EBC

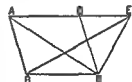
Therefore etc

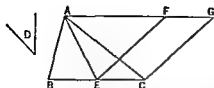
Q E D

PROPOSITION 42

To construct in a given rectilineal angle a parallelogram equal to a given triangle

Let ABC be the given triangle and D the given rectilineal angle
thus it is required to construct in the rectilineal angle D a parallelogram equal





to the triangle ABC

Let BC be bisected at E , and let AE be joined,
on the straight line EC , and at the point
 E on it let the angle CEF be constructed
equal to the angle D , [I 23]

through A let AG be drawn parallel to EC and [I 31]

through C let CG be drawn parallel to EF

Then $FECG$ is a parallelogram

And since BE is equal to EC ,

the triangle ABE is also equal to the triangle AEC

for they are on equal bases BE, EC and in the same parallels BC, AG , [I 38]

therefore the triangle ABC is double of the triangle AEC

But the parallelogram $FECG$ is also double of the triangle AEC , for it has
the same base with it and is in the same parallels with it, [I 41]

therefore the parallelogram $FECG$ is equal to the triangle ABC

And it has the angle CEF equal to the given angle D

Therefore the parallelogram $FECG$ has been constructed equal to the given
triangle ABC in the angle CEF which is equal to D Q E F

PROPOSITION 43

In any parallelogram the complements of the parallelograms about the diameter are equal to one another

Let $ABCD$ be a parallelogram and AC its diameter

and about AC let EH, FG be parallelograms and BK, KD the so-called complements,

I say that the complement BK is equal to the complement KD

For since $ABCD$ is a parallelogram, and AC its diameter,

the triangle ABC is equal to the triangle ACD [I 34]

Again since EH is a parallelogram, and AK is
its diameter,

the triangle AEK is equal to the triangle AHK

For the same reason

the triangle KFC is also equal to KGC

Now since the triangle AEK is equal to the
triangle AHK ,

and KFC to KGC

the triangle AEK together with KGC is equal to the triangle AHK together
with KFC [C.N 2]

And the whole triangle ABC is also equal to the whole ADC ,

therefore the complement BK which remains is equal to the complement KD
which remains [C.N 3]

Therefore etc

Q E D

PROPOSITION 44

To a given straight line to apply in a given rectilineal angle a parallelogram equal to a given triangle

Let AB be the given straight line C the given triangle and D the given recti-
lineal angle,

line GH in the angle GHM which is equal to E [I 44]

Then since the angle E is equal to each of the angles HAF GHM
the angle HAF is also equal to the angle GHM [CN 1]

Let the angle AHG be added to each,

therefore the angles FAH AHG are equal to the angles KHG , GHM

But the angles FAH , KHG are equal to two right angles, [I 29]

therefore the angles AHG GHM are also equal to two right angles

Thus with a straight line GH , and at the point H on it two straight lines KH HM not lying on the same side make the adjacent angles equal to two right angles,

therefore KH is in a straight line with HM [I 14]

And, since the straight line HG falls upon the parallels AM , FG the alternate angles MHG HGF are equal to one another [I 29]

Let the angle HGL be added to each

therefore the angles MHG HGL are equal to the angles HGF , HGL [CN 2]

But the angles MHG HGL are equal to two right angles, [I 29]

therefore the angles HGF , HGL are also equal to two right angles [CN 1]

Therefore FG is in a straight line with GL [I 14]

And since FA is equal and parallel to HG , [I 34]

and HG to ML also

AF is also equal and parallel to ML , [CN 1 I 30]

and the straight lines AM FL join them (at their extremities), therefore AM FL are also equal and parallel [I 33]

Therefore $AFLM$ is a parallelogram

And since the triangle ABD is equal to the parallelogram FH
and DBC to GM

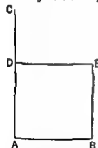
the whole rectilinear figure $ABCD$ is equal to the whole parallelogram $AFLM$

Therefore the parallelogram $AFLM$ has been constructed equal to the given rectilinear figure $ABCD$ in the angle FAM which is equal to the given angle E

Q E F

PROPOSITION 46

On a given straight line to describe a square



Let AB be the given straight line thus it is required to describe a square on the straight line AB

Let AC be drawn at right angles to the straight line AB from the point A on it [I 11] and let AD be made equal to AB

through the point D let DE be drawn parallel to AB
and through the point B let BE be drawn parallel to AD [I 31]

Therefore $ADFB$ is a parallelogram

therefore AB is equal to DE and AD to BE [I 31]

But AB is equal to AD

therefore the four straight lines BA AD DE EB are equal to one another,
therefore the parallelogram $ADEB$ is equilateral

I say next that it is also right angled

For since the straight line AD falls upon the parallels AB DE
the angles BAD ADE are equal to two right angles

But the angle BAD is right,

therefore the angle ADE is also right

And in parallelogrammic areas the opposite sides and angles are equal to one another, [1 34]

therefore each of the opposite angles ABE , BED is also right

Therefore $ADEB$ is right-angled

And it was also proved equilateral

Therefore it is a square, and it is described on the straight line AB $Q E T$

PROPOSITION 47

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle

Let ABC be a right angled triangle having the angle BAC right,

I say that the square on BC is equal to the squares on BA , AC

For let there be described on BC the square $BDEC$, and on BA , AC the squares GB , HC [1 46]

through A let AL be drawn parallel to either BD or CE and let AD , FC be joined

Then since each of the angles BAC , BAG is right it follows that with a straight line BA and at the point A on it the two straight lines AC , AG not lying on the same side make the adjacent angles equal to two right angles,

therefore CA is in a straight line with AG [1 14]

For the same reason

BA is also in a straight line with AH

And since the angle DBC is equal to the angle FBA for each is right

let the angle ABC be added to each

therefore the whole angle DBA is equal to the whole angle FBC [CN 2]

And since DB is equal to BC and FB to BA

the two sides AB , BD are equal to the two sides FB , BC respectively,

and the angle ABD is equal to the angle FBC

therefore the base AD is equal to the base FC ,

and the triangle ABD is equal to the triangle FBC [1 4]

Now the parallelogram BL is double of the triangle ABD for they have the same base BD and are in the same parallels BD , AL [1 41]

And the square GB is double of the triangle FBC

for they again have the same base FB and are in the same parallels FB , GC [1 41]

[But the doubles of equals are equal to one another]

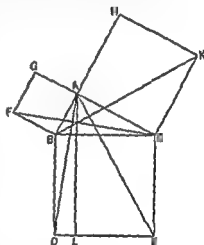
Therefore the parallelogram BL is also equal to the square GB

Similarly if AE , BK be joined

the parallelogram CL can also be proved equal to the square HC ,

therefore the whole square $BDEC$ is equal to the two squares GB , HC [CN 2]

And the square $BDEC$ is described on BC ,



and the squares GB, HC on BA, AC

Therefore the square on the side BC is equal to the squares on the sides BA, AC

Therefore etc

Q E D

PROPOSITION 48

If in a triangle the square on one of the sides be equal to the squares on the remaining two sides of the triangle, the angle contained by the remaining two sides of the triangle is right

For in the triangle ABC let the square on one side BC be equal to the squares on the sides BA, AC ,

I say that the angle BAC is right

For let AD be drawn from the point A at right angles to the straight line BC , let AD be made equal to BA , and let DC be joined

Since DA is equal to AB ,

the square on DA is also equal to the square on AB

Let the square on AC be added to each

therefore the squares on DA, AC are equal to the squares on BA, AC

But the square on DC is equal to the squares on DA, AC , for the angle DAC is right, [I 47]
and the square on BC is equal to the squares on BA, AC , for this is the hypothesis

therefore the square on DC is equal to the square on BC
so that the side DC is also equal to BC

And since DA is equal to AB ,

and AC is common,

the two sides DA, AC are equal to the two sides BA, AC

and the base DC is equal to the base BC

therefore the angle DAC is equal to the angle BAC [I 8]

But the angle DAC is right,

therefore the angle BAC is also right

Therefore etc

Q E D



BOOK TWO

DEFINITIONS

1 Any rectangular parallelogram is said to be *contained* by the two straight lines containing the right angle

2 And in any parallelogrammic area let any one whatever of the parallelograms about its diameter with the two complements be called a *gnomon*

BOOK II PROPOSITIONS

PROPOSITION 1

If there be two straight lines and one of them be cut into any number of segments whatever the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments

Let A BC be two straight lines, and let BC be cut at random at the points D E

I say that the rectangle contained by A BC is equal to the rectangle contained by A BD that contained by A DE and that contained by A EC

For let BF be drawn from B at right angles to BC ,

let BG be made equal to A

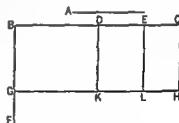
[1 11]

through G let GH be drawn parallel to BC

[1 3]

and through D E C let DK EL , CH be drawn parallel to BG

[1 31]



Then BH is equal to BK DL EH

Now BH is the rectangle A BC for it is contained by GB BC and BG is equal to A

BK is the rectangle A BD for it is contained by GB BD and BG is equal to A

and DL is the rectangle A DE for DK that is BG is equal to A [1 34]

Similarly also EH is the rectangle A EC

Therefore the rectangle A BC is equal to the rectangle A , BD the rectangle A , DE and the rectangle A EC

Therefore etc

Q E D

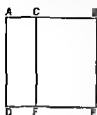
PROPOSITION 2

If a straight line be cut at random the rectangle contained by the whole and both of the segments is equal to the square on the whole

For let the straight line AB be cut at random at the point C

I say that the rectangle contained by AB BC together with the rectangle contained by BA AC is equal to the square on AB

For let the square $ADEB$ be described on AB [I 46] and let CF be drawn through C parallel to either AD or BE [I 31]



Then AE is equal to AF , CE

Now AE is the square on AB ,

AF is the rectangle contained by BA , AC for it is contained by DA , AC , and AD is equal to AB ,

and CE is the rectangle AB , BC for BE is equal to AB

Therefore the rectangle BA , AC together with the rectangle AB , BC is equal to the square on AB

Therefore etc

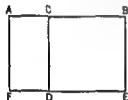
Q E D

PROPOSITION 3

If a straight line be cut at random the rectangle contained by the whole and one of the segments is equal to the rectangle contained by the segments and the square on the aforesaid segment

For let the straight line AB be cut at random at C ,

I say that the rectangle contained by AB , BC is equal to the rectangle contained by AC , CB together with the square on BC



For let the square $CDEB$ be described on CB , [I 46]

let ED be drawn through to F ,

and through A let AF be drawn parallel to either CD or BE [I 31]

Then AE is equal to AD , CE

Now AE is the rectangle contained by AB , BE and BE is equal to BC for it is contained by AB , BE and BE is equal to BC

AD is the rectangle AC , CB for DC is equal to CB ,

and DB is the square on CB

Therefore the rectangle contained by AB , BC is equal to the rectangle contained by AC , CB together with the square on BC

Therefore etc

Q E D

PROPOSITION 4

If a straight line be cut at random the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments

For let the straight line AB be cut at random at C ,

I say that the square on AB is equal to the squares on AC , CB and twice the rectangle contained by AC , CB

For let the square $ADEB$ be described on AB

[I 46]

let BD be joined

through C let CF be drawn parallel to either AD or EB ,

and through G let HK be drawn parallel to either AB or DE [I 31]

Then since CF is parallel to AD and BD has fallen on them

the exterior angle CGB is equal to the interior and opposite angle ADB [I 29]

But the angle ADB is equal to the angle ABD

since the side BA is also equal to AD ,

[I 5]

therefore the angle CGB is also equal to the angle GBC ,

so that the side BC is also equal to the side CG

[1 6]

But CB is equal to GA , and CG to AB ,

[1 34]

therefore GA is also equal to AB ,

therefore $CGAB$ is equilateral

I say next that it is also right-angled

For since CG is parallel to BA

the angles ABC GCB are equal to two right angles

[1 29]

But the angle ABC is right

therefore the angle BCG is also right

so that the opposite angles CGA , GAB are also right

[1 34]

Therefore $CGAB$ is right angled,

and it was also proved equilateral,

therefore it is a square

and it is described on CB

For the same reason

HF is also a square,

and it is described on HG that is AC

[1 31]

Therefore the squares HF AC are the squares on AC , CB

Now, since AG is equal to GE

and AG is the rectangle AC , CB , for GC is equal to CB ,

therefore GE is also equal to the rectangle AC , CB

Therefore AG , GE are equal to twice the rectangle AC , CB

But the squares HF , CA are also the squares on AC , CB therefore the four areas HF CA , AG GE are equal to the squares on AC , CB and twice the rectangle contained by AC , CB

But HF , CA , AG , GE are the whole $ADFB$

which is the square on AB

Therefore the square on AB is equal to the squares on AC , CB and twice the rectangle contained by AC , CB

Therefore etc

Q E D

PROPOSITION 5

If a straight line be cut into equal and unequal segments the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half

For let a straight line AB be cut into equal segments at C and into unequal segments at D ,

I say that the rectangle contained by AD DB together with the square on CD is equal to the square on CB

For let the square $CEFB$ be described on CB ,

[1 46]

and let BE be joined

through D let DG be drawn parallel to either CE or BF

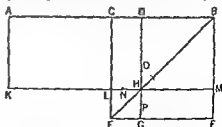
through H again let AM be drawn parallel to either AB or EF

and again through A let AK be drawn parallel to either CL or BM

[1 31]

Then, since the complement CH is equal to the complement HT ,

[1 43]



let DM be added to each.

therefore the whole CM is equal to the whole DF

But CM is equal to AL .

since AC is also equal to CB

therefore AL is also equal to DF

[1 36]

Let CH be added to each.

therefore the whole AM is equal to the gnomon NOP

But AH is the rectangle $AD \cdot DB$, for DH is equal to DB .

therefore the gnomon NOP is also equal to the rectangle $AD \ DB$

Let LG , which is equal to the square on CD , be added to each therefore the gnomon NOP and LG are equal to the rectangle contained by AD , DB and the square on CD

But the gnomon NOP and LG are the whole square $CEFB$ which is described on CB .

therefore the rectangle contained by AD DB together with the square on CD is equal to the square on CB

Therefore etc

Q E D

PROPOSITION 6

If a straight line be bisected and a straight line be added to it in a straight line the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line

For let a straight line AB be bisected at the point C , and let a straight line BD be added to it in a straight line.

I say that the rectangle contained by AD , DB together with the square on CB is equal to the square on CD .

For let the square $CEFD$ be described on CD

ix 467

and let DE be joined.

through the point B let BG be drawn parallel to either EC or DF

through the point H let KM be drawn parallel to either AB or EF

and further through A let AK be drawn parallel to either CL or DM (r 31)

Then since AC is equal to CB ,

AL is also equal to CH

[1 36]

But CH is equal to HF

[I 43]

Therefore AI is also equal to HF

Let CM be added to each,

therefore the whole AM is equal to the gnomon NOP

But AM is the rectangle $AD \ DB$,

for DM is equal to DB .

therefore the gnomon NOP is also equal to the rectangle $AD \cdot DB$

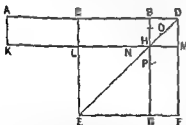
Let LG which is equal to the square on BC be added to each, therefore the rectangle contained by AD DB together with the square on CB is equal to the gnomon NOP and LG .

But the gnomon NOP and LG are the whole square $CEFD$ which is described on CD .

therefore the rectangle contained by AD , DB together with the square on CB is equal to the square on CD

Therefore etc

Q E D



PROPOSITION 7

If a straight line be cut at random the square on the whole and that on one of the segments both together are equal to twice the rectangle contained by the whole and the said segment and the square on the remaining segment

For let a straight line AB be cut at random at the point C ,

I say that the squares on AB , BC are equal to twice the rectangle contained by AB , BC and the square on CA

For let the square $ADEB$ be described on AB , [I 46]
and let the figure be drawn

Then since AG is equal to GE [I 43] let CF be added to each,

therefore the whole AF is equal to the whole CE

Therefore AF , CE are double of AG

But AF , CE are the gnomon KLM and the square CF
therefore the gnomon KLM and the square CF are double of AG

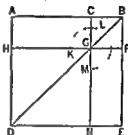
But twice the rectangle AB , BC is also double of AG
for BF is equal to BC ,
therefore the gnomon KLM and the square CF are equal to twice the rectangle AB , BC

Let DG which is the square on AC , be added to each,
therefore the gnomon KLV and the squares BG , GD are equal to twice the rectangle contained by AB , BC and the square on AC

But the gnomon KLM and the squares BG , GD are the whole $ADEB$ and CF ,

which are squares described on AB , BC ,
therefore the squares on AB , BC are equal to twice the rectangle contained by AB , BC together with the square on AC

Therefore etc



Q E D

PROPOSITION 8

If a straight line be cut at random four times the rectangle contained by the whole and one of the segments together with the square on the remaining segment is equal to the square described on the whole and the aforesaid segment as on one straight line

For let a straight line AB be cut at random at the point C

I say that four times the rectangle contained by AB , BC together with the square on AC is equal to the square described on AB , BC as on one straight line

For let [the straight line] BD be produced in a straight line [with AB] and let BD be made equal to CB ,

let the square $AEFD$ be described on AD and let the figure be drawn

Then since CB is equal to BD while CB is equal to GK , and BD to KN ,
therefore GK is also equal to KN

For the same reason

QR is also equal to RP

And, since BC is equal to BD and GK to KN

therefore CK is also equal to KN and GR to RN [I 36]

But CK is equal to RN for they are complements of the parallelogram CP [I 43]

therefore AD is also equal to GR

therefore the four areas DK CK GR RN are equal to one another
Therefore the four are quadruple of CA

Again since CB is equal to BD

while BD is equal to BA that is CG

and CB is equal to GK that is GQ

therefore CG is also equal to GQ

And since CG is equal to GQ and QR to RP ,

AG is also equal to MQ and QL to RF [I 36]

But MQ is equal to QL for they are complements of the parallelogram ML [I 43]

therefore AG is also equal to RF ,

therefore the four areas AG MQ QL RF are equal to one another

Therefore the four are quadruple of AG

But the four areas CA AD GR RN were proved to be quadruple of CK
therefore the eight areas which contain the gnomon STU , are quadruple of AA

Now, since AA is the rectangle AB BD for BK is equal to BD

therefore four times the rectangle AB BD is quadruple of AK

But the gnomon STU was also proved to be quadruple of AK

therefore four times the rectangle AB , BD is equal to the gnomon STU

Let OH which is equal to the square on AC be added to each

therefore four times the rectangle AB BD together with the square on AC is equal to the gnomon STU and OH

But the gnomon STU and OH are the whole square $A E F D$

which is described on AD

therefore four times the rectangle AB BD together with the square on AC is equal to the square on AD

But BD is equal to BC

therefore four times the rectangle contained by AB BC together with the square on AC is equal to the square on AD that is to the square described on AB and BC as on one straight line

Therefore etc

Q E D

PROPOSITION 9

If a straight line be cut into equal and unequal segments the squares on the unequal segments of the whole are double of the square on the half and of the square on the straight line between the points of section

For let a straight line AB be cut into equal segments at C and into unequal segments at D

I say that the squares on AD DB are double of the squares on AC CD

For let CE be drawn from C at right angles to AB and let it be made equal to either AC or CB

let EA EB be joined

let DF be drawn through D parallel to EC

and FG through F parallel to AB

and let AF be joined

Then, since AC is equal to CF

the angle EAC is also equal to the angle AEC

And since the angle at C is right

the remaining angles EAC AEC are equal to one right angle [1 32]

And they are equal,

therefore each of the angles CEA , CAE is half a right angle

For the same reason

each of the angles CFB , EBC is also half a right angle

therefore the whole angle AEB is right

And since the angle GEF is half a right angle

and the angle EGF is right for it is equal to the interior and opposite angle ECB

the remaining angle EFG is half a right angle

therefore the angle GEF is equal to the angle EFG ,

so that the side FG is also equal to GF

[1 29]

[1 32]

[1 6]

Again since the angle at B is half a right angle

and the angle FDB is right, for it is again equal to the interior and opposite angle ECB ,

the remaining angle BFD is half a right angle

therefore the angle at B is equal to the angle DFB ,

so that the side FD is also equal to the side DB

[1 29]

[1 32]

[1 6]

Now since AC is equal to CE

the square on AC is also equal to the square on CE

therefore the squares on AC CE are double of the square on AC

But the square on EA is equal to the squares on AC CE for the angle ACE is right

[1 47]

therefore the square on EA is double of the square on AC

Again since EG is equal to GF

the square on EG is also equal to the square on GF ,

therefore the squares on EG GF are double of the square on GF

But the square on EF is equal to the squares on EG GF

therefore the square on EF is double of the square on GF

But GF is equal to CD

[1 34]

therefore the square on EF is double of the square on CD

But the square on EA is also double of the square on AC

therefore the squares on AE , EF are double of the squares on AC , CD

And the square on AF is equal to the squares on AE EF for the angle AEF is right

[1 47]

therefore the square on AF is double of the squares on AC CD

But the squares on AD , DF are equal to the square on AF for the angle at D is right,

[1 47]

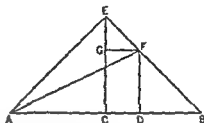
therefore the squares on AD DF are double of the squares on AC CD

And DF is equal to DB

therefore the squares on AD , DB are double of the squares on AC CD

Therefore etc

Q E D

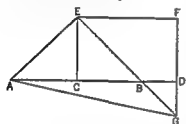


PROPOSITION 10

If a straight line be bisected and a straight line be added to it in a straight line the square on the whole with the added straight line and the square on the added straight line both together are double of the square on the half and of the square described on the straight line made up of the half and the added straight line as on one straight line

For let a straight line AB be bisected at C , and let a straight line BD be added to it in a straight line.

I say that the squares on AD , DB are double of the squares on AC , CD .



For let CE be drawn from the point C at right angles to AB [I 11] and let it be made equal to either AC or CB [I 3].

let EA EB be joined

through E let EF be drawn parallel to AD
and through D let FD be drawn parallel to
 CE [31]

Then since a straight line EF falls on the parallel straight lines EC FD

the angles CEF ETD are equal to two right angles [I 29]

therefore the angles FEB EFD are less than two right angles

But straight lines produced from angles less then two right angles meet

[1 Post 5]

therefore $EB \parallel FD$ if produced in the direction B, D will meet

Let them be produced and meet at G

and let AG be joined

Then since AC is equal to CE ,

the angle EAC is also equal to the angle AEC . [I 5]

and the angle at C is right

therefore each of the angles EAC AEC is half a right angle [I 32]

For the same reason

each of the angles CLB EBC is also half a right angle

therefore the angle AEB is right

And since the angle EBC is half a right angle

the angle DBG is also half a right angle [I 15]

But the angle BDG is also right.

for it is equal to the angle DCE they being alternate, [I 29]

therefore the remaining angle DGB is half a right angle, [I 32]

therefore the angle DGB is equal to the angle DBG

so that the side BD is also equal to the side GD [1 6]

Again, since the angle EGF is half a right angle

and the angle at F is right for it is equal to the opposite angle the angle at C [1 34]

the remaining angle $FEG =$ half a right angle

therefore the angle $\angle GF$ is equal to the angle $\angle FEG$

so that the side GF is also equal to the side EF [1 6]

Now since the square on EC is equal to the square on CA

the squares on EC CA are double of the square on CA

But the square on EA is equal to the squares on EC CA , [I 47]

therefore the square on EA is double of the square on AC [C.N. 1]
 Again since FG is equal to EF ,

the square on FG is also equal to the square on FE ,

therefore the squares on GF, FE are double of the square on EF

But the square on EG is equal to the squares on GF, FE , [1 47]

therefore the square on EG is double of the square on EF

And EF is equal to CD [1 34]

therefore the square on EG is double of the square on CD

But the square on EA was also proved double of the square on AC ,

therefore the squares on AF, EG are double of the squares on AC, CD

And the square on AG is equal to the squares on AE, EG [1 47]

therefore the square on AG is double of the squares on AC, CD

But the squares on AD, DG are equal to the square on AG , [1 47]

therefore the squares on AD, DG are double of the squares on AC, CD

And DG is equal to DB

therefore the squares on AD, DB are double of the squares on AC, CD

Therefore etc

Q E D

PROPOSITION 11

To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment

Let AB be the given straight line,
 thus it is required to cut AB so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment

For let the square $ABDC$ be described on AB , [1 46]

let AC be bisected at the point E and let BF be joined

let CA be drawn through to F and let EF be made equal to BE

let the square FHI be described on AF and let GH be drawn through to K

I say that AB has been cut at H so as to make the rectangle contained by AB, BH equal to the square on AH

For since the straight line AC has been bisected at E , and FA is added to it

the rectangle contained by CF, FA together with the square on AE is equal to the square on EF [1 6]

But EF is equal to EB

therefore the rectangle CF, FA together with the square on AE is equal to the square on EB

But the squares on BA, AE are equal to the square on EB for the angle at A is right [1 47]

therefore the rectangle CF, FA together with the square on AE is equal to the squares on BA, AE

Let the square on AE be subtracted from each

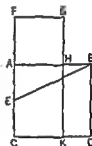
therefore the rectangle CF, FA which remains is equal to the square on AB

Now the rectangle CF, FA is FK for AF is equal to FG ,

and the square on AB is AD

therefore FK is equal to AD

Let AK be subtracted from each,



therefore FH which remains is equal to HD

And HD is the rectangle $AB \cdot BH$, for AB is equal to BD ,

and FH is the square on AH

therefore the rectangle contained by $AB \cdot BH$ is equal to the square on HA
therefore the given straight line AB has been cut at H so as to make the rectangle contained by AB, BH equal to the square on HA □ E F

PROPOSITION 12

In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle

Let ABC be an obtuse-angled triangle having the angle BAC obtuse, and let BD be drawn from the point B perpendicular to CA produced,

I say that the square on BC is greater than the squares on BA, AC by twice the rectangle contained by $CA \cdot AD$

For, since the straight line CD has been cut at random at the point A ,
the square on DC is equal to the squares on $CA \cdot AD$
and twice the rectangle contained by CA, AD [I 4]

Let the square on DB be added to each
therefore the squares on CD, DB are equal to the squares on $CA \cdot AD \cdot DB$ and twice the rectangle CA, AD

But the square on CB is equal to the squares on CD, DB , for the angle at D is right, [I 47]

and the square on AB is equal to the squares on $AD \cdot DB$ [I 47]
therefore the square on CB is equal to the squares on $CA \cdot AB$ and twice the rectangle contained by $CA \cdot AD$,

so that the square on CB is greater than the squares on $CA \cdot AB$ by twice the rectangle contained by $CA \cdot AD$

Therefore etc

Q E D

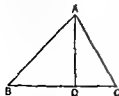
PROPOSITION 13

In acute angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle

Let ABC be an acute-angled triangle having the angle at B acute, and let AD be drawn from the point A perpendicular to BC ,

I say that the square on AC is less than the squares on $CB \cdot BA$ by twice the rectangle contained by $CB \cdot BD$

For since the straight line CB has been cut at random at D
the squares on $CB \cdot BD$ are equal to twice the rectangle contained by $CB \cdot BD$ and the square on DC



Let the square on DA be added to each,
therefore the squares on CB BD DA are equal to twice the rectangle contained by CB , BD and the squares on AD DC

But the square on AB is equal to the squares on BD , DA , for the angle at D is right, [1 47]

and the square on AC is equal to the squares on AD , DC ,
therefore the squares on CB , BA are equal to the square on AC and twice the rectangle CB , BD ,
so that the square on AC alone is less than the squares on CB , BA by twice the rectangle contained by CB BD

Therefore etc

Q E D

PROPOSITION 14

To construct a square equal to a given rectilinear figure

I let A be the given rectilinear figure,
thus it is required to construct a square equal to the rectilinear figure A

For let there be constructed the rectangular parallelogram BD equal to the rectilinear figure A

[1 45]

Then, if BE is equal to ED , that which was enjoined will have been done for a square BD

has been constructed equal to the rectilinear figure A

But if not one of the straight lines BE ED is greater

Let BE be greater, and let it be produced to F ,

let EF be made equal to ED and let BF be bisected at G

With centre G and distance one of the straight lines GB GF let the semi circle BHF be described let DE be produced to H and let GH be joined

Then since the straight line BF has been cut into equal segments at G and into unequal segments at E

the rectangle contained by BE EF together with the square on EG is equal to the square on GF [11 5]

But GF is equal to GH ,
therefore the rectangle BE , EF together with the square on GE is equal to the square on GH

But the squares on HE EG are equal to the square on GH , [1 47]
therefore the rectangle BE , EF together with the square on GE is equal to the squares on HE , EG

Let the square on GE be subtracted from each
therefore the rectangle contained by BE , EF which remains is equal to the square on EH

But the rectangle BE EF is BD for EF is equal to LD

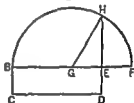
therefore the parallelogram BD is equal to the square on HE

And BD is equal to the rectilinear figure A

Therefore the rectilinear figure A is also equal to the square which can be described on EH

Therefore a square namely that which can be described on EH , has been constructed equal to the given rectilinear figure A

Q E F



BOOK THREE

DEFINITIONS

1 *Equal circles* are those the diameters of which are equal or the radii of which are equal

2 A straight line is said to *touch a circle* which, meeting the circle and being produced, does not cut the circle

3 *Circles* are said to *touch one another* which meeting one another do not cut one another

4 In a circle straight lines are said to be *equally distant from the centre* when the perpendiculars drawn to them from the centre are equal

■ And that straight line is said to be at a *greater distance* on which the greater perpendicular falls

6 A *segment of a circle* is the figure contained by a straight line and a circumference of a circle

7 An *angle of a segment* is that contained by a straight line and a circumference of a circle

8 An *angle in a segment* is the angle which when a point is taken on the circumference of the segment and straight lines are joined from it to the extremities of the straight line which is the *base of the segment*, is contained by the straight lines so joined

9 And when the straight lines containing the angle cut off a circumference, the angle is said to *stand upon* that circumference

10 A *sector of a circle* is the figure which when an angle is constructed at the centre of the circle, is contained by the straight lines containing the angle and the circumference cut off by them

11 *Similar segments of circles* are those which admit equal angles or in which the angles are equal to one another

BOOK III PROPOSITIONS

PROPOSITION 1

To find the centre of a given circle

Let ABC be the given circle

thus it is required to find the centre of the circle ABC

Let a straight line AB be drawn through it at random, and let it be bisected at the point D

from D let DC be drawn at right angles to AB and let it be drawn through to E let CE be bisected at F

I say that F is the centre of the circle ABC

For suppose it is not but if possible let G be the centre

and let G 1, GD GB be joined

Then since AD is equal to DB and DG is common
the two sides AD DG are equal to the two sides BD DG respectively,
and the base GA is equal to the base GB , for they are
radii,

therefore the angle ADG is equal to the angle GDB

[1 8]

But when a straight line set up on a straight line
makes the adjacent angles equal to one another each of
the equal angles is right, [1 Def 10]

therefore the angle GDB is right

But the angle FDB is also right
therefore the angle FDB is equal to the angle GDB ,
the greater to the less which is impossible

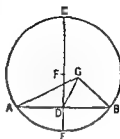
Therefore G is not the centre of the circle ABC

Similarly we can prove that neither is any other point except F

Therefore the point F is the centre of the circle ABC

PROPOSITION From this it is manifest that if in a circle a straight line cut a
straight line into two equal parts and at right angles, the centre of the circle is
on the cutting straight line

Q E F



PROPOSITION 2

If on the circumference of a circle two points be taken at random, the straight line
joining the points will fall within the circle

Let ABC be a circle and let two points A B be taken at random on its cir-
cumference

I say that the straight line joined from A to B will fall within the circle

For suppose it does not but if possible let it fall outside as AEB ,
let the centre of the circle ABC be taken [111 1], and let it be D , let DA DB be
joined and let DFE be drawn through

Then since DA is equal to DB

the angle DAE is also equal to the angle DBE [1 5]

And since one side AEB of the triangle DAE is pro-
duced

the angle DEB is greater than the angle DAE [1 16]

But the angle DAF is equal to the angle DBF

therefore the angle DEB is greater than the angle
 DBE

And the greater angle is subtended by the greater
side,

[1 19]

therefore DB is greater than DE

But DB is equal to DF

therefore DF is greater than DE ,

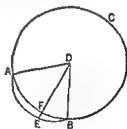
the less than the greater which is impossible

Therefore the straight line joined from A to B will not fall outside the circle

Similarly we can prove that neither will it fall on the circumference itself
therefore it will fall within

Therefore etc

Q E D



PROPOSITION 3

If in a circle a straight line through the centre bisect a straight line not through the centre, it also cuts it at right angles and if it cut it at right angles it also bisects it

Let ABC be a circle and in it let a straight line CD through the centre bisect a straight line AB not through the centre at the point F ,

I say that it also cuts it at right angles

For let the centre of the circle ABC be taken and let it be E let EA , EB be joined



Then, since AF is equal to FB and FE is common,

two sides are equal to two sides,
and the base EA is equal to the base EB ,

therefore the angle AFE

is equal to the angle BFE [I 8]

But when a straight line set up on a straight line makes the adjacent angles equal to one another each of the equal angles is right, [I Def 10]

therefore each of the angles AFE , BFE is right

Therefore CD which is through the centre and bisects AB which is not through the centre also cuts it at right angles

Again let CD cut AB at right angles

I say that it also bisects it that is that AF is equal to FB

For with the same construction

since EA is equal to EB ,

the angle EAF is also equal to the angle EBF [I 5]

But the right angle AFE is equal to the right angle BFE therefore EAF , EBF are two triangles having two angles equal to two angles and one side equal to one side namely EF which is common to them and subtends one of the equal angles,

therefore they will also have the remaining sides equal to the remaining sides [I 26]

therefore AF is equal to FB

Therefore etc

Q E D

PROPOSITION 4

If in a circle two straight lines cut one another which are not through the centre they do not bisect one another

Let $ABCD$ be a circle and in it let the two straight lines AC , BD which are not through the centre cut one another at E

I say that they do not bisect one another

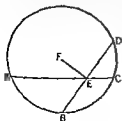
For if possible let them bisect one another so that AE is equal to EC and BE to ED ,

let the centre of the circle $ABCD$ be taken [III 1] and let it be F let FE be joined

Then since a straight line FE through the centre bisects a straight line AC not through the centre

it also cuts it at right angles [III 3]

therefore the angle FEA is right



Again, since \square straight line FE bisects a straight line BD ,
 it also cuts it at right angles,
 therefore the angle FEB is right

[III 3]

But the angle FEA was also proved right,
 therefore the angle FEA \square equal to the angle FEB the less to the greater
 which is impossible

Therefore AC , BD do not bisect one another

Therefore etc

Q E D

PROPOSITION 5

If two circles cut one another they will not have the same centre

For let the circles ABC , CDG cut one another at the points B , C ,

I say that they will not have the same centre

For if possible let it be E , let EC be joined, and let EFG be drawn through
 at random

Then, since the point E is the centre of the circle
 ABC ,

EC \square equal to EF [I Def 15]

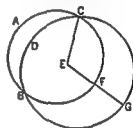
Again, since the point E \square the centre of the circle
 CDG ,

EC \square equal to EG

But EC was proved equal to EF also
 therefore EF is also equal to EG , the less to the
 greater which is impossible

Therefore the point E is not the centre of the circles ABC , CDG

Therefore etc



Q E D

PROPOSITION 6

If two circles touch one another they will not have the same centre

For let the two circles ABC , CDE touch one another at the point C

I say that they will not have the same centre

For if possible let it be F let FC be joined and let FEB be drawn through
 at random

Then since the point F is the centre of the circle
 ABC

FC is equal to FB

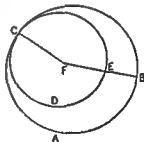
Again since the point F \square the centre of the
 circle CDE ,

FC \square equal to FE

But FC was proved equal to FB
 therefore FE is also equal to FB the less to the
 greater which \square impossible

Therefore F is not the centre of the circles ABC , CDE

Therefore etc



Q E D

PROPOSITION 7

*If on the diameter of a circle a point be taken which is not the centre of the circle
 and from the point straight lines fall upon the circle that will be greatest on which
 the centre is the remainder of the same diameter will be least and of the rest the*

nearer to the straight line through the centre is always greater than the more remote and only two equal straight lines will fall from the point on the circle one on each side of the least straight line

Let $ABCD$ be a circle and let AD be a diameter of it, on AD let a point F be taken which is not the centre of the circle let E be the centre of the circle

and from F let straight lines FB , FC , FG fall upon the circle $ABCD$,
I say that FA is greatest FD is least, and of the rest FB is greater than FC , and FC than FG

For let BE CE GE be joined

Then, since in any triangle two sides are greater than the remaining one

[I 20]

EB EF are greater than BF

But AE is equal to BE

therefore AF is greater than BF

Again since BE is equal to CE and FE is common

the two sides BE EF are equal to the two sides CE EF

But the angle BCE is also greater than the angle CEF

therefore the base BF is greater than the base CF

[I 24]

For the same reason

CF is also greater than FG

Again since GF , FE are greater than EG

and EG is equal to ED

GF , FE are greater than ED

Let EF be subtracted from each,

therefore the remainder GF is greater than the remainder FD

Therefore FA is greatest FD is least, and FB is greater than FC and FC than FG

I say also that from the point F only two equal straight lines will fall on the circle $ABCD$, one on each side of the least FD

For on the straight line EF and at the point E on it let the angle FEH be constructed equal to the angle GEF [I 23] and let FH be joined

Then since GE is equal to EH

and EF is common

the two sides GE EF are equal to the two sides HE EF

and the angle GEF is equal to the angle HEF

therefore the base GF is equal to the base FH

[I 4]

I say again that another straight line equal to FG will not fall on the circle from the point F

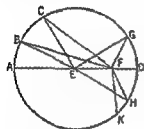
For if possible let FK so fall

Then since FA is equal to FG and FH to FG

FK is also equal to FH

the nearer to the straight line through the centre being thus equal to the more remote which is impossible

Therefore another straight line equal to GF will not fall from the point F upon the circle



therefore only one straight line will so fall

Therefore etc

Q. E. D.

PROPOSITION 8

If a point be taken outside a circle and from the point straight lines be drawn through to the circle one of which is through the centre and the others are drawn at random, then of the straight lines which fall on the concave circumference that through the centre is greatest while of the rest the nearer to that through the centre is always greater than the more remote but of the straight lines falling on the convex circumference that between the point and the diameter is least while of the rest the nearer to the least is always less than the more remote and only two equal straight lines will fall on the circle from the point one on each side of the least

Let ABC be a circle and let a point D be taken outside ABC , let there be drawn through from it straight lines DA, DE, DF, DC , and let DA be through the centre,

I say that, of the straight lines falling on the concave circumference $AEFC$ the straight line DA through the centre is greatest

while DE is greater than DF and DF than DC ,
but of the straight lines falling on the convex circumference $HLKG$ the straight line DG between the point and the diameter AG is least and the nearer to the least DG is always less than the more remote namely DK than DL and DI than DH

For let the centre of the circle ABC be taken [III 1] and let it be M , let ME, MF, MC, MA, ML, MH be joined

Then since AM is equal to EM , let MD be added to each

therefore AD is equal to $FM + MD$

But EM, MD are greater than ED

therefore AD is also greater than ED

Again since ME is equal to MF

and MD is common

therefore $EM + MD$ are equal to $FM + MD$

and the angle EMD is greater than the angle FMD

therefore the base ED is greater than the base FD

[I 24]

Similarly we can prove that FD is greater than CD therefore DA is greatest while DE is greater than DF and DF than DC

Next since MA, AD are greater than MD

[I 20]

and MG is equal to MA

therefore the remainder AD is greater than the remainder GD

so that GD is less than AD

And since on MD one of the sides of the triangle MLD two straight lines MA, AD were constructed meeting within the triangle

therefore MA, AD are less than $ML + LD$

[I 21]

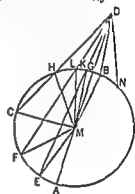
and MA is equal to ML ,

therefore the remainder DA is less than the remainder DL

Similarly we can prove that DL is also less than DH

therefore DG is least while DK is less than DL and DL than DH

I say also that only two equal straight lines will fall from the point D on the



circle, one on each side of the least DG

On the straight line MD , and at the point M on it, let the angle DMB be constructed equal to the angle KMD , and let DB be joined

Then, since MA is equal to MB ,

and MD is common

the two sides KM , MD are equal to the two sides BM , MD respectively,

and the angle KMD is equal to the angle BMD ,

therefore the base DA is equal to the base DB [I 4]

I say that no other straight line equal to the straight line DA will fall on the circle from the point D

For, if possible let a straight line so fall and let it be DN

Then, since DK is equal to DN ,

while DK is equal to DB

DB is also equal to DN ,

that is, the nearer to the least DG equal to the more remote which was proved impossible

Therefore no more than two equal straight lines will fall on the circle ABC from the point D , one on each side of DG the least

Therefore etc

Q E D

PROPOSITION 9

If a point be taken within a circle, and more than two equal straight lines fall from the point on the circle, the point taken is the centre of the circle

Let ABC be a circle and D a point within it, and from D let more than two equal straight lines, namely DA , DB , DC , fall on the circle ABC ,

I say that the point D is the centre of the circle ABC

For let AB , BC be joined and bisected at the points E , F , and let ED , FD be joined and drawn through to the points G , K , H , L

Then since AE is equal to EB and ED is common

the two sides AE , ED are equal to the two sides BE , ED

and the base DA is equal to the base DB

therefore the angle AED is equal to the angle BED [I 8]

Therefore each of the angles AED , BED is right [I Def 10]

therefore GK cuts AB into two equal parts and at right angles

And since if in a circle a straight line cut a straight line into two equal parts and at right angles the centre of the circle is on the cutting straight line [III 1 Por]

the centre of the circle is on GK

For the same reason

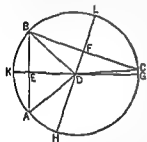
the centre of the circle ABC is also on HL

And the straight lines GK , HL have no other point common but the point D

therefore the point D is the centre of the circle ABC

Therefore etc

Q E D



PROPOSITION 10

A circle does not cut a circle at more points than two

For if possible let the circle ABC cut the circle DEF at more points than two namely B, C, F, H ,

let BH, BG be joined and bisected at the points K, L ,
and from K, L let AC, LM be drawn at right angles to BH, BG and carried through to the points A, E

Then since in the circle ABC a straight line AC cuts a straight line BH into two equal parts and at right angles

the centre of the circle ABC is on AC

[III 1, Por]

Again since in the same circle ABC a straight line LM cuts a straight line BG into two equal parts and at right angles

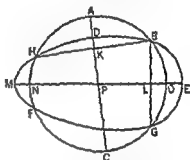
the centre of the circle ABC is on LM

But it was also proved to be on AC , and
the straight lines AC, LM meet at no point
except at P ,
therefore the point P is the centre of the
circle ABC

Similarly we can prove that P is also the centre of the circle DEF ,
therefore the two circles ABC, DEF which cut one another have the same centre P which is impossible

Therefore etc

[III 5]
Q E D



PROPOSITION 11

If two circles touch one another internally and their centres be taken the straight line joining their centres, if it be also produced will fall on the point of contact of the circles

For let the two circles ABC, ADE touch one another internally at the point A and let the centre F of the circle ABC and the centre G of ADE , be taken,

I say that the straight line joined from G to F
and produced will fall on A

For suppose it does not but if possible let it fall as FGH and let AF, AG be joined

Then since AG, GF are greater than FA that
is than FH

let FG be subtracted from each
therefore the remainder AG is greater than the
remainder GH

But AG is equal to GD ,

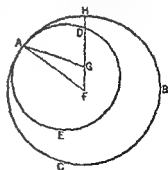
therefore GD is also greater than GH
the less than the greater which is impossible

Therefore the straight line joined from F to
 G will not fall outside,

therefore it will fall at A on the point of contact

Therefore etc

Q E D



PROPOSITION 12

If two circles touch one another externally, the straight line joining their centres will pass through the point of contact

For let the two circles ABC , ADE touch one another externally at the point A , and let the centre F of ABC and the centre G of ADE , be taken

I say that the straight line joined from F to G will pass through the point of contact at A

For suppose it does not but, if possible, let it pass as $FCDG$, and let AF , AG be joined

Then since the point F is the centre of the circle ABC ,

FA is equal to FC

Again, since the point G is the centre of the circle ADE

GA is equal to GD

But FA was also proved equal to FC , therefore FA , AG are equal to FC , GD so that the whole FG is greater than FA , AG ,

but it is also less [I 20] which is impossible

Therefore the straight line joined from F to G will not fail to pass through the point of contact at A ,

therefore it will pass through it

Therefore etc

Q E D

PROPOSITION 13

A circle does not touch a circle at more points than one whether it touch it internally or externally

For, if possible let the circle $ABDC$ touch the circle $EBFD$ first internally, at more points than one namely D , B

Let the centre G of the circle $ABDC$ and the centre H of $EBFD$ be taken

Therefore the straight line joined from G to H will fall on B , D [III 11]

Let it so fall as $BGHD$

Then since the point G is the centre of the circle $ABDC$

BG is equal to GD ,

therefore BG is greater than HD

therefore BH is much greater than HD

Again since the point H is the centre of the circle $EBFD$

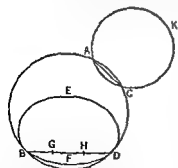
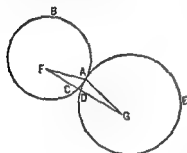
BH is equal to HD ,

but it was also proved much greater than it which is impossible

Therefore a circle does not touch a circle internally at more points than one

I say further that neither does it so touch it externally

For if possible let the circle ACK touch the circle $ABDC$ at more points than one namely A , C ,



and let AC be joined

Then, since on the circumference of each of the circles $ABDC$, ACK two points A , C have been taken at random, the straight line joining the points will fall within each circle, [III 2]
but it fell within the circle $ABDC$ and outside ACK [III Def 3] which is absurd

Therefore a circle does not touch a circle externally at more points than one
And it was proved that neither does it so touch it internally

Therefore etc

Q E D

PROPOSITION 14

In a circle equal straight lines are equally distant from the centre and those which are equally distant from the centre are equal to one another

Let $ABDC$ be a circle and let AB , CD be equal straight lines in it,

I say that AB , CD are equally distant from the centre

For let the centre of the circle $ABDC$ be taken [III 1] and let it be E , from E let EF , EG be drawn perpendicular to AB , CD and let AF , EC be joined

Then since a straight line EF through the centre cuts a straight line AB not through the centre at right angles it also bisects it [III 3]

Therefore AF is equal to FB ,

therefore AB is double of AF

For the same reason

CD is also double of CG ,

and AB is equal to CD

therefore AF is also equal to CG

And since AE is equal to EC ,

the square on AE is also equal to the square on EC

But the squares on AF , FE are equal to the square on AE for the angle at F is right,

and the squares on EG , GC are equal to the square on EC , for the angle at G is right, [I 47]

therefore the squares on AF , FE are equal to the squares on CG , GF ,
of which the square on AF is equal to the square on CG for AF is equal to CG ,

therefore the square on FE which remains is equal to the square on EG

therefore FE is equal to EG

But in a circle straight lines are said to be equally distant from the centre when the perpendiculars drawn to them from the centre are equal [III Def 4],

therefore AB , CD are equally distant from the centre

Next let the straight lines AB , CD be equally distant from the centre that is let EF be equal to EG

I say that AB is also equal to CD

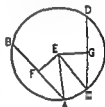
For with the same construction we can prove similarly, that AB is double of AF and CD of CG

And since AE is equal to CE

the square on AE is equal to the square on CE

But the squares on EF , FA are equal to the square on AE and the squares on EG , GC equal to the square on CE [I 47]

Therefore the squares on EF , FA are equal to the squares on EG , GC ,
of which the square on EF is equal to the square on EG for EF is equal to EG ,



therefore the square on AF which remains is equal to the square on CG ,
 therefore AF is equal to CG
 And AB is double of AF , and CD double of CG
 therefore AB is equal to CD

Therefore etc

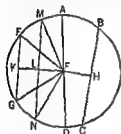
Q E D

PROPOSITION 15

Of straight lines in a circle the diameter is greatest and of the rest the nearer to the centre is always greater than the more remote

Let $ABCD$ be a circle, let AD be its diameter and E the centre, and let BC be nearer to the diameter AD , and FG more remote

I say that AD is greatest and BC greater than FG
 For from the centre E let EH , EA be drawn perpendicular to BC FG



Then since BC is nearer to the centre and FG more remote EK is greater than EH [III Def 5]

Let EL be made equal to EH through L let IM be drawn at right angles to EA and carried through to N , and let ME EN , FE , EG be joined

Then since EH is equal to EL

BC is also equal to MN

[III 14]

Again since AE is equal to EM and ED to EN ,

AD is equal to ME , EN

But ME , EN are greater than MN ,

[I 20]

and MN is equal to BC ,

therefore AD is greater than BC

And since the two sides ME EN are equal to the two sides FE EG

and the angle MEN greater than the angle FEG

therefore the base MN is greater than the base FG

[I 24]

But MN was proved equal to BC

Therefore the diameter AD is greatest and BC greater than FG

Therefore etc

Q E D

PROPOSITION 16

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle and into the space between the straight line and the circumference another straight line cannot be interposed further the angle of the semi circle is greater and the remaining angle less, than any acute rectilineal angle

Let ABC be a circle about D as centre and AB as diameter,

I say that the straight line drawn from A at right angles to AB from its extremity will fall outside the circle

For suppose it does not but if possible let it fall within as CA and let DC be joined

Since DA is equal to DC

the angle DAC is also equal to the angle ACD

[I 5]

But the angle DAC is right

therefore the angle ACD is also right

thus, in the triangle ACD the two angles DAC ACD are equal to two right angles which is impossible

[I 17] /

Therefore the straight line drawn from the point *A* at right angles to *BA* will not fall within the circle

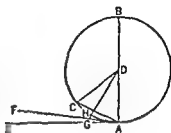
Similarly we can prove that neither will it fall on the circumference,

therefore it will fall outside

Let it fall as *AE*,

I say next that into the space between the straight line *AE* and the circumference *CHA* another straight line cannot be interposed

For if possible let another straight line be so interposed, as *FA* and let *DG* be drawn from the point *D* perpendicular to *FA*



Then, since the angle *AGD* is right,

and the angle *DAG* is less than a right angle,

AD is greater than *DG*

[10]

But *DA* is equal to *DH*,

therefore *DH* is greater than *DG*, the less than the greater which is impossible

Therefore another straight line cannot be interposed into the space between the straight line and the circumference

I say further that the angle of the semicircle contained by the straight line *BA* and the circumference *CHA* is greater than any acute rectilinear angle and the remaining angle contained by the circumference *CHA* and the straight line *AE* is less than any acute rectilinear angle

For if there is any rectilinear angle greater than the angle contained by the straight line *BA* and the circumference *CHA* and any rectilinear angle less than the angle contained by the circumference *CHA* and the straight line *AE*, then into the space between the circumference and the straight line *AE* a straight line will be interposed such as will make an angle contained by straight lines which is greater than the angle contained by the straight line *BA* and the circumference *CHA*, and another angle contained by straight lines which is less than the angle contained by the circumference *CHA* and the straight line *AE*

But such a straight line cannot be interposed

therefore there will not be any acute angle contained by straight lines which is greater than the angle contained by the straight line *BA* and the circumference *CHA* nor yet any acute angle contained by straight lines which is less than the angle contained by the circumference *CHA* and the straight line *AE* —

PROBEM From this it is manifest that the straight line drawn at right angles to the diameter of a circle from its extremity touches the circle Q E D

PROPOSITION 17

From a given point to draw a straight line touching a given circle

Let *A* be the given point and *BCD* the given circle thus it is required to draw from the point *A* a straight line touching the circle *BCD*

For let the centre *F* of the circle be taken, [III 1]
let *AE* be joined and with centre *E* and distance *EA* let the circle *AFG* be described,

from *D* let *DF* be drawn at right angles to *EA*,

and let *EF* *AB* be joined

I say that AB has been drawn from the point A touching the circle BCD
 For, since E is the centre of the circles BCD AFG

EA is equal to EF , and ED to EB ,

therefore the two sides AE , EB are equal to the two sides FE , ED ,
 and they contain a common angle, the angle at E ,
 therefore the base DF is equal to the base AB ,
 and the triangle DEF is equal to the triangle BEA ,
 and the remaining angles to the remaining angles

[I 4]

therefore the angle EDF is equal to the angle EBA

But the angle EDF is right,

therefore the angle EBA is also right

Now EB is a radius

and the straight line drawn at right angles to the di-
 ameter of a circle from its extremity, touches the
 circle,

[III 16 Por]

therefore AB touches the circle BCD

Therefore from the given point A the straight line AB has been drawn
 touching the circle BCD

Q E F

PROPOSITION 18

If a straight line touch a circle, and a straight line be joined from the centre to the point of contact the straight line so joined will be perpendicular to the tangent

For let a straight line DE touch the circle ABC at the point C , let the centre F of the circle ABC be taken, and let FC be joined from F to C ,

I say that FC is perpendicular to DE

For, if not let FG be drawn from F perpen-
 dicular to DE

Then since the angle FGC is right

the angle FCG is acute

[I 17]

and the greater angle is subtended by the great-
 er side,

[I 19]

therefore FC is greater than FG

But FC is equal to FB ,

therefore FB is also greater than FG

the less than the greater which is impossible

Therefore FG is not perpendicular to DE

Similarly we can prove that neither is any other straight line except FC ,
 therefore FC is perpendicular to DE

Therefore etc

Q E D

PROPOSITION 19

If a straight line touch a circle and from the point of contact a straight line be drawn at right angles to the tangent the centre of the circle will be on the straight line so drawn

For let a straight line DE touch the circle ABC at the point C and from C
 let CA be drawn at right angles to DE

I say that the centre of the circle is on CA

For suppose it is not but if possible let F be the centre

and let CF be joined

Since a straight line DE touches the circle ABC ,
and FC has been joined from the centre to the
point of contact,

FC is perpendicular to DE , [III 18]
therefore the angle FCE is right

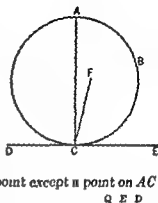
But the angle ACE is also right,
therefore the angle FCE is equal to the angle
 ACE ,

the less to the greater which is impossible

Therefore F is not the centre of the circle ABC

Similarly we can prove that neither is any other point except a point on AC

Therefore etc



PROPOSITION 20

In a circle the angle at the centre is double of the angle at the circumference when
the angles have the same circumference as base

Let ABC be a circle let the angle BEC be an angle at its centre, and the
angle BAC an angle at the circumference and let them have the same circum-
ference BC as base

I say that the angle BEC is double of the angle
 BAC

For let AE be joined and drawn through to F

Then since EA is equal to EB ,
the angle EAB is also equal to the angle EBA , [I 5]
therefore the angles EAB , EBA are double of the
angle EAB

But the angle BEF is equal to the angles EAB
 EBA [I 32]
therefore the angle BEF is also double of the angle
 EAB

For the same reason

the angle FEC is also double of the angle EAC

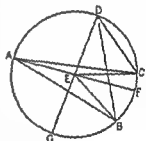
Therefore the whole angle BEC is double of the whole angle BAC

Again let another straight line be inflected and let there be another angle
 BDC , let DE be joined and produced to G

Similarly then we can prove that the angle GEC is double of the angle EDC ,
of which the angle GEB is double of the angle EDB

therefore the angle BEC which remains is double of the angle BDC

Therefore etc



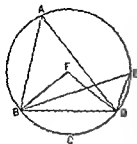
PROPOSITION 21

In a circle the angles in the same segment are equal to one another

Let $ABCD$ be a circle and let the angles BAD BED be angles in the same
segment $BAED$

I say that the angles BAD BED are equal to one another

For let the centre of the circle $ABCD$ be taken and let it be F , let BF , FD
be joined



Now, since the angle BFD is at the centre,
and the angle BAD at the circumference
and they have the same circumference BCD as base
therefore the angle BFD is double of the angle
 BAD [III 20]

For the same reason
the angle BFD is also double of the angle BED
therefore the angle BAD is equal to the angle BED
Therefore etc Q E D

PROPOSITION 22

The opposite angles of quadrilaterals in circles are equal to two right angles

Let $ABCD$ be a circle and let $ABCD$ be a quadrilateral in it,
I say that the opposite angles are equal to two right angles

Let AC BD be joined

Then since in any triangle the three angles are equal
to two right angles [I 32]

the three angles CAB ABC , BCA of the triangle ABC
are equal to two right angles

But the angle CAB is equal to the angle BDC for they
are in the same segment $BADC$ [III 21]

and the angle ACB is equal to the angle ADB for they
are in the same segment $ADCB$

therefore the whole angle ADC is equal to the angles BAC ACB

Let the angle ABC be added to each

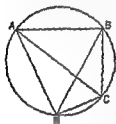
therefore the angles ABC , BAC ACB are equal to the angles ABC ADC

But the angles ABC BAC , ACB are equal to two right angles,

therefore the angles ABC ADC are also equal to two right angles

Similarly we can prove that the angles BAD , DCB are also equal to two
right angles

Therefore etc Q E D



PROPOSITION 23

*On the same straight line there cannot be constructed two similar and unequal
segments of circles on the same side*

For if possible on the same straight line AB let two similar and unequal
segments of circles ACB ADB be constructed on the
same side

let ACD be drawn through and let CB DB be joined

Then since the segment ACB is similar to the seg-
ment ADB ,

and similar segments of circles are those which admit equal angles [III Def 11]
the angle ACB is equal to the angle ADB , the exterior to the interior which is
impossible [I 16]

Therefore etc Q E D



PROPOSITION 21

Similar segments of circles on equal straight lines are equal to one another

For let AEB , CFD be similar segments of circles on equal straight lines AB , CD ,

I say that the segment AEB is equal to the segment CFD

I or, if the segment AEB be applied to CFD and if the point A be placed on C and the straight line AB on CD ,

the point B will also coincide with the point D , because AB is equal to CD ,



and AB coinciding with CD ,

the segment AEB will also coincide with CFD

For if the straight line AB coincide with CD but the segment AEB do not coincide with CFD

it will either fall within it or outside it or it will fall away as CGD and a circle cuts a circle at more points than two which is impossible [III 10]

Therefore if the straight line AB be applied to CD the segment AEB will not fail to coincide with CFD also,

therefore it will coincide with it and will be equal to it

Therefore etc

Q E D

PROPOSITION 22

Given a segment of a circle to describe the complete circle of which it is a segment

Let ABC be the given segment of a circle, thus it is required to describe the complete circle belonging to the segment ABC that is of which it is a segment

For let AC be bisected at D , let DB be drawn from the point D at right angles to AC and let AB be joined

the angle ABD is then greater than equal to or less than the angle BAD

First let it be greater

and on the straight line BA and at the point A on it, let the angle BAE be constructed equal to the angle ABD , let DB be drawn through to E and let EC be joined

Then since the angle ABE is equal to the angle BAE the straight line EB is also equal to EA [I 6]

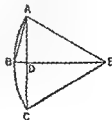
And since AD is equal to DC and DE is common the two sides AD , DE are equal to the two sides CD , DE respectively

and the angle ADE is equal to the angle CDE for each is right,

therefore the base AE is equal to the base CE

But EA was proved equal to EB

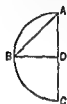
therefore BE is also equal to CE ,



therefore the three straight lines AE , EB , EC are equal to one another

Therefore the circle drawn with centre E and distance one of the straight lines AE , EB , EC will also pass through the remaining points and will have been completed [III 9]

Therefore, given a segment of a circle, the complete circle has been described



And it is manifest that the segment ABC is less than a semicircle because the centre E happens to be outside it

Similarly, even if the angle ABD be equal to the angle BAD ,

AD being equal to each of the two BD , DC the three straight lines DA , DB , DC will be equal to one another,

D will be the centre of the completed circle

and ABC will clearly be a semicircle

But, if the angle ABD be less than the angle BAD , and if we construct on the straight line BA and at the point A on it an angle equal to the angle ABD the centre will fall on DB within the segment ABC , and the segment ABC will clearly be greater than a semicircle



Therefore, given a segment of a circle, the complete circle has been described Q E F

PROPOSITION 26

In equal circles equal angles stand on equal circumferences, whether they stand at the centres or at the circumferences

Let ABC , DEF be equal circles, and in them let there be equal angles, namely at the centres the angles BGC , EHF and at the circumferences the angles BAC , EDF

I say that the circumference BAC is equal to the circumference ELF

For let BC , EF be joined

Now, since the circles ABC , DEF are equal

the radii are equal

Thus the two straight lines BG , GC are equal to the two straight lines EH , HF , and the angle at G is equal to the angle at H

therefore the base BC is equal to the base EF [I 4]

And since the angle at A is equal to the angle at D

the segment BAC is similar to the segment EDF [III Def 11] and they are upon equal straight lines

But similar segments of circles on equal straight lines are equal to one another, [III 24]

therefore the segment BAC is equal to EDF

But the whole circle ABC is also equal to the whole circle DEF , therefore the circumference BAC which remains is equal to the circumference ELF

Therefore etc

Q E D

PROPOSITION 27

In equal circles angles standing on equal circumferences are equal to one another, whether they stand at the centres or at the circumferences

For in equal circles ABC , DEF , on equal circumferences BC EF , let the angles BGC , EHF stand at the centres G H , and the angles BAC , EDF at the circumferences,

I say that the angle BGC is equal to the angle EHF ,
and the angle BAC is equal to the angle EDF

For, if the angle BGC is unequal to the angle EHF ,

one of them is greater

Let the angle BGC be greater and on the straight line BG , and at the point G on it let the angle BGA be constructed equal to the angle EHF [I 23]

Now equal angles stand on equal circumferences, when they are at the centres

therefore the circumference BA is equal to the circumference EF [III 26]

But EF is equal to BC ,

therefore BA is also equal to BC the less to the greater which is impossible

Therefore the angle BGC is not unequal to the angle EHF ,

therefore it is equal to it

And the angle at A is half of the angle BGC

and the angle at D half of the angle EHF , [III 20]

therefore the angle at A is also equal to the angle at D

Therefore etc

Q E D

PROPOSITION 28

In equal circles equal straight lines cut off equal circumferences the greater equal to the greater and the less to the less

Let ABC , DEF be equal circles and in the circles let AB , DE be equal straight lines cutting off ACB , DFE as greater circumferences and AGB , DHE as lesser

I say that the greater circumference ACB is equal to the greater circumference DFE and the less circumference AGB to DHE

For let the centres K L of the circles be taken and let AK KB DL LE be joined

Now since the circles are equal
the radii are also equal
therefore the two sides AK KB are equal
to the two sides DL LE ,

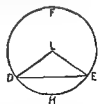
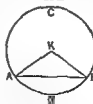
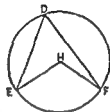
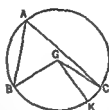
and the base AB is equal to the base DE

therefore the angle AKB is equal to the angle DLE [I 8]

But equal angles stand on equal circumferences when they are at the centres [III 26]

therefore the circumference AGB is equal to DHE

And the whole circle ABC is also equal to the whole circle DEF ,



therefore the circumference ACB which remains is also equal to the circumference DFE which remains

Therefore etc

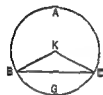
Q E D

PROPOSITION 29

In equal circles equal circumferences are subtended by equal straight lines

Let ABC DEF be equal circles and in them let equal circumferences BGC EHF be cut off and let the straight lines BC , EF be joined

I say that BC is equal to EF



For let the centres of the circles be taken and let them be K L let BK KC EL LF be joined

Now, since the circumference BGC is equal to the circumference EHF , the angle BKC is also equal to the angle ELF [III 27]

And since the circles ABC DEF are equal

the radii are also equal

therefore the two sides BK , KC are equal to the two sides EL , LF and they contain equal angles

therefore the base BC is equal to the base EF [I 4]

Therefore etc

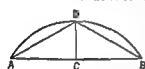
Q E D

PROPOSITION 30

To bisect a given circumference

Let ADB be the given circumference,

thus it is required to bisect the circumference ADB



Let AB be joined and bisected at C from the point C let CD be drawn at right angles to the straight line AB and let AD DB be joined

Then since AC is equal to CB and CD is common

the two sides AC , CD are equal to the two sides BC , CD

and the angle ACD is equal to the angle BCD for each is right

therefore the base AD is equal to the base DB [I 4]

But equal straight lines cut off equal circumferences the greater equal to the greater, and the less to the less [III 25]

and each of the circumferences AD , DB is less than a semicircle

therefore the circumference AD is equal to the circumference DB

Therefore the given circumference has been bisected at the point D

Q E D

PROPOSITION 31

In a circle the angle in the semicircle is right that in a greater segment less than a right angle and that in a less segment greater than a right angle and further the angle of the greater segment is greater than a right angle and the angle of the less segment less than a right angle

Let $ABCD$ be a circle let BC be its diameter and E its centre and AC , AD , DC be joined

I say that the angle BAC in the semicircle BAC is right
the angle ABC in the segment ABC greater than the semicircle is less than a right angle,

and the angle ADC in the segment ADC less than the semicircle is greater than a right angle

Let AE be joined, and let BA be carried through to F

Then since BE is equal to EA
the angle ABE is also equal to the angle BAE
[I 5]

Again since CE is equal to EA ,
the angle ACE is also equal to the angle CAE
[I 5]

Therefore the whole angle BAC is equal to the two angles ABC, ACB
But the angle FAC exterior to the triangle ABC is also equal to the two angles ABC, ACB ,
[I 32]

therefore the angle BAC is also equal to the angle FAC ,
therefore each is right, [I Def 10]

therefore the angle BAC in the semicircle BAC is right

Next since in the triangle ABC the two angles ABC, BAC are less than two right angles [I 17]

and the angle BAC is a right angle,

the angle ABC is less than a right angle,

and it is the angle in the segment ABC greater than the semicircle

Next since $ABCD$ is a quadrilateral in a circle
and the opposite angles of quadrilaterals in circles are equal to two right angles [III 22]

while the angle ABC is less than a right angle

therefore the angle ADC which remains is greater than a right angle

and it is the angle in the segment ADC less than the semicircle

I say further that the angle of the greater segment namely that contained by the circumference ABC and the straight line AC , is greater than a right angle

and the angle of the less segment namely that contained by the circumference ADC and the straight line AC is less than a right angle

This is at once manifest

For since the angle contained by the straight lines BA, AC is right
the angle contained by the circumference ABC and the straight line AC is greater than a right angle

Again since the angle contained by the straight lines AC, AF is right
the angle contained by the straight line CA and the circumference ADC is less than a right angle

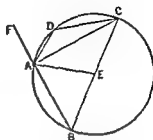
Therefore etc

Q E D

PROPOSITION 32

If a straight line touch a circle and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle

For let a straight line EF touch the circle $ABCD$ at the point B , and from



the point B let there be drawn across, in the circle $ABCD$ a straight line BD cutting it,

I say that the angles which BD makes with the tangent EF will be equal to the angles in the alternate segments of the circle that is that the angle FBD is equal to the angle constructed in the segment BAD and the angle EBD is equal to the angle constructed in the segment DCB

For let BA be drawn from B at right angles to EF ,
let \square point C be taken at random on the circumference BD ,

and let AD DC , CB be joined

Then, since a straight line EF touches the circle $ABCD$ at B ,
and BA has been drawn from the point of contact at right angles to the tangent the centre of the circle $ABCD$ is on BA [III 19]

Therefore BA is a diameter of the circle $ABCD$

therefore the angle ADB , being an angle in a semicircle is right [III 31]

Therefore the remaining angles BAD ABD are equal to one right angle [I 32]

But the angle ABF is also right,

therefore the angle ABF is equal to the angles BAD ABD

Let the angle ABD be subtracted from each

therefore the angle DBF which remains is equal to the angle BAD in the alternate segment of the circle

Next since $ABCD$ is a quadrilateral in a circle,

its opposite angles are equal to two right angles [III 22]

But the angles DBF DBE are also equal to two right angles,

therefore the angles DBF DBE are equal to the angles BAD BCD ,

of which the angle BAD was proved equal to the angle DBF ,

therefore the angle DBE which remains is equal to the angle DCB in the alternate segment DCB of the circle

Therefore etc

Q E D

PROPOSITION 33

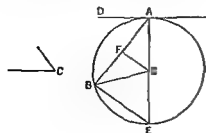
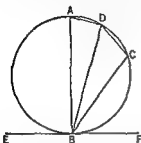
On a given straight line to describe a segment of a circle admitting an angle equal to a given rectilineal angle

Let AB be the given straight line and the angle at C the given rectilineal angle,

thus it is required to describe on the given straight line AB a segment of a circle admitting an angle equal to the angle at C

The angle at C is then acute or right or obtuse

First let it be acute and as in the first figure on the straight line AB and at the point A let the angle BAD be constructed equal to the angle at C



therefore the angle BAD is also acute

Let AE be drawn at right angles to DA , let AB be bisected at F , let FG be drawn from the point F at right angles to AB , and let GB be joined

Then, since AF is equal to FB

and FG is common

the two sides AF FG are equal to the two sides BF FG ,

and the angle AFG is equal to the angle BFG ,

therefore the base AG is equal to the base BG

[I 4]

Therefore the circle described with centre G and distance GA will pass through B also

Let it be drawn, and let it be ABE ,

let EB be joined

Now since AD is drawn from A the extremity of the diameter AE , at right angles to AE

therefore AD touches the circle ABE

[III 16 Por]

Since then a straight line AD touches the circle ABE

and from the point of contact at A a straight line AB is drawn across in the circle ABE

the angle DAB is equal to the angle AEB in the alternate segment of the circle

[III 32]

But the angle DAB is equal to the angle at C ,

therefore the angle at C is also equal to the angle AEB

Therefore on the given straight line AB the segment AEB of a circle has been described admitting the angle AEB equal to the given angle the angle at C

Next let the angle at C be right and let it be again required to describe on AB a segment of a circle admitting an angle equal to the right angle at C

Let the angle BAD be constructed equal to the right angle at C as is the case in the second figure

let AB be bisected at F and with centre F and distance either FA or FB let the circle AEB be described

Therefore the straight line AD touches the circle ABE because the angle at A is right

[III 16 Por]

And the angle BAD is equal to the angle in the segment AEB for the latter too is itself a right angle being an angle in a semicircle

[III 31]

But the angle BAD is also equal to the angle at C

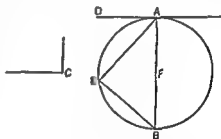
Therefore the angle AEB is also equal to the angle at C

Therefore again the segment AEB of a circle has been described on AB admitting an angle equal to the angle at C

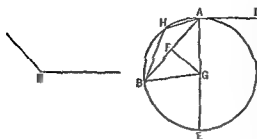
Next let the angle at C be obtuse

and on the straight line AB and at the point A let the angle BAD be constructed equal to it as is the case in the third figure

let AB be drawn at right angles to AD let AB be again bisected at F let FG be drawn at right angles to AB and let GB be joined



Then, since AF is again equal to FB ,
and FG is common.



the two sides AF, FG are equal to
the two sides BF, FG ,
and the angle AFG is equal to the
angle BFG ,
therefore the base AG is equal to
the base BG [I. 4]

Therefore the circle described with centre G and distance GA will pass through B also, let it so pass, as AEB

Now, since AD is drawn at right angles to the diameter AE from its extremity,

AD touches the circle AEB [III 10. For]

And AB has been drawn across from the point of contact at A ,
therefore the angle BAD is equal to the angle constructed in the alternate seg-
ment AHB of the circle

But the angle BAD is equal to the angle at C

Therefore the angle in the segment AHB is also equal to the angle at C

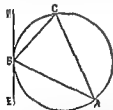
Therefore on the given straight line AB the segment AHB of a circle has been described admitting an angle equal to the angle at C Q.E.F.

PROPOSITION 34

From a given circle to cut off a segment admitting an angle equal to a given rectilineal angle

Let ABC be the given circle and the angle at D the given rectilineal angle, thus it is required to cut off from the circle ABC a segment admitting an angle equal to the given rectilineal angle, the angle at D .

Let EF be drawn touching ABC at the point B and on the straight line FB and at the point B on it let the angle FBC be constructed equal to the angle



Then since a straight line EF touches the circle ABC ,

and BC has been drawn across from the point of contact at B

the angle FBC is equal to the angle constructed in the alternate segment BAC' [III 32]

But the angle FBC is equal to the angle at D .

therefore the angle in the segment BAC is equal to the angle at D

Therefore from the given circle ABC the segment BAC has been cut off admitting an angle equal to the given rectilineal angle the angle at D $Q E F$

PROPOSITION 35

If in a circle two straight lines cut one another the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other

For in the circle $ABCD$ let the two straight lines AC BD cut one another at the point E

I say that the rectangle contained by AE , EC is equal to the rectangle contained by DE , EB

If now AC , BD are through the centre so that E is the centre of the circle $ABCD$

it is manifest that, AE EC DE EB being equal the rectangle contained by AE , EC is also equal to the rectangle contained by DE , EB

Next let AC DB not be through the centre, let the centre of $ABCD$ be taken, and let it be F from F let FG , FH be drawn perpendicular to the straight lines AC , DB , and let FB FC , FE be joined

Then since a straight line GF through the centre cuts a straight line AC not through the centre at right angles

it also bisects it,

[III 3]

therefore AG is equal to GC

Since then, the straight line AC has been cut into equal parts at G and into unequal parts at E

the rectangle contained by AE EC together with the square on EG is equal to the square on GC , [II 5]

Let the square on GF be added

therefore the rectangle AE , EC together with the squares on GF , GF is equal to the squares on CG GF

But the square on FE is equal to the squares on EG GF , and the square on FC is equal to the squares on CG GF , [I 47]

therefore the rectangle AE , EC together with the square on FE is equal to the square on FC

And FC is equal to FB

therefore the rectangle AE EC together with the square on EF is equal to the square on FB

For the same reason also

the rectangle DE EB together with the square on FE is equal to the square on FB

But the rectangle AE EC together with the square on FE was also proved equal to the square on FB ,

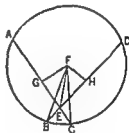
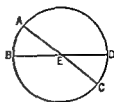
therefore the rectangle AE EC together with the square on FE is equal to the rectangle DE EB together with the square on FE

Let the square on FE be subtracted from each,

therefore the rectangle contained by AE EC which remains is equal to the rectangle contained by DE EB

Therefore etc

Q E D



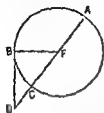
PROPOSITION 36

If a point be taken outside a circle and from it there fall on the circle two straight lines and if one of them cut the circle and the other touch it the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference will be equal to the square on the tangent

For let D be taken outside the circle ABC and from D let the two

straight lines DCA , DB fall on the circle ABC , let DCA cut the circle ABC and let DB touch it

I say that the rectangle contained by AD DC is equal to the square on DB



Then DCA is either through the centre or not through the centre

First let it be through the centre and let F be the centre of the circle ABC let FB be joined,

therefore the angle FBD is right [III 18]

And, since AC has been bisected at F and CD is added to it,

the rectangle AD , DC together with the square on FC is equal to the square on FD [II 6]

But FC is equal to FB ,
therefore the rectangle AD , DC together with the square on FB is equal to the square on FD

And the squares on FB , BD are equal to the square on FD [I 47]
therefore the rectangle AD , DC together with the square on FB is equal to the squares on FB , BD

Let the square on FB be subtracted from each
therefore the rectangle AD DC which remains is equal to the square on the tangent DB

Again let DCA not be through the centre of the circle ABC ,
let the centre E be taken, and from E let EF be drawn perpendicular to AC ,
let EB , EC , ED be joined

Then the angle EBD is right [III 18]

And since a straight line EF through the centre cuts a straight line AC not through the centre at right angles,

it also bisects it [III 3]

therefore AF is equal to FC

Now, since the straight line AC has been bisected at the point F and CD is added to it
the rectangle contained by AD DC together with the square on FC is equal to the square on FD [II 6]

Let the square on FE be added to each
therefore the rectangle AD , DC together with the squares on CF FE is equal to the squares on FD FE

But the square on EC is equal to the squares on CF FE for the angle EFC is right [I 47]

and the square on ED is equal to the squares on DF FE
therefore the rectangle AD , DC together with the square on EC is equal to the square on ED

And EC is equal to EB
therefore the rectangle AD DC together with the square on EB is equal to the square on ED

But the squares on EB BD are equal to the square on ED for the angle EBD is right [I 47]

therefore the rectangle AD, DC together with the square on EB is equal to the squares on EB, BD

Let the square on EB be subtracted from each,

therefore the rectangle AD, DC which remains is equal to the square on DB

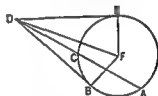
Therefore etc

Q E D

PROPOSITION 37

If a point be taken outside a circle and from the point there fall on the circle two straight lines, if one of them cut the circle, and the other fall on it and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference be equal to the square on the straight line which falls on the circle the straight line which falls on it will touch the circle

For let a point D be taken outside the circle ABC , from D let the two straight lines DCA, DB fall on the circle ABC , let DCA cut the circle and DB fall on it and let the rectangle AD, DC be equal to the square on DB



I say that DB touches the circle ABC

For let DF be drawn touching ABC , let the centre of the circle ABC be taken, and let it be F let FE, FB, FD be joined

Thus the angle FED is right

[III 18]

Now since DE touches the circle ABC and DCA cuts it the rectangle AD, DC is equal to the square on DE

[III 36]

But the rectangle AD, DC was also equal to the square on DB

therefore the square on DE is equal to the square on DB

therefore DE is equal to DB

And FE is equal to FB ,

therefore the two sides DE, EF are equal to the two sides DB, BF ,

and FD is the common base of the triangles,

therefore the angle DEF is equal to the angle DBF

[I 8]

But the angle DEF is right

therefore the angle DBF is also right

And FB produced is a diameter

and the straight line drawn at right angles to the diameter of a circle from its extremity touches the circle

[III 16, Por]

therefore DB touches the circle

Similarly this can be proved to be the case even if the centre be on AC

Therefore etc

Q E D

BOOK FOUR

DEFINITIONS

1 A rectilineal figure is said to be *inscribed in a rectilineal figure* when the respective angles of the inscribed figure lie on the respective sides of that in which it is inscribed

■ Similarly a figure is said to be *circumscribed about a figure* when the respective sides of the circumscribed figure pass through the respective angles of that about which it is circumscribed

3 A rectilineal figure is said to be *inscribed in a circle* when each angle of the inscribed figure lies on the circumference of the circle

4 A rectilineal figure is said to be *circumscribed about a circle*, when each side of the circumscribed figure touches the circumference of the circle

5 Similarly a circle is said to be *inscribed in a figure* when the circumference of the circle touches each side of the figure in which it is inscribed

6 A circle is said to be *circumscribed about a figure* when the circumference of the circle passes through each angle of the figure about which it is circumscribed

7 A straight line is said to be *fitted into a circle* when its extremities are on the circumference of the circle

BOOK IV PROPOSITIONS

PROPOSITION I

Into a given circle to fit a straight line equal to a given straight line which is not greater than the diameter of the circle

Let ABC be the given circle, and D the given straight line not greater than the diameter of the circle,

thus it is required to fit into the circle ABC a straight line equal to the straight line D

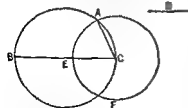
Let a diameter BC of the circle ABC be drawn

Then if BC is equal to D that which was enjoined will have been done for BC has been fitted into the circle ABC equal to the straight line D

But if BC is greater than D
let CE be made equal to D , and with centre C and distance CE let the circle EAF be described

let CA be joined

Then since the point C is the centre of the circle EAF ,



CA is equal to *CE*

But *CE* is equal to *D*,

therefore *D* is also equal to *CA*

Therefore into the given circle *ABC* there has been fitted *CA* equal to the given straight line *D* Q E F

PROPOSITION 2

In a given circle to inscribe a triangle equiangular with a given triangle

Let *ABC* be the given circle, and *DEF* the given triangle, thus it is required to inscribe in the circle *ABC* a triangle equiangular with the triangle *DEF*

Let *GH* be drawn touching the circle *ABC* at *A* [III 16 Por], on the straight line *AH*, and at the point *A* on it let the angle *HAC* be constructed equal to the angle *DEF*,

and on the straight line *AG* and at the point *A* on it, let the angle *GAB* be constructed equal to the angle *DFE*, [I 23]

let *BC* be joined

Then since a straight line *AH* touches the circle *ABC*, and from the point of contact at *A* the straight line *AC* is drawn across in the circle,

therefore the angle *HAC* is equal to the angle *ABC* in the alternate segment of the circle [III 32]

But the angle *HAC* is equal to the angle *DEF*,

therefore the angle *ABC* is also equal to the angle *DEF*

For the same reason

the angle *ACB* is also equal to the angle *DFE*,

therefore the remaining angle *BAC* is also equal to the remaining angle *EDF* [I 32]

Therefore in the given circle there has been inscribed a triangle equiangular with the given triangle Q E F

PROPOSITION 3

About a given circle to circumscribe a triangle equiangular with a given triangle

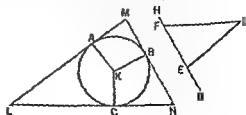
Let *ABC* be the given circle and *DEF* the given triangle, thus it is required to circumscribe about the circle *ABC* a triangle equiangular with the triangle *DEF*

Let *EF* be produced in both directions to the points *G H*

let the centre *K* of the circle *ABC* be taken [III 1] and let the straight line *AB* be drawn across at random,

on the straight line *AB* and at the point *A* on it, let the angle *BKA* be constructed equal to the angle *DEG*,

and the angle *BKC* equal to the angle *DFH*, [I 23]



and through the points A, B, C let LAM, MBN, NCL be drawn touching the circle ABC [III 16, Por.]

Now since LM, MN, NL touch the circle ABC at the points A, B, C , and LA, KB, AC have been joined from the centre K to the points A, B, C , therefore the angles at the points A, B, C are right [III 18]

And since the four angles of the quadrilateral $AMBA$ are equal to four right angles, inasmuch as $AMBA$ is in fact divisible into two triangles, and the angles KAM, KBM are right

therefore the remaining angles KAB, KMB are equal to two right angles

But the angles DEG, DEF are also equal to two right angles [I 13]

therefore the angles KAB, AMB are equal to the angles DEG, DEF ,

of which the angle KAB is equal to the angle DEG ,

therefore the angle AMB which remains is equal to the angle DEF which remains

Similarly it can be proved that the angle LNB is also equal to the angle DFE ,

therefore the remaining angle MLN is equal to the angle EDF [I 32]

Therefore the triangle LMN is equiangular with the triangle DEF , and it has been circumscribed about the circle ABC

Therefore about a given circle there has been circumscribed a triangle equiangular with the given triangle Q E F

PROPOSITION 4

In a given triangle to inscribe a circle

Let ABC be the given triangle

thus it is required to inscribe a circle in the triangle ABC

Let the angles ABC, ACB be bisected by the straight lines BD, CD [I 9] and let these meet one another at the point D , from D let DE, DF, DG be drawn perpendicular to the straight lines AB, BC, CA

Now since the angle ABD is equal to the angle CBD

and the right angle BED is also equal to the right angle BFD ,

EBD, FBD are two triangles having two angles equal to two angles and one side equal to one side namely that subtending one of the equal angles which is BD common to the triangles

therefore they will also have the remaining sides equal to the remaining sides [I 26]

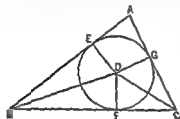
therefore DE is equal to DF

For the same reason

DG is also equal to DF

Therefore the three straight lines DE, DF, DG are equal to one another, therefore the circle described with centre D and distance one of the straight lines DE, DF, DG will pass also through the remaining points and will touch the straight lines AB, BC, CA because the angles at the points E, F, G are right

For, if it cuts them the straight line drawn at right angles to the d



the circle from its extremity will be found to fall within the circle which was proved absurd, [III 16]

therefore the circle described with centre D and distance one of the straight lines DE DF , DG will not cut the straight lines AB , BC CA ,

therefore it will touch them and will be the circle inscribed in the triangle ABC [IV Def 5]

Let it be inscribed as IGI'

Therefore in the given triangle ABC the circle EFG has been inscribed

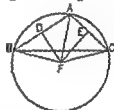
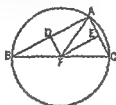
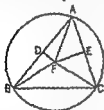
Q E F

PROPOSITION 5

About a given triangle to circumscribe a circle

Let ABC be the given triangle

thus it is required to circumscribe a circle about the given triangle ABC



Let the straight lines AB AC be bisected at the points D , E [I 10], and from the points D E let DF EF be drawn at right angles to AB , AC , they will then meet within the triangle ABC , or on the straight line BC , or outside BC

First let them meet within at F , and let FB , FC FA be joined

Then, since AD is equal to DB ,

and DF is common and at right angles,

therefore the base AF is equal to the base FB [I 4]

Similarly we can prove that

CF is also equal to AF ,

so that FB is also equal to FC

therefore the three straight lines FA FB FC are equal to one another

Therefore the circle described with centre F and distance one of the straight lines FA FB FC will pass also through the remaining points and the circle will have been circumscribed about the triangle ABC

Let it be circumscribed as ABC

Next let DF EF meet on the straight line BC at F , as is the case in the second figure, and let AF be joined

Then similarly we shall prove that the point F is the centre of the circle circumscribed about the triangle ABC

Again let DF , EF meet outside the triangle ABC at F , as is the case in the third figure and let AF BF CF be joined

Then again since AD is equal to DB ,

and DF is common and at right angles,

therefore the base AF is equal to the base BF [I 4]

Similarly we can prove that

CF is also equal to AF

so that BF is also equal to FC ,

therefore the circle described with centre F and distance one of the straight lines FA, FB, FC will pass also through the remaining points and will have been circumscribed about the triangle ABC

Therefore about the given triangle a circle has been circumscribed

Q E F

And it is manifest that, when the centre of the circle falls within the triangle the angle BAC , being in \square segment greater than the semicircle is less than a right angle,
when the centre falls on the straight line BC , the angle BAC , being in \square semicircle is right,
and when the centre of the circle falls outside the triangle, the angle BAC , being in a segment less than the semicircle, is greater than a right angle [III 31]

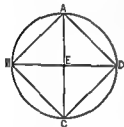
PROPOSITION 6

In \square given circle to inscribe a square

Let $ABCD$ be the given circle,

thus it is required to inscribe a square in the circle $ABCD$

Let two diameters AC, BD of the circle $ABCD$ be drawn at right angles to one another, and let AB, BC, CD, DA be joined



Then since BE is equal to ED for E is the centre and EA is common and at right angles

therefore the base AB is equal to the base AD [I 4]

For the same reason

each of the straight lines BC, CD is also equal to each of the straight lines AB, AD ,

therefore the quadrilateral $ABCD$ is equilateral

I say next that it is also right angled

For, since the straight line BD is a diameter of the circle $ABCD$

therefore BAD is a semicircle

therefore the angle BAD is right

[III 31]

For the same reason

each of the angles ABC, BCD, CDA is also right,

therefore the quadrilateral $ABCD$ is right angled

But it was also proved equilateral

therefore it is a square

[I Def 22]

and it has been inscribed in the circle $ABCD$

Therefore in the given circle the square $ABCD$ has been inscribed Q E F

PROPOSITION 7

About \square given circle to circumscribe a square

Let $ABCD$ be the given circle

thus it is required to circumscribe a square about the circle $ABCD$

Let two diameters AC, BD of the circle $ABCD$ be drawn at right angles to one another and through the points A, B, C, D let FG, GH, HA, AF be drawn touching the circle $ABCD$ [III 16 Por]

Then since FG touches the circle $ABCD$

and EA has been joined from the centre E to the point of contact at A

therefore the angles at A are right

[III 18]

For the same reason

the angles at the points B C D are also right

Now, since the angle AEB is right

and the angle EBG is al^{so} right

therefore GH is parallel to AC [1 28]

For the same reason

AC is also parallel to FK ,

so that GH is also parallel to FK [1 30]

Similarly we can prove that

each of the straight lines GF HA is parallel to BED

Therefore GK , GC AK , FB , BK are parallelograms,
therefore GF is equal to HK , and GH to FK [1 34]

And since AC is equal to BD

and AC is also equal to each of the straight lines GH FK ,

while BD is equal to each of the straight lines GF , HK , [1 34]

therefore the quadrilateral $FGHK$ is equilateral

I say next that it is also right-angled

For, since $GBEA$ is a parallelogram

and the angle AEB is right,

therefore the angle AGB is also right [1 34]

Similarly we can prove that

the angles at H , K , F are also right

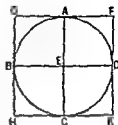
Therefore $FGHK$ is right angled

But it was also proved equilateral,

therefore it is a square

and it has been circumscribed about the circle $ABCD$

Therefore about the given circle a square has been circumscribed $Q.E.D.$



PROPOSITION 8

In a given square to inscribe a circle

Let $ABCD$ be the given square

thus it is required to inscribe a circle in the given square $ABCD$

Let the straight lines AD , AB be bisected at the points E , F respectively [1 10]

through E let EH be drawn parallel to either AB or CD , and through F let FK be drawn parallel to either AD or BC , [1 31]

therefore each of the figures AA KB AH HD AG GC BG GD is a parallelogram and their opposite sides are evidently equal [1 34]

Now since AD is equal to AB

and AE is half of AD , and AF half of AB ,

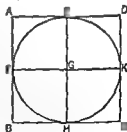
therefore AE is equal to AF ,

so that the opposite sides are al^{so} equal

therefore FG is equal to GE

Similarly we can prove that each of the straight lines GH GK is equal to each of the straight lines FG GE

therefore the four straight lines GE GF GH GK are equal to one another



Therefore the circle described with centre G and distance one of the straight lines GE, GF, GH, GK will pass also through the remaining points

And it will touch the straight lines AB, BC, CD, DA , because the angles at E, F, H, K are right

For, if the circle cuts AB, BC, CD, DA , the straight line drawn at right angles to the diameter of the circle from its extremity will fall within the circle which was proved absurd, [III 16]

therefore the circle described with centre G and distance one of the straight lines GE, GF, GH, GK will not cut the straight lines AB, BC, CD, DA

Therefore it will touch them and will have been inscribed in the square $ABCD$

Therefore in the given square a circle has been inscribed

Q E F

PROPOSITION 9

About a given square to circumscribe a circle

Let $ABCD$ be the given square,

thus it is required to circumscribe a circle about the square $ABCD$

For let AC, BD be joined and let them cut one another at E

Then, since DA is equal to AB and AC is common

therefore the two sides DA, AC are equal to the two sides BA, AC ,

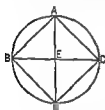
and the base DC is equal to the base BC ,

therefore the angle DAC is equal to the angle BAC

[I 8]

Therefore the angle DAB is bisected by AC

Similarly we can prove that each of the angles ABC, BCD, CDA is bisected by the straight lines AC, DB



Now, since the angle DAB is equal to the angle ABC

and the angle EAB is half the angle DAB ,

and the angle EBA half the angle ABC ,

therefore the angle EAB is also equal to the angle EBA ,

so that the side EA is also equal to EB

[I 6]

Similarly we can prove that each of the straight lines EA, EB, EC, ED is equal to each of the straight lines EC, ED

Therefore the four straight lines EA, EB, EC, ED are equal to one another

Therefore the circle described with centre E and distance one of the straight lines EA, EB, EC, ED will pass also through the remaining points,

and it will have been circumscribed about the square $ABCD$

Let it be circumscribed as $ABCD$

Therefore about the given square a circle has been circumscribed Q E F

PROPOSITION 10

To construct an isosceles triangle having each of the angles at the base double of the remaining one

Let any straight line AB be set out and let it be cut at the point C so that the rectangle contained by AB, BC is equal to the square on CA [II 11]

with centre A and distance AB let the circle BDE be described

and let there be fitted in the circle BDE the straight line BD equal to the straight line AC which is not greater than the diameter of the circle BDE

the triangle FGH , so that the angle CAD is equal to the angle at F and the angles at G, H respectively equal to the angles ACD, CDA , [iv 2]
therefore each of the angles ACD, CDA is also double of the angle CAD



Now let the angles ACD, CDA be bisected respectively by the straight lines CE, DB [I 9] and let AB, BC, DE, EA be joined

Then since each of the angles ACD, CDA is double of the angle CAD and they have been bisected by the straight lines CE, DB ,

therefore the five angles DAC, ACE, ECD, CDB, BDA are equal to one another

But equal angles stand on equal circumferences [III 26]
therefore the five circumferences AB, BC, CD, DE, EA are equal to one another

But equal circumferences are subtended by equal straight lines [III 29]
therefore the five straight lines AB, BC, CD, DE, EA are equal to one another
therefore the pentagon $ABCDE$ is equilateral

I say next that it is also equiangular

For since the circumference AB is equal to the circumference DE , let BCD be added to each
therefore the whole circumference $ABCD$ is equal to the whole circumference $EDCB$

And the angle AED stands on the circumference $ABCD$ and the angle BAE on the circumference $EDCB$,

therefore the angle BAE is also equal to the angle AED [III 27]

For the same reason
each of the angles ABC, BCD, CDE is also equal to each of the angles BAE, AED

therefore the pentagon $ABCDE$ is equiangular

But it was also proved equilateral
therefore in the given circle an equilateral and equiangular pentagon has been inscribed Q E F

PROPOSITION 12

About a given circle to circumscribe an equilateral and equiangular pentagon

Let $ABCDE$ be the given circle
thus it is required to circumscribe an equilateral and equiangular pentagon about the circle $ABCDE$

Let A, B, C, D, E be conceived to be the angular points of the inscribed pentagon so that the circumferences AB, BC, CD, DE, EA are equal [IV 11]
through A, B, C, D, E let GH, HK, KL, LM, MG be drawn touching the circle, [III 16 Por.]

let the centre F of the circle $ABCDE$ be taken [III 1] and let FB, FK, FC, FL, FD be joined

Then since the straight line KL touches the circle $ABCDE$ at C
and FC has been joined from the centre F to the point of contact at C

therefore FC is perpendicular to KL , [III 18]

therefore each of the angles at C is right

For the same reason

the angles at the points B, D are also right

And, since the angle FCA is right

therefore the square on FA = equal to the squares on FC, CA

For the same reason

the square on FA = also equal to the squares on FB, BA ,

so that the squares on FC, CA are equal to the squares on FB, BA ,

of which the square on FC is equal to the square on FB

therefore the square on CA which remains is equal to the square on BA

Therefore BA is equal to CA

And, since FB is equal to FC

and FA common

the two sides BF, FA are equal to the two sides CF, FA , and the base BA equal to the base CA ,

therefore the angle BFA is equal to the angle AFC ,

and the angle BAF to the angle FAC

Therefore the angle BFC is double of the angle AFC ,

and the angle BAC of the angle FAC

For the same reason

the angle CFD is also double of the angle CFL ,

and the angle DLC of the angle FLC

Now since the circumference BC is equal to CD

the angle BFC is also equal to the angle CFD

[III 27]

And the angle BFC is double of the angle AFC , and the angle DFC of the angle LFC

therefore the angle AFC is also equal to the angle LFC

But the angle FCA = also equal to the angle FCL ,
therefore FAC, FLC are two triangles having two angles equal to two angles and one side equal to one side, namely FC which is common to them,
therefore they will also have the remaining sides equal to the remaining sides and the remaining angle to the remaining angle,

[I 20]

therefore the straight line AC is equal to CL ,

and the angle FAC to the angle FLC ,

And, since AC = equal to CL

therefore AL is double of AC

For the same reason it can be proved that

HA is also double of BA

And BA = equal to AC ,

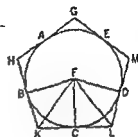
therefore HA is also equal to AL

Similarly each of the straight lines HG, GM, ML can also be proved equal to each of the straight lines HA, AL

therefore the pentagon $GHALM$ is equilateral

I say next that it is also equiangular

For, since the angle FAC is equal to the angle FLC ,



and the angle HKL was proved double of the angle FAC ,
 and the angle KLM double of the angle FLC
 therefore the angle HKL is also equal to the angle KLM

Similarly each of the angles KHG , HQM , GML can also be proved equal to each of the angles HKL , KLM ,
 therefore the five angles GHL , HKL , KLM , LMG , MGH are equal to one another

Therefore the pentagon $GHLKM$ is equiangular

And it was also proved equilateral, and it has been circumscribed about the circle $ABCDE$ Q E F

PROPOSITION 13

In a given pentagon, which is equilateral and equiangular, to inscribe a circle

Let $ABCDE$ be the given equilateral and equiangular pentagon,
 thus it is required to inscribe a circle in the pentagon $ABCDE$

For let the angles BCD , CDE be bisected by the straight lines CF , DF respectively, and from the point F at which the straight lines CF , DF meet one another, let the straight lines FB , FA , FE be joined

Then since BC is equal to CD and CF common,

the two sides BC , CF are equal to the two sides DC , CF ,

and the angle BCF is equal to the angle DCF ,

therefore the base BF is equal to the base DF

and the triangle BCF is equal to the triangle DCF ,
 and the remaining angles will be equal to the remaining angles namely those which the equal sides subtend [14]

Therefore the angle CBF is equal to the angle CDF

And since the angle CDE is double of the angle CDI ,

and the angle CDE is equal to the angle ABC ,

while the angle CDF is equal to the angle CBF

therefore the angle CBA is also double of the angle CBF ,

therefore the angle ABI is equal to the angle FBC

therefore the angle ABC has been bisected by the straight line BF

Similarly it can be proved that

the angles BAE , AED have also been bisected by the straight lines FA , FE respectively

Now let FG , FH , FK , FL , FM be drawn from the point F perpendicular to the straight lines AB , BC , CD , DE , EA

Then, since the angle HCF is equal to the angle KCF

and the right angle FHC is also equal to the angle FKC

FHC , FKC are two triangles having two angles equal to two angles and one side equal to one side, namely FC which is common to them and subtends one of the equal angles

therefore they will also have the remaining sides equal to the remaining sides [15]

therefore the perpendicular FH is equal to the perpendicular FK

Similarly it can be proved that

each of the straight lines FL FM FG is also equal to each of the straight lines FH FA ,

therefore the five straight lines FG FH , FA , FL FM are equal to one another

Therefore the circle described with centre F and distance one of the straight lines FG FH FA FL FM will pass also through the remaining points and it will touch the straight lines AB BC , CD , DE , EA , because the angles at the points G , H A , L , M are right

For if it does not touch them, but cuts them,
it will result that the straight line drawn at right angles to the diameter of the circle from its extremity falls within the circle which was proved absurd

[III 16]

Therefore the circle described with centre F and distance one of the straight lines FG FH , FA , FL , FM will not cut the straight lines AB , BC , CD , DE , EA , therefore it will touch them

Let it be described as $GHIIM$

Therefore in the given pentagon which is equilateral and equiangular a circle has been inscribed

Q E F

PROPOSITION 14

About a given pentagon which is equilateral and equiangular, to circumscribe a circle

Let $ABCDE$ be the given pentagon which is equilateral and equiangular, thus it is required to circumscribe a circle about the pentagon $ABCDE$

Let the angles BCD CDE be bisected by the straight lines CF , DF respectively, and from the point F at which the straight lines meet let the straight lines FB FA FE be joined to the points B , A , E

Then in manner similar to the preceding it can be proved that the angles CBA BAE AED have also been bisected by the straight lines FB , FA FE respectively



Now, since the angle BCD is equal to the angle CDE ,
and the angle FCD is half of the angle BCD
and the angle CDF half of the angle CDE ,
therefore the angle FCD is also equal to the angle CDF ,
so that the side FC is also equal to the side FD

[I 6]

Similarly it can be proved that
each of the straight lines FB FA , FE is also equal to each of the straight lines FC , FD

therefore the five straight lines FA FB FC FD FE are equal to one another

Therefore the circle described with centre F and distance one of the straight lines FA FB FC FD FE will pass also through the remaining points and will have been circumscribed

Let it be circumscribed and let it be $ABCDE$

Therefore about the given pentagon which is equilateral and equiangular, a circle has been circumscribed

Q E F

PROPOSITION 15

In a given circle to inscribe an equilateral and equiangular hexagon

Let $ABCDEF$ be the given circle,
thus it is required to inscribe an equilateral and equiangular hexagon in the circle $ABCDEF$

Let the diameter AD of the circle $ABCDEF$ be drawn,

let the centre G of the circle be taken and with centre D and distance DG let the circle $EGCH$ be described
let EG CG be joined and carried through to the points B , F ,

and let AB BC , CD DE , EF , FA be joined

I say that the hexagon $ABCDEF$ is equilateral and equiangular

For since the point G is the centre of the circle $ABCDEF$,

GE is equal to GD

Again, since the point D is the centre of the circle GCH ,

DF is equal to DG

But GE was proved equal to GD ,

therefore GE is also equal to ED ,

therefore the triangle EGD is equilateral

and therefore its three angles EGD , GDE , DEG are equal to one another inasmuch as in isosceles triangles the angles at the base are equal to one another [I 5]

And the three angles of the triangle are equal to two right angles, [I 32]

therefore the angle EGD is one third of two right angles

Similarly the angle DGC can also be proved to be one-third of two right angles

And since the straight line CG standing on EB makes the adjacent angles EGC , CGB equal to two right angles

therefore the remaining angle CGB is also one-third of two right angles

Therefore the angles EGD DGC CGB are equal to one another

so that the angles vertical to them the angles BGA AGF FGE are equal [I 15]

Therefore the six angles EGD DGC CGB , BGA AGF , FGE are equal to one another

But equal angles stand on equal circumferences [III 26]

therefore the six circumferences AB BC CD DE EF FA are equal to one another

And equal circumferences are subtended by equal straight lines [III 29]

therefore the six straight lines are equal to one another

therefore the hexagon $ABCDEF$ is equilateral

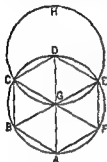
I say next that it is also equiangular

For, since the circumference FA is equal to the circumference ED ,

let the circumference $ABCD$ be added to each

therefore the whole $FABCD$ is equal to the whole $EDCBA$

and the angle FED stands on the circumference $FABCD$



and the angle AFE on the circumference $EDCBA$,
therefore the angle AFE is equal to the angle DEF [III 27]

Similarly it can be proved that the remaining angles of the hexagon $ABCDEF$
are also severally equal to each of the angles AFE , FED ,
therefore the hexagon $ABCDEF$ is equiangular

But it was also proved equilateral

and it has been inscribed in the circle $ABCDEF$

Therefore in the given circle an equilateral and equiangular hexagon has
been inscribed Q E F

PROPOSITION From this it is manifest that the side of the hexagon is equal to the
radius of the circle

And in like manner as in the case of the pentagon if through the points of
division on the circle we draw tangents to the circle, there will be circumscribed
about the circle an equilateral and equiangular hexagon in conformity with
what was explained in the case of the pentagon

And further by means similar to those explained in the case of the pentagon
we can both inscribe a circle in a given hexagon and circumscribe one about it

Q E F

PROPOSITION 16

In a given circle to inscribe a fifteen-angled figure which shall be both equilateral
and equiangular

Let $ABCD$ be the given circle,
thus it is required to inscribe in the circle $ABCD$ a fifteen-angled figure which
shall be both equilateral and equiangular

In the circle $ABCD$ let there be in-
scribed a side AC of the equilateral tri-
angle inscribed in it and a side AB of an
equilateral pentagon
therefore of the equal segments of which
there are fifteen in the circle $ABCD$ there
will be five in the circumference ABC which
is one-third of the circle and there will be
three in the circumference AB which is
one-fifth of the circle

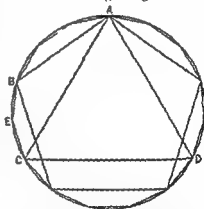
therefore in the remainder BC there will
be two of the equal segments

Let BC be bisected at E , [III 30]

therefore each of the circumferences BE EC is a fifteenth of the circle $ABCD$
If therefore we join BE EC and fit into the circle $ABCD$ straight lines equal
to them and in contiguity a fifteen-angled figure which is both equilateral and
equiangular will have been inscribed in it Q E F

And, in like manner as in the case of the pentagon if through the points of
division on the circle we draw tangents to the circle there will be circumscribed
about the circle a fifteen-angled figure which is equilateral and equi-
angular

And further by proofs similar to those in the case of the pentagon we can
both inscribe a circle in the given fifteen-angled figure and circumscribe one
about it Q E F



BOOK FIVE

DEFINITIONS

- 1 A magnitude is a *part* of a magnitude, the less of the greater, when it measures the greater
- 2 The greater is a *multiple* of the less when it is measured by the less
- 3 A *ratio* is a sort of relation in respect of size between two magnitudes of the same kind
- 4 Magnitudes are said to *have a ratio* to one another which are capable, when multiplied, of exceeding one another
- 5 Magnitudes are said to be *in the same ratio*, the first to the second and the third to the fourth when if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth the former equimultiples alike exceed are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order
- 6 Let magnitudes which have the same ratio be called *proportional*
- 7 When of the equimultiples the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to *have a greater ratio* to the second than the third has to the fourth
- 8 A proportion in three terms is the least possible
- 9 When three magnitudes are proportional the first is said to have to the third the *duplicate ratio* of that which it has to the second
- 10 When four magnitudes are *<continuously>* proportional, the first is said to have to the fourth the *triplicate ratio* of that which it has to the second, and so on continually whatever be the proportion
- 11 The term *corresponding magnitudes* is used of antecedents in relation to antecedents and of consequents in relation to consequents
- 12 *Alternate ratio* means taking the antecedent in relation to the antecedent and the consequent in relation to the consequent
- 13 *Inverse ratio* means taking the consequent as antecedent in relation to the antecedent as consequent
- 14 *Composition of a ratio* means taking the antecedent together with the consequent as one in relation to the consequent by itself
- 15 *Separation of a ratio* means taking the excess by which the antecedent exceeds the consequent in relation to the consequent by itself
- 16 *Conversion of a ratio* means taking the antecedent in relation to the excess by which the antecedent exceeds the consequent
- 17 A ratio *ex aequali* arises when there being several magnitudes and another set equal to them in multitude which taken two and two are in the same proportion as the first is to the last among the first magnitudes, so is the first to the last among the second magnitudes,

Or in other words, it means taking the extreme terms by virtue of the removal of the intermediate terms

18 A *perturbed proportion* arises when, there being three magnitudes and another set equal to them in multitude, as antecedent is to consequent among the first magnitudes so is antecedent to consequent among the second magnitudes while as the consequent is to a third among the first magnitudes so is a third to the antecedent among the second magnitudes

BOOK V PROPOSITIONS

PROPOSITION 1

If there be any number of magnitudes whatever which are respectively equimultiples of any magnitudes equal in multitude, then, whatever multiple one of the magnitudes is of one that multiple also will all be of all

Let any number of magnitudes whatever AB CD be respectively equimultiples of any magnitudes E F equal in multitude,

I say that whatever multiple AB is of E that multiple will AB , CD also be of E F



For since AB is the same multiple of E that CD is of F , as many magnitudes as there are in AB equal to E so many also are there in CD equal to F

Let AB be divided into the magnitudes AG , GB equal to E ,
and CD into CH HD equal to F

then the multitude of the magnitudes AG GB will be equal to the multitude of the magnitudes CH HD

Now since AG is equal to E , and CH to F

therefore AG is equal to E , and AG , CH to E F

For the same reason

GB is equal to F and GB HD to E F

therefore as many magnitudes as there are in AB equal to E so many also are there in AB CD equal to E F

therefore whatever multiple AB is of E , that multiple will AB CD also be of E F

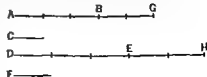
Therefore etc

Q E D

PROPOSITION 2

If a first magnitude be the same multiple of a second that a third is of a fourth and a fifth also be the same multiple of the second that a sixth is of the fourth the sum of the first and fifth will also be the same multiple of the second that the sum of the third and sixth is of the fourth

Let a first magnitude AB be the same multiple of a second C , that a third DE is of a fourth F , and let a fifth BG also be the same multiple of the second C that a sixth EH is of the fourth F



I say that the sum of the first and fifth AG will be the same multiple of the

second, C , that the sum of the third and sixth, DH is of the fourth F

For since AB is the same multiple of C that DE is of F therefore, as many magnitudes as there are in AB equal to C , so many also are there in DE equal to F

For the same reason also
as many as there are in BG equal to C so many are there also in EH equal to F ,
therefore as many as there are in the whole AG equal to C , so many also are there in the whole DH equal to F

Therefore whatever multiple AG is of C , that multiple also is DH of F

Therefore the sum of the first and fifth AG is the same multiple of the second C that the sum of the third and sixth, DH is of the fourth, F

Therefore etc

Q E D

PROPOSITION 3

If a first magnitude be the same multiple of a second that a third is of a fourth, and if equimultiples be taken of the first and third then also ex aequali the magnitudes taken will be equimultiples respectively the one of the second and the other of the fourth

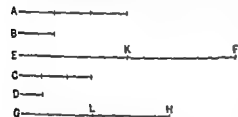
Let a first magnitude A be the same multiple of a second B that a third C is of a fourth D , and let equimultiples EF , GH be taken of A , C ,

I say that EF is the same multiple of B that GH is of D

For, since EF is the same multiple of A that GH is of C , therefore as many magnitudes as there are in EF equal to A so many also are there in GH equal to C

Let EF be divided into the magnitudes EK , KF equal to A and GH into the magnitudes GL , LH equal to C

then the multitude of the magnitudes EK , KF will be equal to the multitude of the magnitudes GL , LH



And since A is the same multiple of B that C is of D ,

while EK is equal to A and GL to C ,
therefore EK is the same multiple of B that GL is of D

For the same reason

KF is the same multiple of B that LH is of D

Since then a first magnitude EK is the same multiple of a second B that a third GL is of a fourth D

and a fifth KF is also the same multiple of the second B that a sixth LH is of the fourth D

therefore the sum of the first and fifth EF is also the same multiple of the second B that the sum of the third and sixth GH is of the fourth D [v 2]

Therefore etc

Q E D

PROPOSITION 4

If a first magnitude have to a second the same ratio as a third to a fourth any equimultiples whatever of the first and third will also have the same ratio to any equimultiples whatever of the second and fourth respectively taken in corresponding order

For let \square first magnitude A have to a second B the same ratio as a third C to a fourth D , and let equmultiples E F be taken of A , C , and G , H other, chance, equmultiples of B , D ,

I say that as E is to G , so is F to H

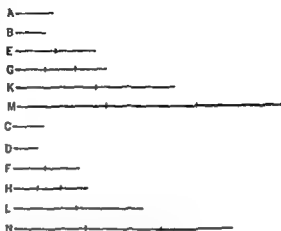
For let equmultiples K , L be taken of E , F , and other, chance equmultiples M N of G , H

Since E is the same multiple of A that F is of C , and equmultiples K , L of E , F have been taken, therefore K \square the same multiple of A that L is of C

[v 3]

For the same reason M is the same multiple of B that N is of D

And since as A is to B , so is C to D



and of A C equmultiples K L have been taken,
and of B D other chance equmultiples M N
therefore if K is in excess of M , L also is in excess of N ,
if it is equal equal, and if less less [v Def 5]
And K L are equmultiples of E , F ,
and M N other chance equmultiples of G , H ,
therefore as E \square to G so is F to H [v Def 5]

Therefore etc

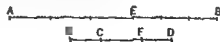
Q E D

PROPOSITION II

If a magnitude be the same multiple of a magnitude that a part subtracted is of a part subtracted the remainder will also be the same multiple of the remainder that the whole is of the whole

For let the magnitude AB be the same multiple of the magnitude CD that the part AE subtracted \square of the part CF subtracted,

I say that the remainder EB is also the same multiple of the remainder FD that the whole AB \square of the whole CD



For whatever multiple AE is of CF let EB be made that multiple of CG
Then, since AE is the same multiple of CF that EB \square of GC

therefore AE is the same multiple of CF that AB is of GF [v 1]

But by the assumption AE is the same multiple of CF that AB \square of CD

Therefore AB is the same multiple of each of the magnitudes GF CD ,

therefore GF is equal to CD

Let CF be subtracted from each,

therefore the remainder GC is equal to the remainder FD

And, since AE is the same multiple of CF that EB is of GC ,

and GC is equal to DF

therefore AE is the same multiple of CF that EB is of FD

But, by hypothesis

AE is the same multiple of CF that AB is of CD ,

therefore EB is the same multiple of FD that AB is of CD

That is the remainder EB will be the same multiple of the remainder FD that the whole AB is of the whole CD

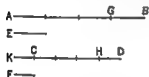
Therefore etc

Q E D

PROPOSITION 6

If two magnitudes be equimultiples of two magnitudes, and any magnitudes subtracted from them be equimultiples of the same, the remainders also are either equal to the same or equimultiples of them

For let two magnitudes AB , CD be equimultiples of two magnitudes E , F , and let AG , CH subtracted from them be equimultiples of the same two E , F ,



I say that the remainders also GB , HD , are either equal to E , F or equimultiples of them

For, first let GB be equal to E ,

I say that HD is also equal to F

For let CK be made equal to F

Since AG is the same multiple of E that CH is of F ,

while GB is equal to E and KC to F ,

therefore AB is the same multiple of E that KH is of F [v 2]

But by hypothesis AB is the same multiple of E that CD is of F ,

therefore KH is the same multiple of F that CD is of F

Since then each of the magnitudes KH , CD is the same multiple of F ,

therefore KH is equal to CD

Let CH be subtracted from each,

therefore the remainder KC is equal to the remainder HD

But F is equal to KC ,

therefore HD is also equal to F

Hence if GB is equal to E , HD is also equal to F

Similarly we can prove that, even if GB be a multiple of E , HD is also the same multiple of F

Therefore etc

Q E D

PROPOSITION 7

Equal magnitudes have to the same the same ratio as also has the same to equal magnitudes

Let A , B be equal magnitudes and C any other chance, magnitude,

I say that each of the magnitudes A , B has the same ratio to C , and C has the same ratio to each of the magnitudes A , B

For let equimultiples D , E of A , B be taken, and of C another, chance multiple F

Then since D is the same multiple of A that E is of B , while A is equal to B , therefore D is equal to E

But F is another, chance, magnitude

If therefore D is in excess of F , E is also in excess of F , if equal to it, equal, and if less less

And D , E are equimultiples of A , B , while F is another, chance multiple of C , therefore as A is to C , so is B to C

[v Def 5]

I say next that C also has the same ratio to each of the magnitudes A , B . For, with the same construction, we can prove similarly that D is equal to E , and F is some other magnitude

If therefore F is in excess of D it is also in excess of E if equal equal, and if less, less

And F is a multiple of C , while D , E are other chance equimultiples of A , B , therefore as C is to A , so is C to B [v Def 5]

Therefore etc

POLEMISM From this it is manifest that if any magnitudes are proportional they will also be proportional inversely

Q E D

PROPOSITION 8

Of unequal magnitudes the greater has to the same a greater ratio than the less has and the same has to the less a greater ratio than it has to the greater

Let AB , C be unequal magnitudes, and let AB be greater let D be another chance magnitude

I say that AB has to D a greater ratio than C has to D and D has to C a greater ratio than it has to AB

For since AB is greater than C let BE be made equal to C

then the less of the magnitudes AE , EB if multiplied will sometime be greater than D [v Def 4]

First let AE be less than EB let AE be multiplied and let

FG be a multiple of it which is greater than D ,

then whatever multiple FG is of AE let GH be made the same multiple of EB and K of C

and let L be taken double of D , M triple of it and successive multiples increasing by one until what is taken is a multiple of D and the first that is greater than K . Let it be taken and let it be N which is quadruple of D and the first multiple of it that is greater than K

Then since K is less than N first

therefore K is not less than M

And since FG is the same multiple of AE that GH is of EB

therefore FG is the same multiple of AE that FH is of AB [v 1]

But FG is the same multiple of AE that K is of C

therefore FH is the same multiple of AB that K is of C ,

therefore FH , K are equimultiples of AB , C

Again since GH is the same multiple of EB that K is of C and EB is equal to C



therefore GH is equal to A

But K is not less than M

therefore neither is GH less than M

And FG is greater than D ,

therefore the whole FH is greater than D M together

But D M together are equal to N inasmuch as M is triple of D , and M , D together are quadruple of D while N is also quadruple of D , whence M D together are equal to N

But FH is greater than M D ,

therefore FH is in excess of N

while K is not in excess of N

And FH , K are equimultiples of AB , C , while N is another chance multiple of D

therefore AB has to D a greater ratio than C has to D [v Def 7]

I say next that D also has to C a greater ratio than D has to AB

For with the same construction we can prove similarly that N is in excess of K while N is not in excess of FH

And N is a multiple of D ,

while FH , K are other, chance equimultiples of AB , C ,

therefore D has to C a greater ratio than D has to AB [v Def 7]

Again let AE be greater than EB

Then the less EB , if multiplied will sometime be greater than D [v Def 4]

Let it be multiplied and let GH be a multiple of EB and greater than D and whatever multiple GH is of EB let FG be made the same multiple of AE and K of C

Then we can prove similarly that FH , K are equimultiples of AB , C and similarly let N be taken a multiple of D but the first that is greater than FG , so that FG is again not less than M But GH is greater than D

therefore the whole FH is in excess of D M that is of N

Now K is not in excess of N inasmuch as FG also which is greater than GH that is than K is not in excess of N

And in the same manner by following the above argument we complete the demonstration

Therefore etc

Q.E.D.

PROPOSITION 9

Magnitudes which have the same ratio to the same are equal to one another and magnitudes to which the same has the same ratio are equal

For let each of the magnitudes A , B have the same ratio to C

I say that A is equal to B

For otherwise each of the magnitudes A , B would not have had the same ratio to C but it has [v 8]

therefore A is equal to B



Again, let C have the same ratio to each of the magnitudes A, B ,

I say that A is equal to B

For otherwise, C would not have had the same ratio to each of the magnitudes A, B , [v 5]

but it has,

therefore A is equal to B

Therefore etc

Q E D

PROPOSITION 10

Of magnitudes which have a ratio to the same, that which has a greater ratio is greater and that to which the same has a greater ratio is less

For let A have to C a greater ratio than B has to C ,

I say that A is greater than B



For if not A is either equal to B or less

Now A is not equal to B ,

for in that case each of the magnitudes A, B would have had the same ratio to C , [v 7]

but they have not,

therefore A is not equal to B

Nor again is A less than B

for in that case A would have had to C a less ratio than B has to C , [v 8]

but it has not

therefore A is not less than B

But it was proved not to be equal either

therefore A is greater than B

Again, let C have to B a greater ratio than C has to A ,

I say that B is less than A

For, if not it is either equal or greater

Now B is not equal to A

for in that case C would have had the same ratio to each of the magnitudes A, B , [v 7]

but it has not

therefore A is not equal to B

Nor again is B greater than A

for in that case C would have had to B a less ratio than it has to A , [v 8]

but it has not

therefore B is not greater than A

But it was proved that it is not equal either

therefore B is less than A

Therefore etc

Q E D

PROPOSITION 11

Ratios which are the same with the same ratio are also the same with one another

For, as A is to B so let C be to D

and as C is to D so let E be to F ,

I say that as A is to B , so is E to F

For of A, C, E let equmultiples G, H, K be taken and of B, D, F other,

| | | |
|-----------|-----------|-----------|
| A _____ | C _____ | E _____ |
| B _____ | D _____ | F _____ |
| G _____ | H _____ | K _____ |
| L _____ | M _____ | N _____ |

chance, equmultiples L, M, N

Then since, as A is to B , so is C to D ,

and of A, C equmultiples G, H have been taken,

and of B, D other chance equmultiples L, M

therefore, if G is in excess of L , H is also in excess of M ,

if equal equal,

and if less less

Again, since as C is to D , so is E to F ,

and of C, E equmultiples H, K have been taken

and of D, F other chance equmultiples M, N

therefore if H is in excess of M , K is also in excess of N ,

if equal equal

and if less less

But we saw that if H was in excess of M , G was also in excess of L , if equal equal, and if less less

so that, in addition, if G is in excess of L , K is also in excess of N ,

if equal equal

and if less less

And G, K are equmultiples of A, E ,

while L, N are other chance equmultiples of B, F ,

therefore as A is to B , so is E to F

Therefore etc

Q E D

PROPOSITION 12

If any number of magnitudes be proportional, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents

Let any number of magnitudes A, B, C, D, E, F be proportional, so that, as A is to B so is C to D and E to F ,

I say that as A is to B so are A, C, E to B, D, F

| | | |
|-----------|-----------|-----------|
| A _____ | B _____ | C _____ |
| D _____ | E _____ | F _____ |
| G _____ | L _____ | |
| H _____ | M _____ | |
| K _____ | N _____ | |

For of A, C, E let equmultiples G, H, K be taken

and of B, D, F other chance equmultiples L, M, N

Then since as A is to B , so is C to D , and E to F ,

and of A, C, E equmultiples G, H, K have been taken

and of B, D, F other chance equmultiples L, M, N ,

therefore if G is in excess of L , H is also in excess of M and K of N ,
if equal equal,

and if less less,
 so that in addition
 if G is in excess of L then G, H, K are in excess of L, M, N ,
 if equal, equal
 and if less less

Now G and G, H, K are equimultiples of A and A, C, E since, if any number of magnitudes whatever are respectively equimultiples of any magnitudes equal in multitude whatever multiple one of the magnitudes is of one that multiple also will all be of all [v 1]

For the same reason

L and L, M, N are also equimultiples of B and B, D, F ,
 therefore as A is to B , so are A, C, E to B, D, F [v Def 5]

Therefore etc

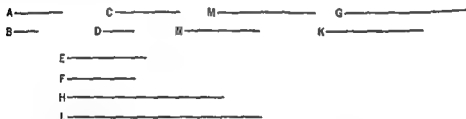
Q E D

PROPOSITION 13

If a first magnitude have to a second the same ratio as a third to a fourth and the third have to the fourth a greater ratio than a fifth has to a sixth the first will also have to the second a greater ratio than the fifth to the sixth

For let a first magnitude A have to a second B the same ratio as a third C has to a fourth D and let the third C have to the fourth D a greater ratio than a fifth E has to a sixth F ,

I say that the first A will also have to the second B a greater ratio than the fifth E to the sixth F



For since there are some equimultiples of C, E ,
 and of D, F other chance, equimultiples such that the multiple of C is in excess of the multiple of D

while the multiple of E is not in excess of the multiple of F , [v Def 7]
 let them be taken

and let G, H be equimultiples of C, E ,

and K, L other chance equimultiples of D, F

so that G is in excess of K but H is not in excess of L

and whatever multiple G is of C let M be also that multiple of A ,

and, whatever multiple K is of D let N be also that multiple of B

Now, since as A is to B so is C to D ,

and of A, C equimultiples M, G have been taken

and of B, D other chance equimultiples N, K

therefore if M is in excess of N G is also in excess of K ,
 if equal equal

and if less less

[v Def 5]

But G is in excess of A therefore M is also in excess of N But H is not in excess of L and M, H are equimultiples of A, E ,and N, L other chance equimultiples of B, F ,therefore A has to B a greater ratio than E has to F [v Def 7]

Therefore etc

Q E D

PROPOSITION 14

If a first magnitude have to a second the same ratio as a third has to a fourth, and the first be greater than the third the second will also be greater than the fourth if equal equal and if less less

For let a first magnitude A have the same ratio to a second B as a third C has to a fourth D , and let A be greater than C ,

I say that B is also greater than D

A —————

C —————

B —————

D —————

For, since A is greater than C and B is another, chance magnitudetherefore A has to B a greater ratio than C has to B [v 8]But as A is to B so is C to D ,therefore C has also to D a greater ratio than C has to B [v 13]

But that to which the same has a greater ratio is less, [v 10]

therefore D is less than B so that B is greater than D Similarly we can prove that if A be equal to C B will also be equal to D and if A be less than C , B will also be less than D

Therefore etc

Q E D

PROPOSITION 15

Parts have the same ratio as the same multiples of them taken in corresponding order

For let AB be the same multiple of C that DE is of F I say that as C is to F so is AB to DE

For since AB is the same multiple of C that DE is of F as many magnitudes as there are in AB equal to C so many are there also in DE equal to F

A — G — H — B

C —————

C — K — L — E

F —————

Let AB be divided into the magnitudes AG, GH, HB equal to C and DE into the magnitudes DK, KL, LE equal to F then the multitude of the magnitudes AG, GH, HB will be equal to the multitude of the magnitudes DK, KL, LE And since AG, GH, HB are equal to one anotherand DK, KL, LE are also equal to one anothertherefore as AG is to DK so is GH to KL and HB to LE [v 7]

Therefore as one of the antecedents is to one of the consequents so will all the antecedents be to all the consequents [v 12]

therefore as AG is to DK so is AB to DE

But AG is equal to C and DA to F ,
therefore, as C is to F , so is AB to DE

Therefore etc

Q E D

PROPOSITION 16

If four magnitudes be proportional, they will also be proportional alternately

Let A, B, C, D be four proportional magnitudes

so that as A is to B , so is C to D ,

I say that they will also be so alternately, that is, as A is to C , so is B to D



For of A, B let equimultiples E, F be taken

and of C, D other, chance, equimultiples G, H

Then, since E is the same multiple of A that F is of B ,

and parts have the same ratio as the same multiples of them. [v 15]

therefore as A is to B , so is E to F

But as A is to B so is C to D ,

therefore also as C is to D so is E to F

[v 11]

Again, since G, H are equimultiples of C, D ,

therefore, as C is to D , so is G to H

[v 15]

But as C is to D so is E to F ,

therefore also as F is to E so is G to H

[v 11]

But if four magnitudes be proportional and the first be greater than the third,

the second will also be greater than the fourth,

if equal equal

and if less less

[v 14]

Therefore if F is in excess of G , F is also in excess of H ,

if equal equal,

and if less less

Now E, F are equimultiples of A, B

and G, H other, chance equimultiples of C, D ,

therefore as A is to C so is B to D

[v Def 5]

Therefore etc

Q E D

PROPOSITION 17

If magnitudes be proportional componendo they will also be proportional separando

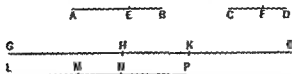
Let AB, BE, CD, DF be magnitudes proportional componendo so that as AB is to BE so is CD to DF ,

I say that they will also be proportional separando, that is as AE is to EB , so is CF to DF

For of AE, FB, CF, FD let equimultiples GH, HK, LM, MN be taken,

and of EB, FD other chance equimultiples AO, NP

Then, since GH is the same multiple of AE that HA is of EB ,
 therefore GH is the same multiple of AE that GA is of AB [v 1]
 But GH is the same multiple of AE that LM is of CF
 therefore GA is the same multiple of AB that LM is of CF



Again, since LM is the same multiple of CF that MN is of FD
 therefore LM is the same multiple of CF that LN is of CD [v 1]
 But LM was the same multiple of CF that GA is of AB
 therefore GA is the same multiple of AB that LN is of CD
 Therefore GA LN are equimultiples of AB , CD
 Again, since HK is the same multiple of EB that MN is of FD ,
 and AO is also the same multiple of EB that NP is of FD ,
 therefore the sum HO is also the same multiple of EB that MP is of FD [v 2]

And, since as AB is to BE so is CD to DF
 and of AB , CD equimultiples GA , LN have been taken
 and of EB , FD equimultiples HO MP
 therefore, if GA is in excess of HO , LN is also in excess of MP ,
 if equal, equal
 and if less, less

Let GH be in excess of KO ,
 then if HA be added to each,
 GA is also in excess of HO

But we saw that if GA was in excess of HO LN was also in excess of MP ,
 therefore LN is also in excess of MP
 and if MN be subtracted from each,
 LM is also in excess of NP

so that if GH is in excess of KO LM is also in excess of NP

Similarly we can prove that

if GH be equal to KO LM will also be equal to NP
 and if less, less

And GH LM are equimultiples of AE CF

while AO NP are other chance equimultiples of EB , FD ,

therefore, as AE is to EB so is CF to FD

Therefore etc

Q E D

PROPOSITION 18

If magnitudes be proportional *separando* they will also be proportional *componendo*

Let AE EB CF FD be magnitudes proportional *separando*, so that as AE is to EB so is CF to FD

I say that they will also be proportional *componendo*, that is as AB is to BE so is CD to FD

For, if CD be not to DF as AB to BE , then, as AB is to BE , so will CD be either to some magnitude less than DF or to a greater

First, let it be in that ratio to a less magnitude DG

Then, since as AB is to BE , so is CD to DG ,

they are magnitudes proportional *componendo*, so that they will also be proportional *separando* [v 17]

Therefore as AE is to EB so is CG to GD

But also by hypothesis

as AE is to EB , so is CF to FD

Therefore also as CG is to GD , so is CF to FD [v 11]

But the first CG is greater than the third CF ,

therefore the second GD is also greater than the fourth FD [v 14]

But it is also less which is impossible

Therefore as AB is to BE so is not CD to a less magnitude than FD

Similarly we can prove that neither is it in that ratio to a greater, it is therefore in that ratio to FD itself

Therefore etc

Q E D

PROPOSITION 19

If as a whole is to a whole so is a part subtracted to a part subtracted the remainder will also be to the remainder as whole to whole

For as the whole AB is to the whole CD , so let the part AE subtracted be to the part CF subtracted,

I say that the remainder EB will also be to the remainder FD as the whole AB to the whole CD

For since as AB is to CD , so is AE to CF ,

alternately also as BA is to AE , so is DC to CF [v 16]

And since the magnitudes are proportional *componendo* they will also be proportional *separando*, [v 17]

that is as BE is to EA so is DF to CF

and alternately

as BE is to DF so is EA to FC [v 16]

But as AE is to CF so by hypothesis is the whole AB to the whole CD

Therefore also the remainder EB will be to the remainder FD as the whole AB is to the whole CD [v 11]

Therefore etc

[PORISM From this it is manifest that if magnitudes be proportional *componendo* they will also be proportional *convertendo*]

Q E D

PROPOSITION 20

If there be three magnitudes and others equal to them in multitude which taken two and two are in the same ratio and if ex aequali the first be greater than the third the fourth will also be greater than the sixth if equal equal and if less less

Let there be three magnitudes $A B C$ and others $D E, F$ equal to them in multitude which taken two and two are in the same ratio so that,

as A is to B so is D to E

as B is to C so is E to F

and let A be greater than C *ex aequali*,

I say that D will also be greater than F , if A is equal to C , equal, and if less, less

A _____
 B _____
 C _____

D _____
 E _____
 F _____

For since A is greater than C ,
 and B is some other magnitude
 and the greater has to the same a greater
 ratio than the less has [v 8]
 therefore A has to B a greater ratio
 than C has to B

But as A is to B , so is D to E ,

and as C is to B inversely so is F to E ,

therefore D has also to E a greater ratio than F has to E [v 13]

But of magnitudes which have a ratio to the same, that which has a greater ratio is greater, [v 10]

therefore D is greater than F

Similarly we can prove that, if A be equal to C , D will also be equal to F ,
 and if less less

Therefore etc

Q E D

PROPOSITION 21

If there be three magnitudes and others equal to them in multitude which taken two and two together are in the same ratio and the proportion of them be perturbed then if *ex aequali* the first magnitude is greater than the third the fourth will also be greater than the sixth if equal equal and if less less

Let there be three magnitudes A, B, C and others D, E, F equal to them in multitude which taken two and two are in the same ratio and let the proportion of them be perturbed so that,

as A is to B so is E to F

and as B is to C so is D to E ,

and let A be greater than C *ex aequali*,

I say that D will also be greater than F if A is equal to C equal, and if less less

A _____
 B _____
 C _____

D _____
 E _____
 F _____

For since A is greater than C
 and B is some other magnitude,
 therefore A has to B a greater
 ratio than C has to B [v 8]

But as A is to B so is E to F

and as C is to B inversely so is E to D

Therefore also E has to F a greater ratio than E has to D [v 13]

But that to which the same has a greater ratio is less [v 10]

therefore F is less than D

therefore D is greater than F

Similarly we can prove that if A be equal to C D will also be equal to F
 and if less less

Therefore etc

Q E D

PROPOSITION 22

If there be any number of magnitudes whatever, and others equal to them in multitude which taken two and two together are in the same ratio they will also be in the same ratio *ex aequali*

Let there be any number of magnitudes A, B, C , and others D, E, F equal to them in multitude which taken two and two together are in the same ratio, so that,

as A is to B so is D to E

and, as B is to C so is E to F ,

I say that they will also be in the same ratio *ex aequali*,

<that is as A is to C so is D to F >

For of A, D let equimultiples G, H be taken

and of B, E other chance, equimultiples K, L ,

and, further of C, F other chance, equimultiples M, N



Then, since as A is to B so is D to E ,

and of A, D equimultiples G, H have been taken,

and of B, E other chance equimultiples K, L ,

therefore as G is to K , so is H to L

[v. 4]

For the same reason also

as K is to M so is L to N

Since then there are three magnitudes G, K, M and others H, L, N equal to them in multitude, which taken two and two together are in the same ratio,

therefore *ex aequali* if G is in excess of M , H is also in excess of N

if equal, equal and if less less

[v. 20]

And G, H are equimultiples of A, D

and M, N other chance equimultiples of C, F

Therefore as A is to C so is D to F

[v. Def. 5]

Therefore etc

Q. E. D.

PROPOSITION 23

If there be three magnitudes and others equal to them in multitude which taken two and two together are in the same ratio and the proportion of them be perturbed they will also be in the same ratio *ex aequali*

Let there be three magnitudes A, B, C , and others equal to them in multitude, which taken two and two together are in the same proportion, namely D, E, F and let the proportion of them be perturbed so that,

as A is to B so is F to E

and,

as B is to C so is D to E

I say that as A is to C so is D to F

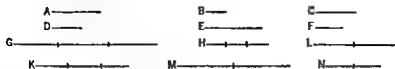
Of A, B, D let equimultiples G, H, K be taken

and of C, E, F other chance equimultiples L, M, N

Then since G, H are equimultiples of A, B

and parts have the same ratio as the same multiples of them, [v 15]
therefore, as A is to B , so is G to H

For the same reason also



as E is to F , so is M to N

And as A is to B , so is E to F ,
therefore also, as G is to H , so is M to N [v 11]

Next, since as B is to C , so is D to E ,
alternately, also, as B is to D , so is C to E [v 16]

And since H, K are equimultiples of B, D
and parts have the same ratio as their equimultiples
therefore as B is to D , so is H to K [v 15]

But as B is to D so is C to E ,
therefore also as H is to K , so is C to E [v 11]

Again, since L, M are equimultiples of C, E ,
therefore as C is to E , so is L to M [v 15]

But, as C is to E so is H to K ,
therefore also as H is to K , so is L to M [v 11]

and alternately as H is to L so is K to M [v 16]

But it was also proved that,

as G is to H , so is M to N

Since then there are three magnitudes G, H, L and others equal to them in multitude A, M, N which taken two and two together are in the same ratio, and the proportion of them is perturbed,

therefore, *ex aequali*, if G is in excess of L , A is also in excess of N ,
if equal equal, and if less, less [v 21]

And G, K are equimultiples of A, D
and L, N of C, F

Therefore as A is to C , so is D to F

Therefore etc

Q E D

PROPOSITION 24

If a first magnitude have to a second the same ratio as a third has to a fourth and also a fifth have to the second the same ratio as a sixth to the fourth the first and fifth added together will have to the second the same ratio as the third and sixth have to the fourth

Let a first magnitude AB have to a second C the same ratio as a third DE has to a fourth F

A ————— B ————— G and let also a fifth BG have to the second C
 C —————
 D ————— E ————— H the same ratio as a sixth EH has to the
 F ————— fourth F ,

I say that the first and fifth added together,
 AG will have to the second C the same ratio
as the third and sixth DH has to the fourth F

For since as BG is to C , so is EH to F

inversely as C is to BG , so $\equiv F$ to EH

Since then, as AB is to C , so is DE to F ,

and as C is to BG so is F to EH ,

therefore *ex aequali* as $AB \equiv$ to BG , so is DE to EH [v 22]

And, since the magnitudes are proportional *separando*, they will also be proportional *componendo*, [v 18]

therefore, as AG is to GB , so $\equiv DH$ to HE

But also, as BG is to C , so is EH to F ,

therefore, *ex aequali* as AG is to C , so is DH to F [v 22]

Therefore etc

Q E D

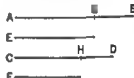
PROPOSITION 25

If four magnitudes be proportional the greatest and the least are greater than the remaining two

Let the four magnitudes AB, CD, E, F be proportional so that as AB is to CD , so is E to F and let AB be the greatest of them and F the least

I say that AB, F are greater than CD, E

For let AG be made equal to E , and CH equal to F



Since as $AB \equiv$ to CD , so is E to F ,

and E is equal to AG and F to CH ,

therefore as AB is to CD so is AG to CH

And since as the whole AB is to the whole CD so is the part AG subtracted to the part CH subtracted the remainder GB will also be to the remainder HD as the whole $AB \equiv$ to the whole CD [v 19]

But AB is greater than CD

therefore GB is also greater than HD

And since AG is equal to E and CH to F ,

therefore AG, F are equal to CH, E

And if GB, HD being unequal and GB greater AG, F be added to GB and CH, E be added to HD ,

it follows that AB, F are greater than CD, E

Therefore etc

Q E D

BOOK SIX

DEFINITIONS

1 *Similar rectilineal figures* are such as have their angles severally equal and the sides about the equal angles proportional

■ A straight line is said to have been *cut in extreme and mean ratio* when, as the whole line is to the greater segment so is the greater to the less

3 The *height* of any figure is the perpendicular drawn from the vertex to the base

BOOK VI PROPOSITIONS

PROPOSITION 1

Triangles and parallelograms which are under the same height are to one another as their bases

Let ABC ACD be triangles and EC CF parallelograms under the same height,

I say that as the base BC is to the base CD so is the triangle ABC to the triangle ACD and the parallelogram EC to the parallelogram CF

For let BD be produced in both directions to the points H , L and let [any number of straight lines] BG , GH be made equal to the base BC , and any number of straight lines DK , KL equal to the base CD ,

let AG , AH , AK , AL be joined

Then since CB , BG , GH are equal to one another

the triangles ABC AGB AHG are also equal to one another [1 28]

Therefore whatever multiple the base HC is of the base BC that multiple also is the triangle AHC of the triangle ABC

For the same reason

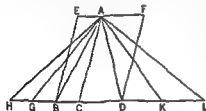
whatever multiple the base LC is of the base CD that multiple also is the triangle AIC of the triangle ACD

and if the base HC is equal to the base CL the triangle AHC is also equal to the triangle ACL

if the base HC is in excess of the base CL the triangle AHC is also in excess of the triangle ACL

and if less less

Thus there being four magnitudes two bases BC CD and two triangles ABC ACD



equimultiples have been taken of the base BC and the triangle ABC , namely the base HC and the triangle AHC and of the base CD and the triangle ADC other chance equimultiples namely the base LC and the triangle ALC ,

and it has been proved that

if the base HC is in excess of the base CL , the triangle AHC is also in excess of the triangle ALC ,

if equal equal, and, if less less

Therefore as the base BC is to the base CD , so \equiv the triangle ABC to the triangle ACD [v Def 5]

Next, since the parallelogram EC is double of the triangle ABC , [i 41] and the parallelogram FC is double of the triangle ACD

while parts have the same ratio as the same multiples of them [v 15] therefore as the triangle ABC is to the triangle ACD so is the parallelogram EC to the parallelogram FC

Since then it was proved that as the base BC is to CD so is the triangle ABC to the triangle ACD

and as the triangle ABC is to the triangle ACD so is the parallelogram EC to the parallelogram FC

therefore also as the base BC is to the base CD so is the parallelogram EC to the parallelogram FC [v 11]

Therefore etc

Q E D

PROPOSITION 2

If a straight line be drawn parallel to one of the sides of a triangle it will cut the sides of the triangle proportionally and if the sides of the triangle be cut proportionally the line joining the points of section will be parallel to the remaining side of the triangle

For let DE be drawn parallel to BC one of the sides of the triangle ABC , I say that as BD is to DA so is CE to EA

For let BE CD be joined
Therefore the triangle BDE is equal to the triangle CDE

for they are on the same base DE and in the same parallels DE BC [i 38]

And the triangle ADE is another area

But equals have the same ratio to the same [v 7]
therefore as the triangle BDE is to the triangle ADE so is the triangle CDE to the triangle ADE

But as the triangle BDE is to ADE so is BD to DA

for being under the same height the perpendicular drawn from E to AB they are to one another as their bases [vi 1]

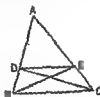
For the same reason also

as the triangle CDE is to ADE so \equiv CE to EA

Therefore also as BD is to DA so \equiv CE to EA [v 11]

Again let the sides AB AC of the triangle ABC be cut proportionally so that as BD \equiv DA so is CE to EA and let DE be joined

I say that DE is parallel to BC



For, with the same construction

since as BD is to DA so is CE to EA

but, as BD is to DA , so is the triangle BDE to the triangle ADE

and as CE is to EA , so is the triangle CDE to the triangle ADE , [vi 1]

therefore also,

as the triangle BDE is to the triangle ADE , so is the triangle CDE to the triangle ADE [v 11]

Therefore each of the triangles BDE CDE has the same ratio to ADE

Therefore the triangle BDE is equal to the triangle CDE , [v 9]

and they are on the same base DF

But equal triangles which are on the same base are also in the same parallels [i 39]

Therefore DE is parallel to BC

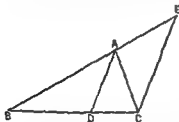
Therefore etc

Q E D

PROPOSITION 3

If an angle of a triangle be bisected and the straight line cutting the angle cut the base also the segments of the base will have the same ratio as the remaining sides of the triangle and, if the segments of the base have the same ratio as the remaining sides of the triangle the straight line joined from the vertex to the point of section will bisect the angle of the triangle

Let ABC be a triangle and let the angle BAC be bisected by the straight line AD ,



I say that as BD is to CD so is BA to AC

For let CE be drawn through C parallel to DA , and let BA be carried through and meet it at E

Then since the straight line AC falls upon the parallels AD EC

the angle ACE is equal to the angle CAD

[i 29]

But the angle CAD is by hypothesis equal to the angle BAD ,

therefore the angle BAD is also equal to the angle ACE

Again since the straight line BAE falls upon the parallels AD EC ,

the exterior angle BAD is equal to the interior angle AEC [i 29]

But the angle ACE was also proved equal to the angle BAD

therefore the angle ACE is also equal to the angle AEC ,

so that the side AE is also equal to the side AC

[i 6]

And since AD has been drawn parallel to EC one of the sides of the triangle BCF

therefore proportionally as BD is to DC , so is BA to AE

But AE is equal to AC ,

[vi 2]

therefore as BD is to DC so is BA to AC

Again let BA be to AC as BD to DC , and let AD be joined,

I say that the angle BAC has been bisected by the straight line AD

For with the same construction,

since as BD is to DC , so is BA to AC

and also as BD is to DC so is BA to AE for AD has been drawn parallel to

EC, one of the sides of the triangle *BCE*

therefore also, as *BA* is to *AC*, so is *BA* to *AE*

[vi 2]

[v 11]

[v 9]

[i 5]

[i 29]

[id]

Therefore *AC* is equal to *AE*

so that the angle *AEC* is also equal to the angle *ACE*

But the angle *AEC* is equal to the exterior angle *BAD*

and the angle *ACE* is equal to the alternate angle *CAD*,

therefore the angle *BAD* is also equal to the angle *CAD*

Therefore the angle *BAC* has been bisected by the straight line *AD*

Therefore etc

Q E D

PROPOSITION 4

In equiangular triangles the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles

Let *ABC*, *DCE* be equiangular triangles having the angle *ABC* equal to the angle *DCE* the angle *BAC* to the angle *CDE* and further the angle *ACB* to the angle *CED*,

I say that in the triangles *ABC*, *DCE* the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles

For let *BC* be placed in a straight line with *CE*

Then since the angles *ABC* *ACB* are less than two right angles [i 17]

and the angle *ACB* is equal to the angle *DEC*,

therefore the angles *ABC* *DEC* are less than two right angles

therefore *BA*, *ED* when produced, will meet [i Post 5]

Let them be produced and meet at *F*

Now since the angle *DCE* is equal to the angle *ABC*,

BF is parallel to *CD*

[i 28]

Again since the angle *ACB* is equal to the angle *DEC*

AC is parallel to *FE*

[i 28]

Therefore *FACD* is a parallelogram,

therefore *FA* is equal to *DC*, and *AC* to *FD*

[i 34]

And since *AC* has been drawn parallel to *FE*, one side of the triangle *FBE*,

therefore as *BA* is to *AF* so is *BC* to *CE*

[vi 2]

But *AF* is equal to *CD*

therefore as *BA* is to *CD*, so is *BC* to *CE*,

and alternately as *AB* is to *BC*, so is *DC* to *CE*

[v 16]

Again since *CD* is parallel to *BF*,

therefore as *BC* is to *CE* so is *FD* to *DE*

[vi 2]

But *FD* is equal to *AC*,

therefore as *BC* is to *CE* so is *AC* to *DE*,

and alternately as *BC* is to *CA*, so is *CE* to *ED*

[v 16]

Since then it was proved that

as *AB* is to *BC* so is *DC* to *CE*

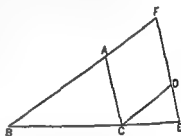
and, as *BC* is to *CA* so is *CE* to *ED*,

therefore *ex aequali* as *BA* is to *AC*, so is *CD* to *DE*

[v 22]

Therefore etc

Q E D



PROPOSITION 5

If two triangles have their sides proportional the triangles will be equiangular and will have those angles equal which the corresponding sides subtend

Let ABC , DEF be two triangles having their sides proportional, so that

as AB is to BC , so is DE to EF ,

as BC is to CA , so is EF to FD ,

and further, as BA is to AC , so is ED to DF ,

I say that the triangle ABC is equiangular with the triangle DEF , and they will have those angles equal which the corresponding sides subtend namely the angle ABC to the angle DEF the angle BCA to the angle EFD and further the angle BAC to the angle EDF

For on the straight line EF , and at the points E , F on it, let there be constructed the angle FEG equal to the angle ABC , and the angle EFG equal to the angle ACB ,

[I 23]

therefore the remaining angle at A is equal to the remaining angle at G

[I 32]

Therefore the triangle ABC is equiangular with the triangle GEF



Therefore in the triangles ABC GEF the sides about the equal angles are proportional and those are corresponding sides which subtend the equal angles [I 4] therefore as AB is to BC , so is GE to EF

But as AB is to BC , so by hypothesis is DE to EF ,

therefore as DE is to EF , so is GE to EF [V 11]

Therefore each of the straight lines DE GE has the same ratio to EF

therefore DE is equal to GE [V 9]

For the same reason

DF is also equal to GF

Since then DE is equal to EG

and EF is common,

the two sides DE , EF are equal to the two sides GE , EF ,

and the base DF is equal to the base FG

therefore the angle DEF is equal to the angle GEF , [I 8]

and the triangle DEF is equal to the triangle GEF ,

and the remaining angles are equal to the remaining angles namely those which the equal sides subtend [I 4]

Therefore the angle DFE is also equal to the angle GFE ,

and the angle EDF to the angle EGF

And, since the angle FED is equal to the angle GEF

while the angle GEF is equal to the angle ABC

therefore the angle ABC is also equal to the angle DEF

For the same reason

the angle ACB is also equal to the angle DFE

and further the angle at A to the angle at D

therefore the triangle ABC is equiangular with the triangle DEF

Therefore etc

Q E D

PROPOSITION 6

If two triangles have one angle equal to one angle and the sides about the equal angles proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend

Let ABC DEF be two triangles having one angle BAC equal to one angle EDF and the sides about the equal angles proportional so that,
as BA is to AC , so is ED to DF ,

I say that the triangle ABC is equiangular with the triangle DEF , and will have the angle ABC equal to the angle DEF , and the angle ACB to the angle DFE

For on the straight line DF and at the points D, F on it, let there be constructed the angle $F DG$ equal to either of the angles BAC, EDF , and the angle $D FG$ equal to the angle ACB , [I 23]

therefore the remaining angle at B is equal to the remaining angle at G [I 32]

Therefore the triangle ABC is equiangular with the triangle DGF

Therefore proportionally as BA is to AC , so is GD to DF [VI 4]

But by hypothesis as BA is to AC so also is ED to DF ,

therefore also as ED is to DF so is GD to DF [V 11]

Therefore ED is equal to DG , [V 9]

and DF is common

therefore the two sides ED, DF are equal to the two sides GD, DF , and the angle EDF is equal to the angle GDF ,



therefore the base EF is equal to the base GF ,

and the triangle DEF is equal to the triangle DGF

and the remaining angles will be equal to the remaining angles namely those which the equal sides subtend [I 4]

Therefore the angle DFG is equal to the angle $D FE$

and the angle DGF to the angle DEF

But the angle DFG is equal to the angle ACB ,

therefore the angle ACB is also equal to the angle $D FE$

And by hypothesis the angle BAC is also equal to the angle EDF ,

therefore the remaining angle at B is also equal to the remaining angle at E , [I 32]

therefore the triangle ABC is equiangular with the triangle DEF

Therefore etc

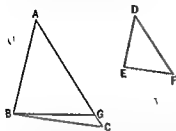
Q E D

PROPOSITION 7

If two triangles have one angle equal to one angle the sides about other angles proportional and the remaining angles either both less or both not less than a right angle, the triangles will be equiangular and will have those angles equal, the sides about which are proportional

Let ABC, DEF be two triangles having one angle equal to one angle the angle BAC to the angle EDF , the sides about other angles ABC, DEF proportional so that as AB is to BC so is DE to EF , and first each of the remaining angles at C, F less than a right angle,

I say that the triangle ABC is equiangular with the triangle DEF , the angle ABC will be equal to the angle DEF , and the remaining angle namely the angle at C , equal to the remaining angle the angle at F



For if the angle ABC is unequal to the angle DEF , one of them is greater

Let the angle ABC be greater, and on the straight line AB , and at the point B on it, let the angle ABG be constructed equal to the angle DEF [I 23]

Then, since the angle A is equal to D , and the angle ABG to the angle DEF , therefore the remaining angle AGB is equal to the remaining angle DFE [I 32]

Therefore the triangle ABG is equiangular with the triangle DEF

Therefore, as AB is to BG , so is DE to EF [VI 4]

But as DE is to EF so by hypothesis is AB to BC , therefore AB has the same ratio to each of the straight lines BC , BG , [V 11]
therefore BC is equal to BG [V 9]

so that the angle at C is also equal to the angle BGC [I 5]

But, by hypothesis the angle at C is less than a right angle, therefore the angle BGC is also less than a right angle, so that the angle AGB adjacent to it is greater than a right angle [I 13]

And it was proved equal to the angle at F , therefore the angle at F is also greater than a right angle

But it is by hypothesis less than a right angle which is absurd

Therefore the angle ABC is not unequal to the angle DEF ,

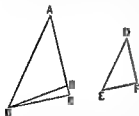
therefore it is equal to it

But the angle at A is also equal to the angle at D

therefore the remaining angle at C is equal to the remaining angle at F [I 32]

Therefore the triangle ABC is equiangular with the triangle DEF

But again, let each of the angles at C , F be supposed not less than a right angle,



I say again that in this case too the triangle ABC is equiangular with the triangle DEF

For, with the same construction, we can prove similarly that

BC is equal to BG

so that the angle at C is also equal to the angle BGC [I 5]

But the angle at C is not less than a right angle, therefore neither is the angle BGC less than a right angle

Thus in the triangle BGC the two angles are not less than two right angles which is impossible [I 17]

Therefore once more the angle ABC is not unequal to the angle DEF , therefore it is equal to it

But the angle at A is also equal to the angle at D

therefore the remaining angle at C is equal to the remaining angle at F [I 32]

Therefore the triangle ABC is equiangular with the triangle DEF

Therefore etc

Q E D

PROPOSITION 8

If in a right angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another

Let ABC be a right-angled triangle having the angle BAC right, and let AD be drawn from A perpendicular to BC ,

I say that each of the triangles ABD , ADC is similar to the whole ABC and further, they are similar to one another

For, since the angle BAC is equal to the angle ADB , for each is right and the angle at B is common to the two triangles ABC and ABD

therefore the remaining angle ACB is equal to the remaining angle BAD , [I 32]

therefore the triangle ABC is equiangular with the triangle ABD

Therefore as BC which subtends the right angle in the triangle ABC is to BA which subtends the right angle in the triangle ABD , so is AB itself which subtends the angle at C in the triangle ABC to BD which subtends the equal angle BAD in the triangle ABD and so also is AC to AD which subtends the angle at B common to the two triangles [VI 4]

Therefore the triangle ABC is both equiangular to the triangle ABD and has the sides about the equal angles proportional

Therefore the triangle ABC is similar to the triangle ABD [VI Def 1]

Similarly we can prove that

the triangle ABC is also similar to the triangle ADC

therefore each of the triangles ABD , ADC is similar to the whole ABC

I say next that the triangles ABD , ADC are also similar to one another

For since the right angle BDA is equal to the right angle ADC ,

and moreover the angle BAD was also proved equal to the angle at C therefore the remaining angle at B is also equal to the remaining angle DAC , [I 32]

therefore the triangle ABD is equiangular with the triangle ADC

Therefore as BD which subtends the angle BAD in the triangle ABD is to DA which subtends the angle at C in the triangle ADC equal to the angle BAD , so is AD itself which subtends the angle at B in the triangle ABD to DC which subtends the angle DAC in the triangle ADC equal to the angle at B , and so also is BA to AC these sides subtending the right angles, [VI 4]

therefore the triangle ABD is similar to the triangle ADC [VI Def 1]

Therefore etc

POLEM. From this it is clear that if in a right angled triangle a perpendicular be drawn from the right angle to the base the straight line so drawn is a mean proportional between the segments of the base Q E D

PROPOSITION 9

From a given straight line to cut off a prescribed part

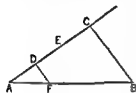
Let AB be the given straight line,

thus it is required to cut off from AB a prescribed part



Let the third part be that prescribed

Let a straight line AC be drawn through from A containing with AB any angle,



let a point D be taken at random on AC , and let DE, EC be made equal to AD [I 3]

Let BC be joined, and through D let DF be drawn parallel to it [I 31]

Then since FD has been drawn parallel to BC one of the sides of the triangle ABC , therefore, proportionally, as CD is to DA , so is BF to FA [VI 2]

But CD is double of DA

therefore BF is also double of FA ,

therefore BA is triple of AF

Therefore from the given straight line AB the prescribed third part AF has been cut off Q E F

PROPOSITION 10

To cut a given uncut straight line similarly to a given cut straight line

Let AB be the given uncut straight line and AC the straight line cut at the points D, E , and let them be so placed as to contain any angle,

let CB be joined and through D, E let DF, EG be drawn parallel to BC , and through D let DH be drawn parallel to AB [I 31]

Therefore each of the figures FH, HB is a parallelogram,

therefore DH is equal to FG and HK to GB

[I 34]

Now since the straight line HE has been drawn parallel to KC , one of the sides of the triangle DKC

therefore, proportionally, as CE is to ED , so is KH to HD [VI 2]

But AH is equal to BG and HD to GF ,

therefore, as CE is to ED so is BG to GF

Again since FD has been drawn parallel to GE , one of the sides of the triangle AGE

therefore proportionally, as ED is to DA so is GF to FA [VI 2]

But it was also proved that

as CE is to ED , so is BG to GF ,

therefore as CE is to ED so is BG to GF

and as ED is to DA so is GF to FA

Therefore the given uncut straight line AB has been cut similarly to the given cut straight line AC Q E F

PROPOSITION 11

To two given straight lines to find a third proportional

Let BA, AC be the two given straight lines and let them be placed so as to contain any angle

thus it is required to find a third proportional to BA, AC

For let them be produced to the points D, E , and let BD be made equal to AC ,

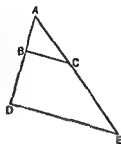
let BC be joined, and through D let DE be drawn parallel to it [I 3]

Since then, BC has been drawn parallel to DE , one of the sides of the triangle ADE , proportionally as AB is to BD , so is AC to CE [I 31]

But BD is equal to AC ,

therefore as AB is to AC , so is AC to CE [VI 2]

Therefore to two given straight lines AB, AC a third proportional to them CE , has been found Q E F

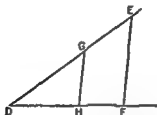


PROPOSITION 12

To three given straight lines to find a fourth proportional

Let A, B, C be the three given straight lines,

thus it is required to find a fourth proportional to A, B, C



Let two straight lines DE, DF be set out containing any angle EDF , let DG be made equal to A , GE equal to B , and further DH equal to C , let GH be joined, and let EF be drawn through E parallel to it [I 31]

Since then, GH has been drawn parallel to EF , one of the sides of the triangle DEF ,

therefore as DG is to GE so is DH to HF [VI 2]

But DG is equal to A , GE to B and DH to C ,

therefore as A is to B so is C to HF

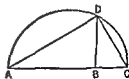
Therefore to the three given straight lines A, B, C a fourth proportional HF has been found Q E F

PROPOSITION 13

To two given straight lines to find a mean proportional

Let AB, BC be the two given straight lines, thus it is required to find a mean proportional to AB, BC

Let them be placed in a straight line, and let the semicircle ADC be described on AC let BD be drawn from the point B at right angles to the straight line AC and let AD, DC be joined



Since the angle ADC is an angle in a semicircle it is right [III 31]

And, since in the right-angled triangle ADC DB has been drawn from the right angle perpendicular to the base,

therefore DB is a mean proportional between the segments of the base, AB BC [vi 8 Por]

Therefore to the two given straight lines AB , BC a mean proportional DB has been found $Q\ E\ F$

PROPOSITION 14

In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional and equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal

Let AB , BC be equal and equiangular parallelograms having the angles at B equal and let DB BE be placed in a straight line

therefore FB , BG are also in a straight line

[I 14]

I say that in AB BC the sides about the equal angles are reciprocally proportional that is to say, that, as DB is to BE , so is GB to BF

For let the parallelogram FE be completed

Since then, the parallelogram AB is equal to the parallelogram BC

and FE is another area

therefore as AB is to FE , so is BC to FE

[v 7]

But, as AB is to FE so is DB to BE

[vi 1]

and as BC is to FE so is GB to BF

[id]

therefore also as DB is to BE so is GB to BF

[v 11]

Therefore in the parallelograms AB BC the sides about the equal angles are reciprocally proportional

Next let GB be to BF as DB to BE

I say that the parallelogram AB is equal to the parallelogram BC

For since as DB is to BE so is GB to BF

while as DB is to BE so is the parallelogram AB to the parallelogram FE

[vi 1]

and as GB is to BF so is the parallelogram BC to the parallelogram FE , [vi 1]

therefore also as AB is to FE so is BC to FE

[v 11]

therefore the parallelogram AB is equal to the parallelogram BC [v 9]

Therefore etc

$Q\ E\ D$

PROPOSITION 15

In equal triangles which have one angle equal to one angle the sides about the equal angles are reciprocally proportional and those triangles which have one angle equal to one angle and in which the sides about the equal angles are reciprocally proportional are equal

Let ABC ADE be equal triangles having one angle equal to one angle namely the angle BAC to the angle DAE

I say that in the triangles ABC , ADE the sides about the equal angles are reciprocally proportional that is to say that

as CA is to AD so is EA to AB

For let them be placed so that CA is in a straight line with AD

therefore EA is also in a straight line with AB

\angle \angle

Let BD be joined

Since, then the triangle ABC is equal to the triangle ADE , and BAD is an other area

therefore as the triangle CAB is to the triangle BAD so is the triangle EAD to the triangle BAD [v 7]

But, as CAB is to BAD , so is CA to AD [vi 1]

and, as EAD is to BAD so is EA to AB [id]

Therefore also as CA is to AD , so is EA to AB [v 11]

Therefore in the triangles ABC , ADE the sides about the equal angles are reciprocally proportional

Next, let the sides of the triangles ABC ADE be reciprocally proportional that is to say, let EA be to AB as CA to AD ,

I say that the triangle ABC is equal to the triangle ADE

For if BD be again joined

since as CA is to AD , so is EA to AB ,

while, as CA is to AD , so is the triangle ABC to the triangle BAD ,

and as EA is to AB , so is the triangle EAD to the triangle BAD , [vi 1]

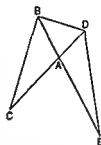
therefore as the triangle ABC is to the triangle BAD , so is the triangle EAD to the triangle BAD [v 11]

Therefore each of the triangles ABC , EAD has the same ratio to BAD

Therefore the triangle ABC is equal to the triangle EAD [v 9]

Therefore etc

Q E D



PROPOSITION 16

If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means and if the rectangle contained by the extremes be equal to the rectangle contained by the means the four straight lines will be proportional

Let the four straight lines AB CD , E F be proportional so that, as AB is to CD so is E to F

I say that the rectangle contained by AB F is equal to the rectangle contained by CD , E



E ———



F ———

Let AG CH be drawn from the points A C at right angles to the straight lines AB , CD and let AG be made equal to F and CH equal to E

Let the parallelograms BG DH be completed

Then since, as AB is to CD so is E to F

while E is equal to CH and F to AG

therefore as AB is to CD , so is CH to AG

Therefore in the parallelograms BG , DH the sides about the equal angles are reciprocally proportional

But those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal [vi 14]

therefore the parallelogram BG is equal to the parallelogram DH

And BG is the rectangle AB, F for AG is equal to F

and DH is the rectangle CD, E , for E is equal to CH ,

therefore the rectangle contained by AB, F is equal to the rectangle contained by CD, E

Next let the rectangle contained by AB, F be equal to the rectangle contained by CD, E ,

I say that the four straight lines will be proportional so that, as AB is to CD so is E to F

For, with the same construction,

since the rectangle AB, F is equal to the rectangle CD, E ,

and the rectangle AB, F is BG , for AG is equal to F

and the rectangle CD, E is DH , for CH is equal to E

therefore BG is equal to DH

And they are equiangular

But in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional [VI 14]

Therefore, as AB is to CD so is CH to AG

But CH is equal to E , and AG to F ,

therefore, as AB is to CD , so is E to F

Therefore etc

Q E D

PROPOSITION 17

If three straight lines be proportional the rectangle contained by the extremes is equal to the square on the mean and, if the rectangle contained by the extremes be equal to the square on the mean the three straight lines will be proportional

Let the three straight lines A, B, C be proportional, so that as A is to B , so is B to C ,

I say that the rectangle contained by A, C is equal to the square on B

A —————

Let D be made equal to B

B —————

D —————

Then since as A is to B , so is B to C

C —————

and B is equal to D

therefore as A is to B so is D to C

But if four straight lines be proportional the rectangle contained by the extremes is equal to the rectangle contained by the means [VI 16]

Therefore the rectangle A, C is equal to the rectangle B, D

But the rectangle B, D is the square on B for B is equal to D ,

therefore the rectangle contained by A, C is equal to the square on B

Next let the rectangle A, C be equal to the square on B I say that as A is to B , so is B to C

For with the same construction

since the rectangle A, C is equal to the square on B

while the square on B is the rectangle B, D for B is equal to D

therefore the rectangle A, C is equal to the rectangle B, D

But if the rectangle contained by the extremes be equal to that contained by the means the four straight lines are proportional [VI 16]

Therefore as A is to B so is D to C

But B is equal to D ,
therefore as A is to B , so is B to C

Therefore etc

Q E D

PROPOSITION 18

On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure

Let AB be the given straight line and CE the given rectilineal figure thus it is required to describe on the straight line AB a rectilineal figure similar and similarly situated to the rectilineal figure CE

Let DF be joined and on the straight line AB , and at the points A, B on it, let the angle GAB be constructed equal to the angle at C , and the angle ABG equal to the angle CDF [I 23]

Therefore the remaining angle CFD is equal to the angle AGB ,

therefore the triangle FCD is equiangular with the triangle GAB

Therefore proportionally, as FD is to GB , so is FC to GA and CD to AB

Again, on the straight line BG , and at the points B, G on it, let the angle BGH be constructed equal to the angle DFE and the angle GBH equal to the angle FDE [I 23]

Therefore the remaining angle at E is equal to the remaining angle at H , [I 32]

therefore the triangle FDE is equiangular with the triangle GBH , therefore, proportionally, as FD is to GB , so is FE to GH , and ED to HB [VI 4]

But it was also proved that as FD is to GB so is FC to GA , and CD to AB , therefore also, as FC is to GA so is CD to AB , and FE to GH , and further ED to HB

And, since the angle CFD is equal to the angle AGB ,

and the angle DFE to the angle BGH ,

therefore the whole angle CFE is equal to the whole angle AGH

For the same reason

the angle CDE is also equal to the angle ABH

And the angle at C is also equal to the angle at A

and the angle at E to the angle at H

Therefore AH is equiangular with CE

and they have the sides about their equal angles proportional

therefore the rectilineal figure AH is similar to the rectilineal figure CE [VI Def 1]

[VI Def 1]

Therefore on the given straight line AB the rectilineal figure AH has been described similar and similarly situated to the given rectilineal figure CE

Q E F

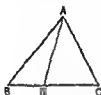
PROPOSITION 19

Similar triangles are to one another in the duplicate ratio of the corresponding sides

Let ABC, DEF be similar triangles having the angle at B equal to the angle

at E , and such that as AB is to BC , so is DE to EF so that BC corresponds to EF [v Def 11]

I say that the triangle ABC has to the triangle DEF a ratio duplicate of that which BC has to EF



For let a third proportional BG be taken to BC , EF , so that as BC is to EF , so is EF to BG , [vi 11]

and let AG be joined

Since then as AB is to BC , so is DE to EF

therefore alternately, as AB is to DE so is BC to EF [v 16]

But, as BC is to EF , so is EF to BG ,

therefore also as AB is to DE , so is EF to BG [v 11]

Therefore in the triangles ABG , DEF the sides about the equal angles are reciprocally proportional

But those triangles which have one angle equal to one angle, and in which the sides about the equal angles are reciprocally proportional are equal, [vi 15]

therefore the triangle ABG is equal to the triangle DEF

Now since, as BC is to EF so is EF to BG , and if three straight lines be proportional the first has to the third a ratio duplicate of that which it has to the second [v Def 9]

therefore BC has to BG a ratio duplicate of that which CB has to EF

But, as CB is to BG so is the triangle ABC to the triangle ABG , [vi 1] therefore the triangle ABC also has to the triangle ABG a ratio duplicate of that which BC has to EF

But the triangle ABG is equal to the triangle DEF therefore the triangle ABC also has to the triangle DEF a ratio duplicate of that which BC has to EF

Therefore etc

ΠΟΡΙΣΜ From this it is manifest that if three straight lines be proportional then, as the first is to the third so is the figure described on the first to that which is similar and similarly described on the second Q E D

PROPOSITION 20

Similar polygons are divided into similar triangles and into triangles equal in multitude and in the same ratio as the wholes and the polygon has to the polygon a ratio duplicate of that which the corresponding side has to the corresponding side

Let $ABCDE$ $FGHKL$ be similar polygons and let AB correspond to FG

I say that the polygons $ABCDE$ $FGHKL$ are divided into similar triangles and into triangles equal in multitude and in the same ratio as the wholes and the polygon $ABCDE$ has to the polygon $FGHKL$ a ratio duplicate of that which AB has to FG

Let BE EC , GL LH be joined

Now, since the polygon $ABCDE$ is similar to the polygon $FGHKL$,

the angle BAE is equal to the angle GFL

and as BA is to AE so is GF to FL [vi Def 1]

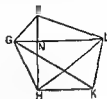
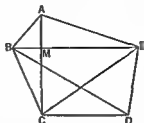
Since then ABE FGL are two triangles having one angle equal to one angle

and the sides about the equal angles proportional,

therefore the triangle ABE is equiangular with the triangle FGL , [vi 6]
so that it is also similar, [vi 4 and Def 1]

therefore the angle ABE is equal to the angle FGL

But the whole angle ABC is also equal to the whole angle FGH because of the similarity of the polygons,
therefore the remaining angle EBC is equal to the angle LGH



And, since because of the similarity of the triangles ABE , FGI

as EB is to BA , so is LG to GF ,

and moreover also because of the similarity of the polygons

as AB is to BC so is FG to GH ,

therefore, *ex aequali* as EB is to BC , so is LG to GH , [v 2^o]

that is the sides about the equal angles EBC , LGH are proportional,

therefore the triangle EBC is equiangular with the triangle LGH , [vi 6]

so that the triangle EBC is also similar to the triangle LGH [vi 4 and Def 1]

For the same reason

the triangle ECD is also similar to the triangle LHK

Therefore the similar polygons $ABCDE$ $FGHLK$ have been divided into similar triangles and into triangles equal in multitude

I say that they are also in the same ratio as the wholes that is in such manner that the triangles are proportional and ABE , EBC , ECD are antecedents while FGL LGH LHK are their consequents and that the polygon $ABCDE$ has to the polygon $FGHLK$ a ratio duplicate of that which the corresponding side has to the corresponding side that is AB to FG

For let AC FH be joined

Then since because of the similarity of the polygons

the angle ABC is equal to the angle FGH ,

and as AB is to BC so is FG to GH

the triangle ABC is equiangular with the triangle FGH [vi 6]

therefore the angle BAC is equal to the angle GFH

and the angle BCA to the angle GHF

And since the angle BAM is equal to the angle GFN ,

and the angle ABM is also equal to the angle FGN ,

therefore the remaining angle AMB is also equal to the remaining angle FNG , [i 32]

therefore the triangle ABM is equiangular with the triangle FGN

Similarly we can prove that

the triangle BMC is also equiangular with the triangle GNH

Therefore proportionally as AM is to MB so is FN to NG

and as BM is to MC so is GN to NH ,

so that in addition *ex aequali*

as AM is to MC , so is FN to NH

But, as AM is to MC so is the triangle ABM to MBC , and AME to EMC , for they are to one another as their bases [vi 1]

Therefore also as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents, [v 12]

therefore, as the triangle AMB is to BMC , so is ABE to CBE

But as AMB is to BMC , so is AM to MC ,

therefore also as AM is to MC , so is the triangle ABE to the triangle EBC

For the same reason also

as FN is to NH , so is the triangle FGL to the triangle GLH

And as AM is to MC , so is FN to NH ,

therefore also, as the triangle ABE is to the triangle BEC , so is the triangle FGL to the triangle GLH ,

and, alternately, as the triangle ABE is to the triangle FGL so is the triangle BEC to the triangle GLH

Similarly we can prove if BD , GK be joined, that as the triangle BEC is to the triangle LGH , so also is the triangle ECD to the triangle LHK

And since as the triangle ABE is to the triangle FGL , so is EBC to LGH , and further ECD to LHK ,

therefore also as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents, [v 12]

therefore as the triangle ABE is to the triangle FGL

so is the polygon $ABCDE$ to the polygon $FGHKL$

But the triangle ABE has to the triangle FGL a ratio duplicate of that which the corresponding side AB has to the corresponding side FG , for similar triangles are in the duplicate ratio of the corresponding sides [vi 19]

Therefore the polygon $ABCDE$ also has to the polygon $FGHKL$ a ratio duplicate of that which the corresponding side AB has to the corresponding side FG

Therefore etc

PROPOSITION Similarly also it can be proved in the case of quadrilaterals that they are in the duplicate ratio of the corresponding sides And it was also proved in the case of triangles therefore also generally similar rectilineal figures are to one another in the duplicate ratio of the corresponding sides

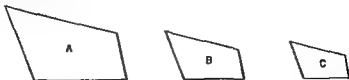
Q E D

PROPOSITION 21

Figures which are similar to the same rectilineal figure are also similar to one another

For let each of the rectilineal figures A , B be similar to C , I say that A is also similar to B

For since A is similar to C ,



it is equiangular with it and has the sides about the equal angles proportional

[vi Def 1]

Again since B is similar to C

it is equiangular with it and has the sides about the equal angles proportional

Therefore each of the figures A, B is equiangular with C and with C has the sides about the equal angles proportional,

therefore $A \equiv$ similar to B

Q E D

PROPOSITION 22

If four straight lines be proportional the rectilineal figures similar and similarly described upon them will also be proportional and if the rectilineal figures similar and similarly described upon them be proportional the straight lines will themselves also be proportional

Let the four straight lines AB, CD, EF, GH be proportional,
so that as AB is to CD , so is EF to GH ,
and let there be described on AB, CD the similar and similarly situated rectilineal figures KAB, LCD

and on EF, GH the similar and similarly situated rectilineal figures MF, NH ,

I say that as KAB is to LCD , so is MF to NH

For let there be taken a third proportional O to AB, CD , and a third proportional P to EF, GH [vi 11]

Then since as AB is to CD , so is EF to GH

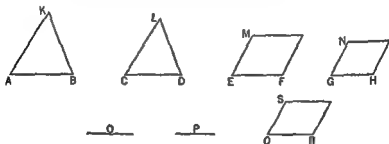
and, as CD is to O so is GH to P ,

therefore *ex aequali* as AB is to O so is EF to P [v 22]

But as AB is to O so is KAB to LCD , [vi 19 Por]

and as EF is to P , so is MF to NH ,

therefore also as KAB is to LCD so is MF to NH [v 11]



Next let MF be to NH as KAB is to LCD ,

I say also that as AB is to CD so is EF to GH

For if EF is not to GH as AB to CD

let EF be to QR as AB to CD [vi 17]

and on QR let the rectilineal figure SR be described similar and similarly situated to either of the two MF, NH [vi 18]

Since then as AB is to CD so is EF to QR
and there have been described on AB, CD the similar and similarly situated figures KAB, LCD

and on EF, QR the similar and similarly situated figures MF, SR ,

therefore as KAB is to LCD so is MF to SR

But also by hypothesis

as KAB is to LCD so is MF to NH

therefore also as MF is to SR so is MF to NH [v 11]

Therefore MF has the same ratio to each of the figures NH , SR ,

therefore NH is equal to SR

[v 9]

But it is also similar and similarly situated to it

therefore GH is equal to QR

And, since, as AB is to CD so is EF to QR ,

while QR is equal to GH

therefore as AB is to CD , so is EF to GH

Therefore etc

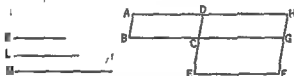
Q E D

PROPOSITION 23

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides

Let AC CF be equiangular parallelograms having the angle BCD equal to the angle ECG ,

I say that the parallelogram AC has to the parallelogram CF the ratio compounded of the ratios of the sides



For let them be placed so that BC is in a straight line with CG ,

therefore DC is also in a straight line with CE

Let the parallelogram DG be completed,

let a straight line A be set out and let it be contrived that

as BC is to CG , so is A to L

and, as DC is to CE so is L to M [vi 12]

Then the ratios of K to L and of L to M are the same as the ratios of the sides namely of BC to CG and of DC to CE

But the ratio of A to M is compounded of the ratio of A to L and of that of L to M ,

so that A has also to M the ratio compounded of the ratios of the sides

Now since, as BC is to CG , so is the parallelogram AC to the parallelogram CH ,

while as BC is to CG so is K to L ,

therefore also as A is to L so is AC to CH [v 11]

Again since as DC is to CE , so is the parallelogram CH to CF [vi 1]

while as DC is to CE so is L to M ,

therefore also, as L is to M so is the parallelogram CH to the parallelogram CF [v 11]

Since then it was proved that, as A is to L so is the parallelogram AC to the parallelogram CH

and as L is to M so is the parallelogram CH to the parallelogram CF

therefore, *ex aequali* as A is to M so is AC to the parallelogram CF

But A has to M the ratio compounded of the ratios of the sides

therefore AC also has to CF the ratio compounded of the ratios of the sides

Therefore etc

Q E D

PROPOSITION 24

In any parallelogram the parallelograms about the diameter are similar both to the whole and to one another

Let $ABCD$ be a parallelogram and AC its diameter, and let EG, HK be parallelograms about AC ,

I say that each of the parallelograms EG, HK is similar both to the whole $ABCD$ and to the other

For, since EF has been drawn parallel to BC one of the sides of the triangle ABC ,

proportionally, as BE is to EA , so is CF to FA

[vi 2]

Again since FG has been drawn parallel to CD , one of the sides of the triangle ACD ,

proportionally as CF is to FA , so is DG to GA

[vi 2]

But it was proved that

as CF is to FA , so also is BE to EA ,

therefore also as BE is to EA so is DG to GA ,

and therefore *componendo*

as BA is to AE , so is DA to AG ,

[v 15]

and alternately,

as BA is to AD , so is EA to AG

[v 16]

Therefore in the parallelograms $ABCD, EG$, the sides about the common angle BAD are proportional

And, since GF is parallel to DC ,

the angle AFG is equal to the angle DCA ,

and the angle DAC is common to the two triangles ADC, AGF ,

therefore the triangle ADC is equiangular with the triangle AGF

For the same reason

the triangle ACB is also equiangular with the triangle AFE

and the whole parallelogram $ABCD$ is equiangular with the parallelogram EG

Therefore proportionally

as AD is to DC , so is AG to GF

as DC is to CA so is GF to FA ,

as AC is to CB , so is AF to FE

and further as CB is to BA , so is FE to EA

And since it was proved that

as DC is to CA so is GF to FA ,

and,

as AC is to CB , so is AF to FE ,

therefore *ex aequali* as DC is to CB so is GF to FE

[v 22]

Therefore in the parallelograms $ABCD, EG$ the sides about the equal angles are proportional,

therefore the parallelogram $ABCD$ is similar to the parallelogram EG

[vi Def 1]

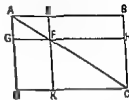
For the same reason

the parallelogram $ABCD$ is also similar to the parallelogram HK

therefore each of the parallelograms EG, HK is similar to $ABCD$

But figures similar to the same rectilineal figure are also similar to one another

[vi 21]

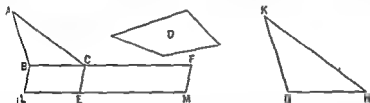


therefore the parallelogram EG is also similar to the parallelogram HK
Therefore etc Q E D

PROPOSITION 25

To construct one and the same figure similar to a given rectilineal figure and equal to another given rectilineal figure

Let ABC be the given rectilineal figure to which the figure to be constructed must be similar and D that to which it must be equal
thus it is required to construct one and the same figure similar to ABC and equal to D



Let there be applied to BC the parallelogram BE equal to the triangle ABC [I 44] and to CF the parallelogram CM equal to D in the angle FCE which is equal to the angle CBL [I 45]

Therefore BC is in a straight line with CF and LE with EM

Now let GH be taken a mean proportional to BC , CF [VI 13] and on GH let AGH be described similar and similarly situated to ABC [VI 18]

Then since, as BC is to GH , so is GH to CF ,
and if three straight lines be proportional as the first is to the third so is the figure on the first to the similar and similarly situated figure described on the second [VI 19 Por]

therefore as BC is to CF so is the triangle ABC to the triangle AGH

But, as BC is to CF so also is the parallelogram BE to the parallelogram EF [VI 1]

Therefore also as the triangle ABC is to the triangle AGH so is the parallelogram BE to the parallelogram EF ,

therefore alternately as the triangle ABC is to the parallelogram BE so is the triangle AGH to the parallelogram EF [V 16]

But the triangle ABC is equal to the parallelogram BE

therefore the triangle AGH is also equal to the parallelogram EF

But the parallelogram EF is equal to D

therefore AGH is also equal to D

And AGH is also similar to ABC

Therefore one and the same figure AGH has been constructed similar to the given rectilineal figure ABC and equal to the other given figure D Q E D

PROPOSITION 26

If from a parallelogram there be taken away a parallelogram similar and similarly situated to the whole and having a common angle with it it is about the same diameter with the whole

For from the parallelogram $ABCD$ let there be taken away the parallelo-

gram AF similar and similarly situated to $ABCD$, and having the angle DAB common with it,

I say that $ABCD$ is about the same diameter with AF

For suppose it is not but, if possible let AHC be the diameter $<$ of $ABCD >$, let GF be produced and carried through to H , and let HK be drawn through H parallel to either of the straight lines AD, BC

[I 31]

Since, then $ABCD$ is about the same diameter with AG , therefore as DA is to AB , so \equiv GA to AK [VI 24]

But also because of the similarity of $ABCD$ EG ,

as DA is to AB , so is GA to AE ,

therefore also as GA is to AK so is GA to AE [V 11]

Therefore GA has the same ratio to each of the straight lines AK, AE

Therefore $AE \equiv$ equal to AK [V 9] the less to the greater which is impossible

Therefore $ABCD$ cannot but be about the same diameter with AF , therefore the parallelogram $ABCD$ is about the same diameter with the parallelogram AF

Therefore etc

Q E D

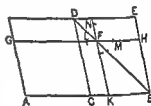
PROPOSITION 27

Of all the parallelograms applied to the same straight line and deficient by parallelogrammic figures similar and similarly situated to that described on the half of the straight line that parallelogram is greatest which is applied to the half of the straight line and is similar to the defect

Let AB be a straight line and let it be bisected at C , let there be applied to the straight line AB the parallelogram AD deficient by the parallelogrammic figure DB described on the half of AB , that is CB ,

I say that of the parallelograms applied to AB and deficient by parallelogrammic figures similar and similarly situated to DB AD is greatest

For let there be applied to the straight line AB the parallelogram AF deficient by the parallelogrammic figure FB similar and similarly situated to DB ,



I say that AD is greater than AF

For since the parallelogram DB is similar to the parallelogram FB , they are about the same diameter [VI 26]

Let their diameter DB be drawn and let the figure be described

Then, since CF is equal to FE [I 43]

and FB is common

therefore the whole CH is equal to the whole AE

But CH is equal to CG since AC is also equal to CB [I 36]

Therefore $GC \equiv$ also equal to EK

Let CF be added to each

therefore the whole AF is equal to the gnomon LMN ,

so that the parallelogram DB , that is, AD , is greater than the parallelogram AF

Therefore etc

Q E D

PROPOSITION 28

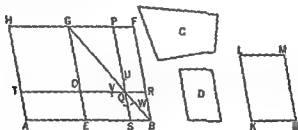
To a given straight line to apply a parallelogram equal to a given rectilineal figure and deficient by a parallelogrammic figure similar to a given one thus the given rectilineal figure must not be greater than the parallelogram described on the half of the straight line and similar to the defect

Let AB be the given straight line C the given rectilineal figure to which the figure to be applied to AB is required to be equal not being greater than the parallelogram described on the half of AB and similar to the defect and D the parallelogram to which the defect is required to be similar, thus it is required to apply to the given straight line AB a parallelogram equal to the given rectilineal figure C and deficient by a parallelogrammic figure which is similar to D

Let AB be bisected at the point E , and on EB let $EBFG$ be described similar and similarly situated to D , [VI 18]

let the parallelogram AG be completed

If then AG is equal to C that which was enjoined will have been done for there has been applied to the given straight line AB the parallelogram AG equal to the given rectilineal figure C and deficient by a parallelogrammic figure GB which is similar to D



But if not let HE be greater than C

Now HE is equal to GB ,

therefore GB is also greater than C

Let $ALMN$ be constructed at once equal to the excess by which GB is greater than C and similar and similarly situated to D [VI 25]

But D is similar to GB ,

therefore KM is also similar to GB

[VI 21]

Let then, KL correspond to GE and LM to GF

Now since GB is equal to C KM ,

therefore GB is greater than KL

therefore also GE is greater than KL and GF than LM

Let GO be made equal to KL and GP equal to LM and let the parallelogram $OGPQ$ be completed

therefore it is equal and similar to KL

Therefore GQ is also similar to GB

[VI 21]

therefore GQ is about the same diameter with GB [VI 26]
 Let GQB be their diameter, and let the figure be described
 Then, since BG is equal to C , AM ,
 and in them GQ is equal to AM
 therefore the remainder the gnomon UV , is equal to the remainder C
 And since PR is equal to OS

let QB be added to each
 therefore the whole PB is equal to the whole OB
 But OB is equal to TE since the side AE is also equal to the side EB , [I 36]
 therefore TE is also equal to PB

Let OS be added to each,
 therefore the whole TS is equal to the whole the gnomon UV
 But the gnomon UV was proved equal to C ,
 therefore TS is also equal to C

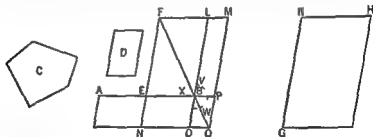
Therefore to the given straight line AB there has been applied the parallelogram ST equal to the given rectilinear figure C and deficient by a parallelogrammic figure QB which is similar to D Q E F

PROPOSITION 29

To a given straight line to apply a parallelogram equal to a given rectilinear figure and exceeding by a parallelogrammic figure similar to a given one

Let AB be the given straight line, C the given rectilinear figure to which the figure to be applied to AB is required to be equal, and D that to which the excess is required to be similar

thus it is required to apply to the straight line AB a parallelogram equal to the rectilinear figure C and exceeding by a parallelogrammic figure similar to D



Let AB be bisected at E
 let there be described on EB the parallelogram BF similar and similarly situated to D
 and let GH be constructed at once equal to the sum of BF , C and similar and similarly situated to D [VI 25]

Let AH correspond to FL and AG to FE

Now since GH is greater than FB

therefore AH is also greater than FL and AG than FE

Let FL , FE be produced

let FLM be equal to AH and FEN to AG

and let MN be completed

therefore FN is both equal and similar to GH

But GH is similar to EL ,

therefore MN is also similar to EL

[VI 21]

therefore EL is about the same diameter with MN

[VI 26]

Let their diameter FO be drawn, and let the figure be described

Since GH is equal to EL , C ,

while GH is equal to MN

therefore MN is also equal to EL , C

Let EL be subtracted from each

therefore the remainder, the gnomon $\backslash H \vee$ is equal to C

Now, since AE is equal to EB ,

AN is also equal to NB [I 36] that is, to LP

[I 43]

Let EO be added to each,

therefore the whole AO is equal to the gnomon $\backslash H \vee$

But the gnomon $VIFV$ is equal to C

therefore AO is also equal to C

Therefore to the given straight line AB there has been applied the parallelogram AO equal to the given rectilineal figure C and exceeding by a parallelogram figure QP which is similar to D since PQ is also similar to EL [VI 24]

Q E F

PROPOSITION 30

To cut a given finite straight line in extreme and mean ratio

Let AB be the given finite straight line

thus it is required to cut AB in extreme and mean ratio



On AB let the square BC be described and let there be applied to AC the parallelogram CD equal to BC and exceeding by the figure AD similar to BC

[VI 29]

Now BC is a square

therefore AD is also a square

And since BC is equal to CD

let CE be subtracted from each,

therefore the remainder BF is equal to the remainder AD

But it is also equiangular with it

therefore in BF , AD the sides about the equal angles are reciprocally proportional,

[VI 14]

therefore as FE is to ED , so is AE to EB

But FE is equal to AB and ED to AE

Therefore, as BA is to AE so is AE to EB

And AB is greater than AE ,

therefore AE is also greater than EB

Therefore the straight line AB has been cut in extreme and mean ratio at E and the greater segment of it is AE

Q E F

PROPOSITION 31

In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle

Let ABC be a right angled triangle having the angle BAC right

I say that the figure on BC is equal to the similar and similarly described figures on BA , AC

Let AD be drawn perpendicular

Then since in the right-angled triangle ABC , AD has been drawn from the right angle at A perpendicular to the base BC , the triangles ABD , ADC adjoining the perpendicular are similar both to the whole ABC and to one another [vi 8]

And since ABC is similar to ABD , therefore as CB is to BA so is AB to BD [vi Def 1]

And since three straight lines are proportional, as the first is to the third so is the figure on the first to the similar and similarly described figure on the second [vi 10 Por]

Therefore as CB is to BD , so is the figure on CB to the similar and similarly described figure on BA

For the same reason also,

as BC is to CD so is the figure on BC to that on CA ,
so that, in addition

as BC is to BD , DC so is the figure on BC to the similar and similarly described figures on BA , AC

But BC is equal to BD , DC , therefore the figure on BC is also equal to the similar and similarly described figures on BA , AC

Therefore etc

Q E D

PROPOSITION 32

If two triangles having two sides proportional to two sides be placed together at one angle so that their corresponding sides are also parallel, the remaining sides of the triangles will be in a straight line

Let ABC , DCE be two triangles having the two sides BA , AC proportional to the two sides DC , DE , so that as AB is to AC so is DC to DE , and AB parallel to DC , and AC to DE

I say that BC is in a straight line with CE

For since AB is parallel to DC and the straight line AC has fallen upon them the alternate angles BAC , ACD are equal to one another [i 29]

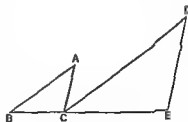
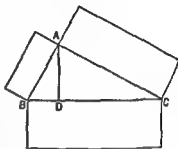
For the same reason the angle CDE is also equal to the angle ACD , so that the angle BAC is equal to the angle CDE

And, since ABC , DCE are two triangles having one angle the angle at A , equal to one angle the angle at D

and the sides about the equal angles proportional,

so that as BA is to AC so is CD to DE

therefore the triangle ABC is equiangular with the triangle DCE , [vi 6]



therefore the angle ABC is equal to the angle DCE

But the angle ACD was also proved equal to the angle BAC ,

therefore the whole angle ACE is equal to the two angles ABC, BAC

Let the angle ACB be added to each,

therefore the angles ACE, ACB are equal to the angles BAC, ACB

But the angles BAC, ABC, ACB are equal to two right angles, [I 32]

therefore the angles ACE, ACB are also equal to two right angles

Therefore with a straight line AC , and at the point C on it the two straight lines BC, CE not lying on the same side make the adjacent angles ACE, ACB equal to two right angles,

therefore BC is in a straight line with CE

[I 14]

Therefore etc

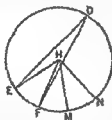
Q E D

PROPOSITION 33

In equal circles angles have the same ratio as the circumferences on which they stand whether they stand at the centres or at the circumferences

Let ABC, DEF be equal circles and let the angles BGC, DHF be angles at their centres G, H , and the angles BAC, EDF angles at the circumferences,

I say that, as the circumference BC is to the circumference EF so is the angle BGC to the angle DHF , and the angle BAC to the angle EDF



For let any number of consecutive circumferences CK, KL be made equal to the circumference BC

and any number of consecutive circumferences FM, MN equal to the circumference EF

and let GA, GL, HM, HN be joined

Then since the circumferences BC, CK, KL are equal to one another

the angles BGC, CGA, AGL are also equal to one another [III 27]

therefore whatever multiple the circumference BL is of BC , that multiple also is the angle BGL of the angle BGC

For the same reason also

whatever multiple the circumference NE is of EF that multiple also is the angle ANE of the angle EHF

If then the circumference BL is equal to the circumference EN the angle BGL is also equal to the angle EHN [III 27]

If the circumference BL is greater than the circumference EN the angle BGL is also greater than the angle EHN

and if less less

There being then four magnitudes two circumferences BC, EN and two angles BGC, EHF

there have been taken of the circumference BC and the angle BGC equimultiples namely the circumference BL and the angle BGL

and of the circumference EF and the angle EHF equimultiples namely the circumference EN and the angle EHN

And it has been proved that

if the circumference BL is in excess of the circumference EN

the angle BGL is also in excess of the angle EHN ,
 if equal, equal,
 and if less less

I therefore as the circumference BC is to EF , so is the angle BGC to the angle EHF [v Def 5]

But as the angle BGC is to the angle EHF , so is the angle BAC to the angle EDF , for they are doubles respectively

Therefore also, as the circumference BC is to the circumference EF , so is the angle BGC to the angle EHF , and the angle BAC to the angle EDF

Therefore etc

■ E D

BOOK SEVEN

DEFINITIONS

- 1 An *unit* is that by virtue of which each of the things that exist is called one
- 2 A *number* is a multitude composed of units
- 3 A number is a *part* of a number the less of the greater, when it measures the greater
- 4 but *parts* when it does not measure it
- 5 The greater number is a *multiple* of the less when it is measured by the less
- 6 An *even number* is that which is divisible into two equal parts
- 7 An *odd number* is that which is not divisible into two equal parts or that which differs by an unit from an even number
- 8 An *even times even number* is that which is measured by an even number according to an even number
- 9 An *even times odd number* is that which is measured by an even number according to an odd number
- 10 An *odd times odd number* is that which is measured by an odd number according to an odd number
- 11 A *prime number* is that which is measured by an unit alone
- 12 Numbers *prime to one another* are those which are measured by an unit alone as a common measure
- 13 A *composite number* is that which is measured by some number
- 14 Numbers *composite to one another* are those which are measured by some number as a common measure
- 15 A number is said to *multiply* a number when that which is multiplied is added to itself as many times as there are units in the other and thus some number is produced
- 16 And when two numbers having multiplied one another make some number the number so produced is called *plane* and its *sides* are the numbers which have multiplied one another
- 17 And when three numbers having multiplied one another make some number the number so produced is *solid* and its *sides* are the numbers which have multiplied one another
- 18 A *square number* is equal multiplied by equal, or a number which is contained by two equal numbers
- 19 And a *cube* is equal multiplied by equal and again by equal or a number which is contained by three equal numbers
- 20 Numbers are *proportional* when the first is the same multiple or the same part or the same parts of the second that the third is of the fourth

21 *Similar plane and solid numbers* are those which have their sides proportional

22 A *perfect number* is that which is equal to its own parts

BOOK VII PROPOSITIONS

PROPOSITION 1

Two unequal numbers being set out and the less being continually subtracted in turn from the greater if the number which is left never measures the one before it until an unit is left the original numbers will be prime to one another

For the less of two unequal numbers AB , CD being continually subtracted from the greater let the number which is left never measure the one before it until an unit is left,

I say that AB , CD are prime to one another that is that an unit alone measures AB , CD

For, if AB , CD are not prime to one another, some number will measure them

Let a number measure them, and let it be E , let CD , measuring BF , leave F less than itself

let AF measuring DG leave GC less than itself,

and let GC measuring FH leave an unit HA

Since then E measures CD and CD measures BF ,

therefore E also measures BF

But it also measures the whole BA ,

therefore it will also measure the remainder AF

But AF measures DG ,

therefore E also measures DG

But it also measures the whole DC

therefore it will also measure the remainder CG

But CG measures FH

therefore E also measures FH

But it also measures the whole FA

therefore it will also measure the remainder the unit HA , though it is a number which is impossible

Therefore no number will measure the numbers AB , CD therefore AB , CD are prime to one another

[VII Def 12]
Q E D



PROPOSITION 2

Given two numbers not prime to one another to find their greatest common measure

Let AB , CD be the two given numbers not prime to one another

Thus it is required to find the greatest common measure of AB , CD

If now CD measures AB —and it also measures itself— CD is a common measure of CD , AB

And it is manifest that it is also the greatest, for no greater number than CD will measure CD

But if CD does not measure AB then the less of the numbers AB , CD being continually subtracted from the greater, some number will be left which will measure the one before it

For an unit will not be left, otherwise AB , CD will be prime to one another [VII 1] which is contrary to the hypothesis

Therefore some number will be left which will measure the one before it

Now let CD , measuring BE , leave EA less than itself,
let EA , measuring DF , leave FC less than itself,
and let CF measure AE

Since then CF measures AE , and AE measures DF ,
therefore CF will also measure DF

But it also measures itself.

therefore it will also measure the whole CD .

But CD measures BE .

therefore CF also measures BE

But it also measures *EA*.

therefore it will also measure the whole BA

But it also measures *CD*

therefore CF measures AB, CD

Therefore CF is a common measure of AB, CD

I say next that it is also the greatest

For if CF is not the greatest common measure of AB , CD , some number which is greater than CF will measure the numbers AB , CD .

Let such a number measure them, and let it be G

Now, since G measures CD while CD measures BE , G also measures BE .

But it also measures the whole *BA*.

therefore it will also measure the remainder AE

But AE measures DF .

therefore G will also measure DF

But it also measures the whole *DC*.

therefore it will also measure the remainder CF that is, the greater will measure the less which is impossible

Therefore no number which is greater than CF will measure the numbers AB, CD

therefore CF is the greatest common measure of AB , CD

PORISM From this it is manifest that if a number measure two numbers it will also measure their greatest common measure Q E D

PROPOSITION 3

Given three numbers not prime to one another to find their greatest common measure

Let A, B, C be the three given numbers not prime to one another

thus it is required to find the greatest common measure of A, B, C

For let the greatest common measure D of the two numbers A B be taken [VII 2]

then D either measures or does not measure C

First let it measure it

But it measures $A \vee B$ also,
therefore D measures A, B, C

therefore D is a common measure of A B C

I say that it is all of the greatest

For if D is not the greatest common measure of A B C some number which

21 *Similar plane and solid numbers* are those which have their sides proportional

22 A *perfect number* is that which is equal to its own parts

BOOK VII PROPOSITIONS

PROPOSITION 1

Two unequal numbers being set out and the less being continually subtracted in turn from the greater if the number which is left never measures the one before it until an unit is left the original numbers will be prime to one another

For the less of two unequal numbers AB , CD being continually subtracted from the greater let the number which is left never measure the one before it until an unit is left,

I say that AB , CD are prime to one another, that is that an unit alone measures AB , CD

For if AB , CD are not prime to one another, some number will measure them

Let a number measure them, and let it be E , let CD , measuring BF , leave FA less than itself

let AF , measuring DG , leave GC less than itself

and let GC measuring FH , leave an unit HA

Since then E measures CD and CD measures BF ,

therefore E also measures BF

But it also measures the whole BA

therefore it will also measure the remainder AF

But AF measures DG ,

therefore E also measures DG

But it also measures the whole DC

therefore it will also measure the remainder CG

But CG measures FH

therefore E also measures FH

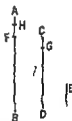
But it also measures the whole FA ,

therefore it will also measure the remainder the unit HA though it is a number which is impossible

Therefore no number will measure the numbers AB , CD , therefore AB , CD are prime to one another

[VII Def 12]

Q E D



PROPOSITION 2

Given two numbers not prime to one another to find their greatest common measure

Let AB , CD be the two given numbers not prime to one another

Thus it is required to find the greatest common measure of AB , CD

If now CD measures AB —and it also measures itself— CD is a common measure of CD , AB

And it is manifest that it is also the greatest, for no greater number than CD will measure CD

But if CD does not measure AB then the less of the numbers AB , CD being continually subtracted from the greater some number will be left which will measure the one before it

For an unit will not be left otherwise AB, CD will be prime to one another [VII 1] which is contrary to the hypothesis

Therefore some number will be left which will measure the one before it

Now let CD measuring BE , leave EA less than itself,

let EA measuring DF , leave FC less than itself,

and let CF measure AE

Since then CF measures AE , and AE measures DF ,

therefore CF will also measure DF

But it also measures itself

therefore it will also measure the whole CD

But CD measures BE ,

therefore CF also measures BE

But it also measures EA

therefore it will also measure the whole BA

But it also measures CD ,

therefore CF measures AB, CD

Therefore CF is a common measure of AB, CD

I say next that it is also the greatest

For if CF is not the greatest common measure of AB, CD some number which is greater than CF will measure the numbers AB, CD

Let such a number measure them and let it be G

Now since G measures CD while CD measures BE G also measures BE

But it also measures the whole BA

therefore it will also measure the remainder AE

But AE measures DF ,

therefore G will also measure DF

But it also measures the whole DC

therefore it will also measure the remainder CF that is the greater will measure the less which is impossible

Therefore no number which is greater than CF will measure the numbers AB, CD

therefore CF is the greatest common measure of AB, CD

PROPOSITION From this it is manifest that if a number measure two numbers it will also measure their greatest common measure Q E D

PROPOSITION 3

Given three numbers not prime to one another to find their greatest common measure

Let A, B, C be the three given numbers not prime to one another

thus it is required to find the greatest common measure of A, B, C

For let the greatest common measure D of the two numbers A, B be taken [VII 2]

then D either measures or does not measure C

First let it measure it

But it measures A, B also

therefore D measures A, B, C

therefore D is a common measure of A, B, C

I say that it is also the greatest

For if D is not the greatest common measure of A, B, C some number which

■ greater than D will measure the numbers A, B, C

Let such a number measure them and let it be E

Since then E measures A, B, C

it will also measure $A, B,$

therefore it will also measure the greatest common measure of A, B

[VII 2, Por]

But the greatest common measure of A, B is $D,$

therefore E measures D , the greater the less which is impossible¹

Therefore no number which is greater than D will measure the numbers $A, B, C,$

therefore D is the greatest common measure of A, B, C

Next let D not measure $C,$

I say first that C, D are not prime to one another

For since A, B, C are not prime to one another some number will measure them

Now that which measures A, B, C will also measure $A, B,$ and will measure D the greatest common measure of A, B

[VII 2 Por]

But it measures C also,

therefore some number will measure the numbers $D, C,$

therefore D, C are not prime to one another

Let then their greatest common measure E be taken

[VII 2]

Then since E measures D

and D measures $A, B,$

therefore E also measures A, B

But it measures C also

therefore E measures A, B, C

therefore E is a common measure of A, B, C

I say next that it is also the greatest

For if E is not the greatest common measure of $A, B, C,$ some number which is greater than E will measure the numbers A, B, C

Let such a number measure them and let it be F

Now since F measures A, B, C

it also measures A, B

therefore it will also measure the greatest common measure of A, B

[VII 2 Por]

But the greatest common measure of A, B is D

therefore F measures D

And it measures C also

therefore F measures D, C

therefore it will also measure the greatest common measure of D, C

[VII 2, Por]

But the greatest common measure of D, C is $E,$

therefore F measures E the greater the less which is impossible

Therefore no number which is greater than E will measure the numbers A, B, C

therefore E is the greatest common measure of A, B, C

Q E D

PROPOSITION 4

Any number is either a part or parts of any number, the less of the greater

Let A, BC be two numbers and let BC be the less,

I say that BC is either a part or parts of A

For A, BC are either prime to one another or not

First, let A, BC be prime to one another

Then if BC be divided into the units in it, each unit of those in BC will be some part of A , so that BC is parts of A

Next let A, BC not be prime to one another, then BC either measures or does not measure A

If now BC measures A , BC is a part of A

But, if not let the greatest common measure D of A, BC be taken [VII 2] and let BC be divided into the numbers equal to D , namely BL, EF, FC

Now since D measures A , D is a part of A

But D is equal to each of the numbers BE, EF, FC ,

therefore each of the numbers BE, EF, FC is also a part of A , so that BC is parts of A

Therefore etc

Q E D

PROPOSITION 5

If a number be a part of a number and another be the same part of another, the sum will also be the same part of the sum that the one is of the one

For let the number A be a part of BC

and another D the same part of another EF that A is of BC

I say that the sum of A, D is also the same part of the sum of BC, EF that A is of BC

For since, whatever part A is of BC , D is also the same part of EF ,

therefore, as many numbers as there are in BC equal to A so many numbers are there also in EF equal to D

Let BC be divided into the numbers equal to A namely BG, GC ,

and EF into the numbers equal to D namely EH, HF ,

then the multitude of BG, GC will be equal to the multitude of EH, HF

And since BG is equal to A , and EH to D ,

therefore BG, EH are also equal to A, D

For the same reason

GC, HF are also equal to A, D

Therefore as many numbers as there are in BC equal to A so many are there also in BC, EF equal to A, D

Therefore whatever multiple BC is of A , the same multiple also is the sum of BC, EF of the sum of A, D

Therefore whatever part A is to BC , the same part also is the sum of A, D of the sum of BC, EF

Q E D

PROPOSITION 6

If a number be parts of a number and another be the same parts of another, the sum will also be the same parts of the sum that the one is of the one

For let the number AB be parts of the number C , and another, DE , the same parts of another, F , that AB is of C ,

I say that the sum of AB , DE is also the same parts of the sum of C , F that AB is of C

For since whatever parts AB is of C , DE is also the same parts of F ,

therefore, as many parts of C as there are in AB , so many parts of F are there also in DE

Let AB be divided into the parts of C , namely AG , GB , and DE into the parts of F , namely DH , HE ,

thus the multitude of AG , GB will be equal to the multitude of DH , HE

And since, whatever part AG is of C , the same part is DH of F also, therefore, whatever part AG is of C , the same part also is the sum of AG , DH of the sum of C , F [VII 5]

For the same reason

whatever part GB is of C , the same part also is the sum of GB , HE of the sum of C , F

Therefore, whatever parts AB is of C , the same parts also is the sum of AB , DE of the sum of C , F Q E D

PROPOSITION 7

If a number be that part of a number, which a number subtracted is of a number subtracted the remainder will also be the same part of the remainder that the whole is of the whole

For let the number AB be that part of the number CD which AE subtracted is of CF subtracted

I say that the remainder EB is also the same part of the remainder FD that the whole AB is of the whole CD



For whatever part AE is of CF , the same part also let EB be of CG
 Now since whatever part AE is of CF the same part also is EB of CG
 therefore whatever part AE is of CF the same part also is AB of GF [VII 5]
 But, whatever part AE is of CF , the same part also, by hypothesis, is AB of CD

therefore whatever part AB is of GF , the same part is it of CD also
 therefore GF is equal to CD

Let CF be subtracted from each

therefore the remainder GC is equal to the remainder FD

Now since whatever part AE is of CF , the same part also is EB of GC , while GC is equal to FD

therefore whatever part AE is of CF the same part also is EB of FD

But whatever part AE is of CF the same part also is AB of CD , therefore also the remainder EB is the same part of the remainder FD that the whole AB is of the whole CD Q E D

PROPOSITION 8

If a number be the same parts of a number that a number subtracted \equiv of a number subtracted, the remainder will also be the same parts of the remainder that the whole is of the whole

For let the number AB be the same parts of the number CD that AE subtracted is of CF subtracted,

I say that the remainder EB is also the same parts of the remainder FD that the whole AB is of the whole CD

For let GH be made equal to AB

Therefore whatever parts $GH \equiv$ of CD , the same parts also is AE of CF

Let GH be divided into the parts of CD namely GK , KH , and AH into the parts of CF , namely AL , LE

thus the multitude of GK , AH will be equal to the multitude of AL , LE

Now since whatever part GK is of CD the same part also is AL of CF , while CD is greater than CF

therefore $GK \equiv$ also greater than AL

Let GM be made equal to AL

Therefore, whatever part GK is of CD , the same part also is GM of CF therefore also the remainder MA is the same part of the remainder FD that the whole GK is of the whole CD [VII 7]

Again since whatever part $AH \equiv$ of CD the same part also is EL of CF , while CD is greater than CF ,

therefore AH is also greater than EL

Let KN be made equal to FL

Therefore whatever part AH is of CD , the same part also \equiv AN of CF , therefore also the remainder NH is the same part of the remainder FD that the whole AH is of the whole CD [VII 7]

But the remainder MK was also proved to be the same part of the remainder FD that the whole GK is of the whole CD , therefore also the sum of MK , NH is the same parts of FD that the whole HG is of the whole CD

But the sum of MA , NH is equal to EB ,

and HG is equal to BA ,

therefore the remainder EB is the same parts of the remainder FD that the whole AB is of the whole CD Q E D

PROPOSITION 9

If a number be a part of a number and another be the same part of another alternately also whatever part or parts the first is of the third the same part or the same parts will the second also be of the fourth

For let the number A be a part of the number BC , and another D the same part of another, EF that A is of BC ,

I say that alternately also whatever part or parts A is of D the same part or parts is BC of EF also

For since whatever part $A \equiv$ of BC the same part also is D of EF



therefore as many numbers as there are in BC equal to A , so many also are there in EF equal to D

Let BC be divided into the numbers equal to A , namely BG GC ,
and EF into those equal to D namely EH , HF ,

thus the multitude of BG GC will be equal to the multitude of EH , HF

Now, since the numbers BG , GC are equal to one another, and the numbers EH HF are also equal to one another

while the multitude of BG , GC is equal to the multitude of EH HF ,
therefore whatever part or parts BG is of EH , the same part or the same parts is GC of HF also,

so that in addition whatever part or parts BG is of EH , the same part also or the same parts in the sum BC of the sum EF [VII 5 6]

But BG is equal to A , and EH to D ,

therefore whatever part or parts A is of D the same part or the same parts is BC of EF also Q E D

PROPOSITION 10

If a number be parts of a number and another be the same parts of another alternately also whatever parts or part the first is of the third the same parts or the same part will the second also be of the fourth

For let the number AB be parts of the number C and another, DE , the same parts of another, F ,

I say that alternately also whatever parts or part AB is of DE the same parts or the same part is C of F also

For since whatever parts AB is of C the same parts also is DE of F

therefore as many parts of C as there are in AB so many parts also of F are there in DE

Let AB be divided into the parts of C namely AG , GB
and DE into the parts of F , namely DH , HE

thus the multitude of AG GB will be equal to the multitude of DH , HE

Now since whatever part AG is of C the same part also is DH of F ,

alternately also whatever part or parts AG is of DH ,

the same part or the same parts is C of F also [VII 9]

For the same reason also,

whatever part or parts GB is of HE , the same part or the same parts is C of F also

so that in addition whatever parts or part AB is of DE , the same parts also or the same part is C of F [VII 5 6]

Q E D

PROPOSITION 11

If as whole is to whole so is a number subtracted to a number subtracted the remainder will also be to the remainder as whole to whole

As the whole AB is to the whole CD so let AE subtracted be to CF subtracted,

I say that the remainder EB is also to the remainder FD as the whole AB to the whole CD

Since, as AB is to CD so is AE to CF ,

$\left. \begin{array}{l} A \\ E \\ B \end{array} \right\} \left. \begin{array}{l} C \\ F \\ D \end{array} \right\}$ whatever part or parts AB is of CD , the same part or the same parts is AE of CF also, [VII Def 20]
 Therefore also the remainder EB is the same part or parts of FD that AB is of CD [VII 7, 8]
 Therefore as EB is to FD , so is AB to CD [VII Def 20]
 Q E D

PROPOSITION 12

If there be as many numbers as we please in proportion, then as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents

Let A, B, C, D be as many numbers as we please in proportion so that, as A is to B , so is C to D

$\left. \begin{array}{l} A \\ B \end{array} \right\} \left. \begin{array}{l} C \\ D \end{array} \right\}$ I say that as A is to B so are C, C to B, D
 For since, as A is to B so is C to D
 whatever part or parts A is of B , the same part or parts is C of D also [VII Def 20]
 Therefore also the sum of A, C is the same part or the same parts of the sum of B, D that A is of B [VII 5, 6]
 Therefore as A is to B , so are A, C to B, D [VII Def 20]
 Q E D

PROPOSITION 13

If four numbers be proportional they will also be proportional alternately

Let the four numbers A, B, C, D be proportional so that as A is to B so is C to D

I say that they will also be proportional alternately so that as A is to C so will B be to D

$\left. \begin{array}{l} A \\ B \end{array} \right\} \left. \begin{array}{l} C \\ D \end{array} \right\}$ For since as A is to B so is C to D
 therefore whatever part or parts A is of B the same part or the same parts is C of D also [VII Def 20]
 Therefore, alternately whatever part or parts A is of C the same part or the same parts is B of D also [VII 10]
 Therefore as A is to C so is B to D [VII Def 20]
 Q E D

PROPOSITION 14

If there be as many numbers as we please and others equal to them in multitude, which taken two and two are in the same ratio they will also be in the same ratio *ex aequali*

Let there be as many numbers as we please A, B, C and others equal to them in multitude D, E, F which taken two and two are in the same ratio so that

$\frac{A}{B} = \frac{D}{E}$ as A is to B so is D to E
 $\frac{B}{C} = \frac{E}{F}$ and as B is to C , so is E to F
 I say that *ex aequali*
 as A is to C so also is D to F

For since as A is to B so is D to E

therefore alternately

as A is to D so is B to E

[VII 13]

Again since as B is to C so is E to F

therefore alternately,

as B is to E , so is C to F

[VII 13]

But as B is to F so is A to D ,

therefore also, as A is to D , so is C to F

Therefore alternately

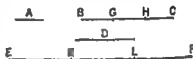
as A is to C , so is D to F

[id]

PROPOSITION 15

If an unit measure any number, and another number measure any other number the same number of times alternately also the unit will measure the third number the same number of times that the second measures the fourth

For let the unit A measure any number BC ,
and let another number D measure any
other number EF the same number of
times,



I say that alternately also the unit A
measures the number D the same number
of times that BC measures EF

For, since the unit A measures the number BC the same number of times
that D measures EF

therefore as many units as there are in BC , so many numbers equal to D are
there in EF also

Let BC be divided into the units in it BG GH , HC ,

and EF into the numbers EK , KL LF equal to D

Thus the multitude of BG , GH HC will be equal to the multitude of EK ,
 KL , LF

And since the units BG , GH HC are equal to one another

and the numbers EK , KL , LF are also equal to one another,

while the multitude of the units BG , GH , HC is equal to the multitude of the
numbers EK , KL , LF ,

therefore as the unit BG is to the number EK , so will the unit GH be to the
number KL and the unit HC to the number LF

Therefore also as one of the antecedents is to one of the consequents so will
all the antecedents be to all the consequents [VII 17]

therefore as the unit BG is to the number EK so is BC to EF

But the unit BG is equal to the unit A ,

and the number EK to the number D

Therefore as the unit A is to the number D so is BC to EF

Therefore the unit A measures the number D the same number of times that
 BC measures EF

Q.E.D.

PROPOSITION 16

*If two numbers by multiplying one another make certain numbers the numbers so
produced will be equal to one another*

Let A B be two numbers and let A by multiplying B make C , and B by
multiplying A make D ,

I say that C is equal to D

For since A by multiplying B has made C

therefore B measures C according to the units in A

But the unit E also measures the number A according to the units in it
therefore the unit E measures A the same number of times that B measures C

_____ A
 _____ B
 _____ C
 _____ D
 _____ E

Therefore alternately, the unit E
measures the number B the same num-
ber of times that A measures C [VII 15]

Again, since B by multiplying A has
made D ,
therefore A measures D according to the
units in B

But the unit E also measures B according to the units in it
therefore the unit E measures the number B the same number of times that A
measures D

But the unit E measured the number B the same number of times that A
measures C ,

therefore A measures each of the numbers C, D the same number of times

Therefore C is equal to D

Q E D

PROPOSITION 17

*If a number by multiplying two numbers make certain numbers the numbers so
produced will have the same ratio as the numbers multiplied*

For let the number A be multiplying the two numbers B, C make D, E

I say that as B is to C , so is D to E

For since A by multiplying B has made D

therefore B measures D according to the units in A

But the unit F also mea-
sures the number A according
to the units in it

therefore the unit F measures
the number A the same num-
ber of times that B measures D

_____ A
 _____ B _____ C
 _____ D _____ E
 _____ F

Therefore as the unit F is to the number A so is B to D [VII Def 20]

For the same reason

as the unit F is to the number A so also is C to E

therefore also as B is to D so is C to E

Therefore alternately as B is to C so is D to E

[VII 13]

Q E D

PROPOSITION 18

*If two numbers by multiplying any number make certain numbers the numbers so
produced will have the same ratio as the multipliers*

For let two numbers A, B by multiplying any number C make D, E

I say that as A is to B so is D to E

_____ C
 _____ A
 _____ B
 _____ D
 _____ E

For since A by multiplying C has made D
therefore also C by multiplying A has made D

[VII 16]

For the same reason also

C by multiplying B has made E

Therefore the number C by multiplying the two numbers A, B has made D, E

Therefore as A is to B so is D to E

[VII 17]

Q E D

PROPOSITION 19

If four numbers be proportional the number produced from the first and fourth will be equal to the number produced from the second and third and, if the number produced from the first and fourth be equal to that produced from the second and third the four numbers will be proportional

Let A, B, C, D be four numbers in proportion so that
as A is to B , so is C to D ,
and let A by multiplying D make E , and let B by multiplying C make F ,
I say that E is equal to F

For let A by multiplying C make G
Since then A by multiplying C has made G and by multiplying D has made E
the number A by multiplying the two numbers C, D
has made G, E

Therefore, as C is to D so is G to E [VII 17]
But as C is to D , so is A to B ,
therefore also as A is to B , so is G to E
Again since A by multiplying C has made G
but further, B has also by multiplying C made F ,
the two numbers A, B by multiplying a certain number C have made G, F

Therefore as A is to B , so is G to F [VII 18]
But further as A is to B , so is G to E also,
therefore also, as G is to E , so is G to F
Therefore G has to each of the numbers E, F the same ratio,
therefore E is equal to F [cf v 9]

Again let E be equal to F
I say that as A is to B so is C to D
For, with the same construction

since E is equal to F
therefore as G is to E , so is G to F [cf v 1]
But as G is to E so is C to D , [VII 17]
and as G is to F so is A to B [VII 18]
Therefore also, as A is to B so is C to D Q.E.D.

PROPOSITION 20

The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times the greater the greater and the less the less

For let CD, EF be the least numbers of those which have the same ratio with A, B ,

I say that CD measures A the same number of times that EF measures B
Now CD is not parts of A

For, if possible let it be so
therefore EF is also the same parts of B that CD is of A

Therefore as many parts of A as there are in CD so many parts of B are there also in EF [VII 13 and Def 20]

Let CD be divided into the parts of A namely CG GD and EF into the parts of B namely EH , HF

thus the multitude of CG , GD will be equal to the multitude of EH HF

Now since the numbers CG , GD are equal to one another and the numbers EH , HF are also equal to one another while the multitude of CG , GD is equal to the multitude of EH , HF ,

therefore as CG is to EH so is GD to HF

Therefore also as one of the antecedents is to one of the consequents so will all the antecedents be to all the consequents [VII 12]

Therefore as CG is to EH , so is CD to EF

Therefore CG EH are in the same ratio with CD , EF , being less than they which is impossible, for by hypothesis CD EF are the least numbers of those which have the same ratio with them

Therefore CD is not parts of A ,

therefore it is a part of it

[VII 4]

And EF is the same part of B that CD is of A

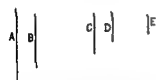
[VII 13 and Def 20]

therefore CD measures A the same number of times that EF measures B

Q E D

PROPOSITION 21

Numbers prime to one another are the least of those which have the same ratio with them



Let A B be numbers prime to one another

I say that A , B are the least of those which have the same ratio with them

For if not there will be some numbers less than A B which are in the same ratio with A , B

Let them be C , D

Since then the least numbers of those which have the same ratio measure those which have the same ratio the same number of times the greater the greater and the less the less that is, the antecedent the antecedent and the consequent the consequent, [VII 20]

therefore C measures A the same number of times that D measures B

Now, as many times as C measures A so many units let there be in E

Therefore D also measures B according to the units in E

And since C measures A according to the units in E

therefore E also measures A according to the units in C [VII 16]

For the same reason

E also measures B according to the units in D [VII 16]

Therefore E measures A , B which are prime to one another which is impossible [VII Def 12]

Therefore there will be no numbers less than A , B which are in the same ratio with A B

Therefore A B are the least of those which have the same ratio with them

Q E D

PROPOSITION 22

The least numbers of those which have the same ratio with them are prime to one another

Let A, B be the least numbers of those which have the same ratio with them,

I say that A, B are prime to one another

For, if they are not prime to one another some number will measure them

Let some number measure them and let it be C

And as many times as C measures A , so many units let there be in D

and as many times as C measures B so many units let there be in E

Since C measures A according to the units in D

therefore C by multiplying D has made A [vii Def 15]

For the same reason also

C by multiplying E has made B

Thus the number C by multiplying the two numbers D, E has made A, B ,

therefore as D is to E so is A to B , [vii 17]

therefore D, E are in the same ratio with A, B being less than they which is impossible

Therefore no number will measure the numbers A, B

Therefore A, B are prime to one another

Q E D

PROPOSITION 23

If two numbers be prime to one another the number which measures the one of them will be prime to the remaining number

Let A, B be two numbers prime to one another, and let any number C measure A ,

I say that C, B are also prime to one another

For if C, B are not prime to one another

some number will measure C, B

Let a number measure them and let it be D

Since D measures C and C measures A therefore D also measures A

But it also measures B

therefore D measures A, B which are prime to one another which is impossible [vii Def 12]

Therefore no number will measure the numbers C, B

Therefore C, B are prime to one another

Q E D

PROPOSITION 24

If two numbers be prime to any number their product also will be prime to the same

For let the two numbers A, B be prime to any number C and let A by multiplying B make D

I say that C, D are prime to one another

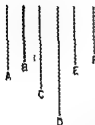
For if C, D are not prime to one another some number will measure C, D

Let E number measure them and let it be E

Now, since C, A are prime to one another

and a certain number E measures C ,
therefore A E are prime to one another [vii 23]

As many times then, as E measures D so many units let there be in F ,



therefore F also measures

D according to the units in E [vii 16]

Therefore E by multiplying F has made D [vii Def 15]

But further, A by multiplying B has also made D ,
therefore the product of E , F is equal to the product of A , B

But if the product of the extremes be equal to that of
the means the four numbers are proportional [vii 19]

therefore as E is to A , so is B to F

But A E are prime to one another

numbers which are prime to one another are also the least of those which have
the same ratio [vii 21]

and the least numbers of those which have the same ratio with them measure
those which have the same ratio the same number of times the greater the
greater and the less the less that is the antecedent the antecedent and the
consequent the consequent [vii 20]

therefore E measures B

But it also measures C ,
therefore E measures B , C which are prime to one another which is impossible
[vii Def 12]

Therefore no number will measure the numbers C D

Therefore C , D are prime to one another

Q E D

PROPOSITION 25

If two numbers be prime to one another the product of one of them into itself will
be prime to the remaining one

Let A B be two numbers prime to one another

and let A by multiplying itself make C



I say that B C are prime to one another

For let D be made equal to A

Since A B are prime to one another and A is equal to D

therefore D B are also prime to one another

Therefore each of the two numbers D , A is prime to B

therefore the product of D A will also be prime to B [vii 24]

But the number which is the product of D A is C

Therefore C B are prime to one another

Q E D

PROPOSITION 26

If two numbers be prime to two numbers both to each their products also will be
prime to one another



For let the two numbers A B be prime
to the two numbers C D both to each,
and let A by multiplying B make E and let
 C by multiplying D make F

I say that E F are prime to one another

For, since each of the numbers A B is prime to C
 therefore the product of A B will also be prime to C [VII 24]
 But the product of A , B is E
 therefore E , C are prime to one another
 For the same reason
 E D are also prime to one another
 Therefore each of the numbers C , D is prime to E
 Therefore the product of C , D will also be prime to E [VII 24]
 But the product of C , D is F
 Therefore E , F are prime to one another Q E D

PROPOSITION 27

If two numbers be prime to one another and each by multiplying itself make a certain number, the products will be prime to one another and if the original numbers by multiplying the products make certain numbers the latter will also be prime to one another [and this is always the case with the extremes]

Let A , B be two numbers prime to one another
 let A by multiplying itself make C , and by multiplying C make D
 and let B by multiplying itself make E , and by multiplying E make F ,

I say that both C , E and D F are prime to one another
 For since A B are prime to one another and A by multiplying itself has made C

therefore C , B are prime to one another [VII 20]

Since then C , B are prime to one another

and B by multiplying itself has made E

therefore C E are prime to one another [id]

Again since A B are prime to one another

and B by multiplying itself has made E ,

therefore A E are prime to one another [id]

Since then the two numbers A C are prime to the two numbers B E , both to each

therefore also the product of A C is prime to the product of B , E [VII 26]

And the product of A C is D and the product of B , E is F

Therefore D F are prime to one another Q E D

PROPOSITION 28

If two numbers be prime to one another the sum will also be prime to each of them and if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another

For let two numbers AB BC prime to one another be added

I say that the sum AC is also prime to each of the numbers AB BC

For if CA AB are not prime to one another

some number will measure CA AB

Let a number measure them and let it be D

Since then D measures CA AB

therefore it will also measure the remainder BC



But it also measures BA ,
therefore D measures AB, BC which are prime to one another which is impossible [VII Def 12]

Therefore no number will measure the numbers CA, AB , therefore CA, AB are prime to one another

For the same reason

AC, CB are also prime to one another

Therefore CA is prime to each of the numbers AB, BC

Again let CA, AB be prime to one another,

I say that AB, BC are also prime to one another

For if AB, BC are not prime to one another,

some number will measure AB, BC

Let a number measure them and let it be D

Now since D measures each of the numbers AB, BC it will also measure the whole CA

Put it also measures AB

therefore D measures CA, AB which are prime to one another

which is impossible

[VII Def 12]

Therefore no number will measure the numbers AB, BC

Therefore AB, BC are prime to one another

Q E D

PROPOSITION 29

Any prime number is prime to any number which it does not measure

Let A be a prime number and let it not measure B

I say that B, A are prime to one another

For if B, A are not prime to one another,

some number will measure them

_____A

_____B

_____C

Let C measure them

Since C measures B

and A does not measure B

therefore C is not the same with A

Now since C measures B, A

therefore it also measures A which is prime though it is not the same with it which is impossible

Therefore no number will measure B, A

Therefore A, B are prime to one another

Q E D

PROPOSITION 30

If two numbers by multiplying one another make some number and any prime number measure the product it will also measure one of the original numbers

For let the two numbers A, B by multiplying one another make C and let any prime number D measure C

I say that D measures one of the numbers A, B

For let it not measure A

Now D is prime

therefore A, D are prime to one another [VII 29]

And as many times as D measures C so many units let there be in E

Since then D measures C according to the units in E

therefore D by multiplying E has made C [VII Def 15]

Further A by multiplying B has also made C ,

therefore the product of D, E is equal to the product of A, B

Therefore as D is to A , so is B to E [VII 19]

But D, A are prime to one another,

primes are also least [VII 21]

and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less that is the antecedent the antecedent and the consequent the consequent, [VII 20]

therefore D measures B

Similarly we can also show that if D does not measure B , it will measure A

Therefore D measures one of the numbers A, B Q E D

PROPOSITION 31

Any composite number is measured by some prime number

Let A be a composite number,

I say that A is measured by some prime number

For since A is composite

some number will measure it

Let a number measure it and let it be B

A _____

Now, if B is prime, what was enjoined will have been done

B _____

C _____

But if it is composite some number will measure it

Let a number measure it, and let it be C

Then since C measures B ,

and B measures A

therefore C also measures A

And if C is prime what was enjoined will have been done

But if it is composite some number will measure it

Thus if the investigation be continued in this way some prime number will be found which will measure the number before it which will also measure A

For if it is not found an infinite series of numbers will measure the number A each of which is less than the other

which is impossible in numbers

Therefore some prime number will be found which will measure the one before it which will also measure A

Therefore any composite number is measured by some prime number

Q E D

PROPOSITION 32

Any number either is prime or is measured by some prime number

Let A be a number

I say that A either is prime or is measured by some prime number A _____

If now A is prime that which was enjoined will have been done

But if it is composite some prime number will measure it [VII 31]

Therefore any number either is prime or is measured by some prime number

Q E D

PROPOSITION 33

Given as many numbers as we please, to find the least of those which have the same ratio with them

Let A, B, C be the given numbers, as many as we please,

thus it is required to find the least of those which have the same ratio with A, B, C

A, B, C are either prime to one another or not

Now, if A, B, C are prime to one another, they are the least of those which have the same ratio with them [VII 21]

But if not, let D the greatest common measure of A, B, C be taken [VII 3]

and, as many times as D measures the numbers A, B, C respectively, so many units let there be in the numbers E, F, G respectively

Therefore the numbers E, F, G measure the numbers A, B, C respectively according to the units in D [VII 16]

Therefore E, F, G measure A, B, C the same number of times

therefore E, F, G are in the same ratio with A, B, C [VII Def 20]

I say next that they are the least that are in that ratio

For if E, F, G are not the least of those which have the same ratio with A, B, C , there will be numbers less than E, F, G which are in the same ratio with A, B, C

Let them be H, K, L , therefore H measures A the same number of times that the numbers K, L measure the numbers B, C respectively

Now, as many times as H measures A , so many units let there be in M

therefore the numbers K, L also measure the numbers B, C respectively according to the units in M

And since H measures A according to the units in M

therefore M also measures A according to the units in H [VII 16]

For the same reason

M also measures the numbers B, C according to the units in the numbers K, L respectively,

Therefore M measures A, B, C

Now since H measures A according to the units in M

therefore H by multiplying M has made A [VII Def 15]

For the same reason also

E by multiplying D has made A

Therefore the product of E, D is equal to the product of H, M

Therefore as E is to H so is M to D [VII 19]

But E is greater than H

therefore M is also greater than D

And it measures A, B, C

which is impossible for by hypothesis D is the greatest common measure of A, B, C

Therefore there cannot be any numbers less than E, F, G which are in the same ratio with A, B, C

Therefore E, F, G are the least of those which have the same ratio with A, B, C Q E D

PROPOSITION 34

Given two numbers to find the least number which they measure

Let A, B be the two given numbers,

thus it is required to find the least number which they measure

Now A, B are either prime to one another or not

First let A, B be prime to one another, and let A

by multiplying B make C ,

therefore also B by multiplying A has made C

Therefore A, B measure C

I say next that it is also the least number they measure

For if not, A, B will measure some number which is less than C

Let them measure D

Then as many times as A measures D so many units let there be in E , and as many times as B measures D , so many units let there be in F ,

therefore A by multiplying E has made D

and B by multiplying F has made D

[VII Def 15]

therefore the product of A, E is equal to the product of B, F

Therefore as A is to B , so is F to E

[VII 19]

But A, B are prime

primes are also least

[VII 21]

and the least measure the numbers which have the same ratio the same number of times the greater the greater and the less the less

[VII 20]

therefore B measures E as consequent consequent

And since A by multiplying B, E has made C, D

therefore as B is to E so is C to D

[VII 17]

But B measures E ,

therefore C also measures D the greater the less

which is impossible

Therefore A, B do not measure any number less than C

therefore C is the least that is measured by A, B

Next, let A, B not be prime to one another

and let F, E the least numbers of those which have the same ratio with A, B , be taken, [VII 33]

therefore the product of A, E is equal to the product of B, F [VII 19]

And let A by multiplying E make C

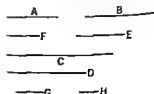
therefore also B by multiplying F has made C ,

therefore A, B measure C

I say next that it is also the least number that they measure

For, if not A, B will measure some number which is less than C

Let them measure D



And, as many times as A measures D , so many units let there be in G
 and as many times as B measures D so many units let there be in H

Therefore A by multiplying G has made D ,

and B by multiplying H has made D

Therefore the product of A, G is equal to the product of B, H ,

therefore, as A is to B , so is H to G [VII 19]

But as A is to B , so is F to E

Therefore also as F is to E so is H to G

But F, E are least

and the least measure the numbers which have the same ratio the same number of times the greater the greater and the less the less [VII 20]

therefore E measures G

And since A by multiplying E G has made C, D

therefore as E is to G , so is C to D [VII 17]

But E measures G

therefore C also measures D , the greater the less

which is impossible

Therefore A, B will not measure any number which is less than C

Therefore C is the least that is measured by A, B Q E D

PROPOSITION 35

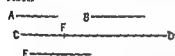
If two numbers measure any number the least number measured by them will also measure the same

For let the two numbers A, B measure any number CD

and let E be the least that they measure

I say that E also measures CD

For if E does not measure CD let E measuring DF leave CF less than itself



Now since A, B measure E

and E measures DF ,

therefore A, B will also measure DF

But they also measure the whole CD ,

therefore they will also measure the remainder CF which is less than E

which is impossible

Therefore E cannot fail to measure CD

therefore it measures it

Q E D

PROPOSITION 36

Given three numbers to find the least number which they measure

Let A, B, C be the three given numbers,

thus it is required to find the least number which they measure

A —————

B —————

C —————

D —————

E —————

Let D the least number measured by the two numbers A, B be taken [VII 34]

Then C either measures or does not measure D

First let it measure it

But A, B also measure D

therefore A, B, C measure D

I say next that it is also the least that they measure

For, if not, A, B, C will measure some number which is less than D

Let them measure E

Since A, B, C measure E ,

therefore also A, B measure E .

Therefore the least number measured by A, B will also measure E [VII 35]

But D is the least number measured by A, B ,

therefore D will measure E , the greater the less
which is impossible

Therefore A, B, C will not measure any number which is less than D ,

therefore D is the least that A, B, C measure

Again let C not measure D ,

and let E the least number measured by C, D , be
taken [VII 34]

Since A, B measure D

and D measures E

therefore also A, B measure E

But C also measures E

therefore also A, B, C measure E

I say next that it is also the least that they measure

For if not, A, B, C will measure some number which is less than E

Let them measure F

Since A, B, C measure F

therefore also A, B measure F ,

therefore the least number measured by A, B will also measure F [VII 35]

But D is the least number measured by A, B ,

therefore D measures F

But C also measures F ,

therefore D, C measure F

so that the least number measured by D, C will also measure F . .

But E is the least number measured by C, D ,

therefore E measures F the greater the less
which is impossible

Therefore A, B, C will not measure any number which is less than E

Therefore E is the least that is measured by A, B, C Q. E. D.

PROPOSITION 37

If a number be measured by any number the number which is measured will have a part called by the same name as the measuring number

For let the number A be measured by any number B

I say that A has a part called by the same name as B

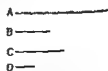
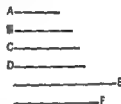
For as many times as B measures A so many units let
there be in C

Since B measures A according to the units in C ,

and the unit D also measures the number C according
to the units in it

therefore the unit D measures the number C the same number of times as B
measures A

Therefore alternately the unit D measures the number B the same number
of times as C measures A [VII 15]



therefore, whatever part the unit D is of the number B , the same part is C of A also

But the unit D is a part of the number B called by the same name as it, therefore C is also a part of A called by the same name as B , so that A has a part C which is called by the same name as B Q E D

PROPOSITION 38

If a number have any part whatever, it will be measured by a number called by the same name as the part

For let the number A have any part whatever, B

and let C be a number called by the same name as the part B ,

A _____

I say that C measures A

For since B is a part of A called by the same name

as C ,

and the unit D is also a part of C called by the same name as it

therefore whatever part the unit D is of the number C

the same part is B of A also

therefore the unit D measures the number C the same number of times that B measures A

Therefore, alternately the unit D measures the number B the same number of times that C measures A [VII 15]

Therefore C measures A Q E D

PROPOSITION 39

To find the number which is the least that will have given parts

Let A, B, C be the given parts

thus it is required to find the number which is the least that will have the parts A, B, C

Let D, E, F be numbers called by the same name as the parts A, B, C

A B C

D _____ E

_____ F

and let G the least number measured by D, E, F , be taken [VII 36]

Therefore G has parts called by the same name as D, E, F [VII 37]

But A, B, C are parts called by the same name as D, E, F

therefore G has the parts A, B, C

I say next that it is also the least number that has

For if not there will be some number less than G which will have the parts A, B, C

Let it be H

Since H has the parts A, B, C

therefore H will be measured by numbers called by the same name as the parts A, B, C [VII 38]

But D, E, F are numbers called by the same name as the parts A, B, C ,

therefore H is measured by D, E, F

And it is less than G which is impossible

Therefore there will be no number less than G that will have the parts A, B, C Q E D

BOOK EIGHT

PROPOSITION I

If there be as many numbers as we please in continued proportion and the extremes of them be prime to one another, the numbers are the least of those which have the same ratio with them

Let there be as many numbers as we please A, B, C, D , in continued proportion and let the extremes of them A, D be prime to one another

I say that A, B, C, D are the least of those which have the same ratio with them

For if not let E, F, G, H be less than A, B, C, D and in the same ratio with them

Now since A, B, C, D are in the same ratio with E, F, G, H and the multitude of the numbers A, B, C, D is equal to the multitude of the numbers E, F, G, H ,

therefore *ex aequali*
as A is to D so is F to H [VII 14]

But A, D are prime

primes are also least [VII 21]

and the least numbers measure those which have the same ratio the same number of times the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent [VI 20]

Therefore A measures E , the greater the less
which is impossible

Therefore E, F, G, H which are less than A, B, C, D are not in the same ratio with them

Therefore A, B, C, D are the least of those which have the same ratio with them Q. E. D.

PROPOSITION II

To find numbers in continued proportion as many as may be prescribed and the least that are in a given ratio

I let the ratio of A to B be the given ratio in least numbers, thus it is required to find numbers in continued proportion as many as may be prescribed and the least that are in the ratio of A to B

Let four be prescribed

let A by multiplying itself make C and by multiplying B let it make D

let B by multiplying itself make E ,

further let A by multiplying C, D, E make F, G, H

and let B by multiplying E make A

Now, since A by multiplying itself has made C ,

and by multiplying B has made D

therefore, as A is to B , so is C to D

[VII 17]

Again since A by multiplying B has made D

and B by multiplying itself has made E

therefore the numbers A, B by multiplying B have made the numbers D, E respectively

Therefore as A is to B so is D to E ,

[VII 18]

But as A is to B , so is C to D ,

therefore also as C is to D so is D to E

And since A by multiplying C D has made F, G ,

therefore as C is to D , so is F to G

[VII 17]

But as C is to D , so was A to B ,

therefore also as A is to B , so is F to G

Again since A by multiplying D, E has made G, H ,

therefore, as D is to E , so is G to H

[VII 17]

But as D is to E , so is A to B

Therefore also as A is to B , so is G to H

And since A, B by multiplying E have made H, K ,

therefore as A is to B , so is H to K

[VII 18]

But as A is to B so is F to G and G to H

Therefore also as F is to G so is G to H and H to K ,

therefore C, D, E and F, G, H, K are proportional in the ratio of A to B

I say next that they are the least numbers that are so

For since A, B are the least of those which have the same ratio with them, and the least of those which have the same ratio are prime to one another,

[VII 22]

therefore A, B are prime to one another

And the numbers A, B by multiplying themselves respectively have made the numbers C, E and by multiplying the numbers C, E respectively have made the numbers F, K

therefore C, E and F, K are prime to one another respectively

[VII 27]

But if there be as many numbers as we please in continued proportion and the extremes of them be prime to one another they are the least of those which have the same ratio with them

[VIII 1]

Therefore C, D, E and F, G, H, K are the least of those which have the same ratio with A, B

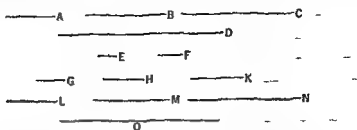
Q E D

PROPOSITION 3
If as many numbers as we please in continued proportion be the least of those which have the same ratio with them the extremes of them are prime to one another

PROPOSITION 3

If as many numbers as we please in continued proportion be the least of those which have the same ratio with them the extremes of them are prime to one another

Let as many numbers as we please A, B, C, D in continued proportion be the least of those which have the same ratio with them,



I say that the extremes of them A, D are prime to one another

For let two numbers E, F , the least that are in the ratio of A, B, C, D be taken, [VII 33]

then three others G, H, K with the same property,

and others more by one continually [VIII 2]

until the multitude taken becomes equal to the multitude of the numbers A, B, C, D

Let them be taken and let them be L, M, N, O

Now, since E, F are the least of those which have the same ratio with them they are prime to one another [VII 2^o]

And since the numbers E, F by multiplying themselves respectively have made the numbers G, K and by multiplying the numbers G, K respectively have made the numbers L, O [VIII 2 For]

therefore both G, K and L, O are prime to one another [VII 27]

And since A, B, C, D are the least of those which have the same ratio with them

while L, M, N, O are the least that are in the same ratio with A, B, C, D and the multitude of the numbers A, B, C, D is equal to the multitude of the numbers L, M, N, O

therefore the numbers A, B, C, D are equal to the numbers L, M, N, O respectively

therefore A is equal to L and D to O

And L, O are prime to one another

Therefore A, D are also prime to one another

Q.E.D.

PROPOSITION 4

Given as many ratios as we please in least numbers to find numbers in continued proportion which are the least in the given ratios

Let the given ratios in least numbers be that of A to B , that of C to D and that of E to F

thus it is required to find numbers in continued proportion which are the least that are in the ratio of A to B in the ratio of C to D and in the ratio of E to F

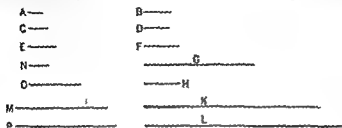
Let G the least number measured by B, C be taken [VII 34]

And as many times as B measures G so many times also let A measure H , and as many times as C measures G so many times also let D measure K

Now E either measures or does not measure K

First let it measure it

And as many times as E measures K , so many times let F measure L also
 Now since A measures H the same number of times that B measures G
 therefore as A is to B , so is H to G [VII Def 20, VII 13]



For the same reason also

as C is to D so is G to K

and further as E is to F , so is K to L ,

therefore H, G, K, L are continuously proportional in the ratio of A to B , in the ratio of C to D and in the ratio of E to F

I say next that they are also the least that have this property

For, if H, G, K, L are not the least numbers continuously proportional in the ratios of A to B , of C to D and of E to F , let them be N, O, M, P

Then since, as A is to B , so is N to O

while A, B are least

and the least numbers measure those which have the same ratio the same number of times the greater the greater and the less the less that is the antecedent the antecedent and the consequent the consequent,

therefore B measures O

[VII 20]

For the same reason

C also measures O ,

therefore B, C measure O

therefore the least number measured by B, C will also measure O [VII 35]

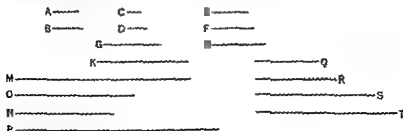
But G is the least number measured by B, C ,

therefore G measures O the greater the less

which is impossible

Therefore there will be no numbers less than H, G, K, L which are continuously in the ratio of A to B , of C to D and of E to F

Next let E not measure K



Let M the least number measured by E, A be taken

And as many times as K measures M so many times let H, G measure N, O respectively

and as many times as E measures M so many times let F measure P also
 Since H measures N the same number of times that G measures O ,
 therefore, as H is to G , so is N to O [VII 13 and Def 20]
 But as H is to G , so is A to B ,
 therefore also as A is to B , so is N to O
 For the same reason also
 as C is to D , so is O to M
 Again since E measures M the same number of times that F measures P ,
 therefore as L is to F so is M to P [VII 13 and Def 20]
 therefore N, O, M, P are continuously proportional in the ratios of A to B , of
 C to D and of E to F
 I say next that they are also the least that are in the ratios A, B, C, D, E, F
 For if not there will be some numbers less than N, O, M, P continuously
 proportional in the ratios A, B, C, D, E, F
 Let them be Q, R, S, T
 Now since Q is to R , so is A to B
 while A, B are least
 and the least numbers measure those which have the same ratio with them the
 same number of times the antecedent the antecedent and the consequent the
 consequent, [VII 20]

therefore B measures R

For the same reason C also measures R ,

therefore B, C measure R

Therefore the least number measured by B, C will also measure R [VII 35]

But G is the least number measured by B, C ,

therefore G measures R

And as G is to R so is A to B [VII 13]

therefore A also measures S

But E also measures S

therefore E, A measure S

Therefore the least number measured by E, A will also measure S [VII 35]

But M is the least number measured by E, A ,

therefore M measures S the greater the less

which is impossible

Therefore there will not be any numbers less than N, O, M, P continuously
 proportional in the ratios of A to B of C to D and of E to F ,
 therefore N, O, M, P are the least numbers continuously proportional in the
 ratios A, B, C, D, E, F Q E D

PROPOSITION 5

Plane numbers have to one another the ratio compounded of the ratios of their sides

Let A, B be plane numbers and let the numbers C, D be the sides of A and
 E, F of B

I say that A has to B the ratio compounded of the ratios of the sides

For the ratios being given which C has to E and D to F , let the least num
 bers G, H, K that are continuously in the ratios C, E, D, F be taken so that

as C is to E so is G to H

and as D is to F so is H to K [VIII 4]

And let D by multiplying E make L

Now since D by multiplying C has made A , and by multiplying E has made L ,

therefore as C is to E , so is A to L [VI 17]

But as C is to E so is G to H ,

therefore also as G is to H , so is A to L

_____ A

_____ B

_____ C _____ D

_____ E _____ F

_____ G

_____ H

_____ K

_____ L

Again since E by multiplying D has made L , and further by multiplying F has made B

therefore, as D is to F so is L to B [VII 17]

But as D is to F , so is H to K ,

therefore also as H is to K , so is L to B

But it was also proved that

as G is to H , so is A to L ,

therefore *ex aequali*,

as G is to K so is A to B [VII 14]

But G has to K the ratio compounded of the ratios of the sides therefore A also has to B the ratio compounded of the ratios of the sides

Q E D

PROPOSITION 6

If there be as many numbers as we please in continued proportion, and the first do not measure the second neither will any other measure any other

Let there be as many numbers as we please A, B, C, D, E in continued proportion and let A not measure B ,

I say that neither will any other measure any other

_____ A

_____ B

_____ C

_____ D

_____ E

_____ F

_____ G

_____ H

Now it is manifest that A, B, C, D, E do not measure one another in order for A does not even measure B

I say then that neither will any other measure any other

For if possible let A measure C

And however many A, B, C are let as many numbers F, G, H the least of those which have

the same ratio with A, B, C be taken [VI 33]

Now, since F, G, H are in the same ratio with A, B, C , and the multitude of the numbers A, B, C is equal to the multitude of the numbers F, G, H

therefore, *ex aequali* as A is to C so is F to H [VII 14]

And since, as A is to B so is F to G

while A does not measure B ,

therefore neither does F measure G , [VII Def 20]

therefore F is not an unit for the unit measures any number

Now F, H are prime to one another [VIII 3]

And as F is to H so is A to C

therefore neither does A measure C

Similarly we can prove that neither will any other measure any other

Q E D

PROPOSITION 7

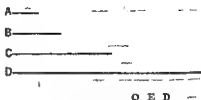
If there be as many numbers as we please in continued proportion, and the first measure the last it will measure the second also

Let there be as many numbers as we please, A, B, C, D , in continued proportion, and let A measure D ,
I say that A also measures B

For if A does not measure B , neither will any other of the numbers measure any other [VIII 6]

But A measures D

Therefore A also measures B



PROPOSITION 8

If between two numbers there fall numbers in continued proportion with them then however many numbers fall between them in continued proportion so many will also fall in continued proportion between the numbers which have the same ratio with the original numbers

Let the numbers C, D fall between the two numbers A, B in continued proportion with them, and let E be made in the same ratio to F as A is to B ,

I say that as many numbers as have fallen between A, B in continued proportion so many will also fall between E, F in continued proportion

For as many as A, B, C, D are in multitude let so many numbers G, H, K, L the least of those which have the same ratio with A, C, D, B , be taken [VII 33]
therefore the extremes of them G, L are prime to one another [VIII 3]



Now since A, C, D, B are in the same ratio with G, H, K, L and the multitude of the numbers A, C, D, B is equal to the multitude of the numbers G, H, K, L

therefore *ex aequali* as A is to B so is G to L [VII 14]

But, as A is to B so is E to F

therefore also as G is to L so is E to F

But G, L are prime,

primes are also least [VII 21]

and the least numbers measure those which have the same ratio the same number of times the greater the greater and the less the less that is the antecedent the antecedent and the consequent the consequent [VII 20]

Therefore G measures E the same number of times as L measures F

Next as many times as G measures E so many times let H, K also measure M, N respectively

therefore G, H, K, L measure E, M, N, F the same number of times

Therefore G, H, K, L are in the same ratio with E, M, N, F [VII Def 20]

But G, H, K, L are in the same ratio with A, C, D, B

therefore A, C, D, B are also in the same ratio with E, M, N, F

But A, C, D, B are in continued proportion

therefore E, M, N, F are also in continued proportion.

Therefore, as many numbers as have fallen between A, B in continued proportion with them, so many numbers have also fallen between E, F in continued proportion. [VIII 2]

PROPOSITION 9

If two numbers be prime to one another and numbers fall between them in continued proportion, then however many numbers fall between them in continued proportion so many will also fall between each of them and an unit in continued proportion.

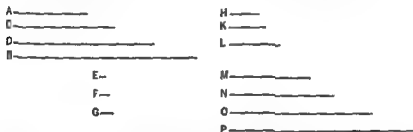
Let A, B be two numbers prime to one another and let C, D fall between them in continued proportion

and let the unit E be set out,

I say that as many numbers as fall between A, B in continued proportion so many will also fall between either of the numbers A, B and the unit in continued proportion

For let two numbers F, G , the least that are in the ratio of A, C, D, B be taken

three numbers H, K, L with the same property, and others more by one continually until their multitude is equal to the multitude of A, C, D, B . [VIII 2]



Let them be taken and let them be M, N, O, P

It is now manifest that F by multiplying itself has made H and by multiplying H has made M while G by multiplying itself has made L and by multiplying L has made P . [VIII 2, Por]

And since M, N, O, P are the least of those which have the same ratio with F, G

and A, C, D, B are also the least of those which have the same ratio with F, G [VIII 1]

while the multitude of the numbers M, N, O, P is equal to the multitude of the numbers A, C, D, B

therefore M, N, O, P are equal to A, C, D, B respectively,

therefore M is equal to A , and P to B

Now since F by multiplying itself has made H

therefore F measures H according to the units in F

But the unit E also measures F according to the units in it
 therefore the unit E measures the number F the same number of times as F measures H

Therefore as the unit I is to the number F so is I to H [VII Def 20]

Again, since F by multiplying H has made M ,
therefore H measures M according to the units in F

But the unit L also measures the number F according to the units in it,
therefore the unit E measures the number F the same number of times as H measures M

Therefore, as the unit E is to the number F , so is H to M

But it was also proved that as the unit C is to the number F so is F to H ,
therefore also as the unit E is to the number F , so is F to H , and H to M

But M is equal to A

therefore, as the unit E is to the number F , so is F to H and H to A

For the same reason also

as the unit E is to the number G so is G to L and L to B

Therefore as many numbers as have fallen between A, B in continued proportion so many numbers also have fallen between each of the numbers A, B and the unit E in continued proportion Q E D

PROPOSITION 10

If numbers fall between each of two numbers and an unit in continued proportion however many numbers fall between each of them and an unit in continued proportion so many also will fall between the numbers themselves in continued proportion

For let the numbers D, E and F, G respectively fall between the two numbers A, B and the unit C in continued proportion

I say that as many numbers as have fallen between each of the numbers A, B and the unit C in continued proportion so many numbers will also fall between A, B in continued proportion

For let D by multiplying I make H and let the numbers D, F by multiplying H make K, L respectively

Now since as the unit C is to the number D so is D to E

therefore the unit C measures the number D the same number of times as D measures E

[VII Def 20]

But the unit C measures the number D according to the units in D

therefore the number D also measures E according to the units in D ,
therefore D by multiplying itself has made E

Again since as C is to the number D so is E to I
therefore the unit C measures the number D the same number of times as E measures A

But the unit C measures the number D according to the units in D ,
therefore E also measures A according to the units in D
therefore D by multiplying E has made A

For the same reason also

F by multiplying itself has made G and by multiplying G has made B



And since D by multiplying itself has made E and by multiplying F has made H ,

therefore as D is to F , so is E to H [VII 17]

For the same reason also

as D is to F , so is H to G [VII 18]

Therefore also as E is to H so is H to G

Again, since D by multiplying the numbers E H has made A K respectively,

therefore as E is to H , so is A to K [VII 17]

But, as E is to H , so is D to F ,

therefore also, as D is to F so is A to K

Again since the numbers D F by multiplying H have made K , L respectively,

therefore, as D is to F , so is K to L [VII 18]

But as D is to F , so is A to K

therefore also as A is to K , so is K to L

Further, since F by multiplying the numbers H G has made L , B respectively

therefore as H is to G , so is L to B , [VII 17]

But, as H is to G so is D to F ,

therefore also as D is to F , so is L to B

But it was also proved that,

as D is to F so is A to K and K to L

therefore also as A is to K , so is K to L and L to B

Therefore A , K , L , B are in continued proportion

Therefore as many numbers as fall between each of the numbers A , B and the unit C in continued proportion so many also will fall between A , B in continued proportion

Q E D

PROPOSITION 11

Between two square numbers there is one mean proportional number and the square has to the square the ratio duplicate of that which the side has to the side

Let A , B be square numbers

and let C be the side of A and D of B

I say that between A B there is one mean proportional number and A has to B the ratio duplicate of that which C has to D

A _____

For let C by multiplying D make E

B _____

Now since A is a square and C is its side

C _____ D _____

therefore C by multiplying itself has made A

E _____

For the same reason also

D by multiplying itself has made B

Since then C by multiplying the numbers C D has made A E respectively

therefore as C is to D , so is A to E [VII 17]

For the same reason also

as C is to D so is E to B

[VII 18]

Therefore also as A is to E so is E to B

Therefore between A B there is one mean proportional number

I say next that A also has to B the ratio duplicate of that which C has to D

For since A E B are three numbers in proportion

therefore A has to B the ratio duplicate of that which A has to E [v Def 9]

But as A is to E , so is C to D

Therefore A has to B the ratio duplicate of that which the side C has to D
Q E D

PROPOSITION 12

Between two cube numbers there are two mean proportional numbers, and the cube has to the cube the ratio triplicate of that which the side has to the side

Let A, B be cube numbers,

and let C be the side of A , and D of B ,

I say that between A, B there are two mean proportional numbers and A has to B the ratio triplicate of that which C has to D

For let C by multiplying

itself make E and by multi

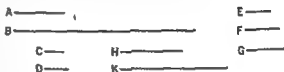
plying D let it make F

let D by multiplying itself

make G

and let the numbers C, D by

multiplying F make H, K respectively



Now since A is a cube and C its side

and C by multiplying itself has made E ,

therefore C by multiplying itself has made E and by multiplying E has made A

For the same reason also

D by multiplying itself has made G and by multiplying G has made B

And, since C by multiplying the numbers C, D has made E, F respectively

therefore, as C is to D so is E to F [vii 17]

For the same reason also,

as C is to D so is F to G [vii 18]

Again since C by multiplying the numbers E, F has made A, H respectively

therefore as E is to F , so is A to H [vii 17]

But as E is to F so is C to D

Therefore also as C is to D , so is A to H

Again since the numbers C, D by multiplying F have made H, K respectively,

therefore as C is to D , so is H to K [vii 18]

Again since D by multiplying each of the numbers F, G has made K, B respectively

therefore as F is to G , so is K to B [vii 17]

But, as F is to G , so is C to D

therefore also as C is to D , so is A to H , H to K and K to B

Therefore H, K are two mean proportionals between A, B

I say next that A also has to B the ratio triplicate of that which C has to D

For since A, H, K, B are four numbers in proportion,

therefore A has to B the ratio triplicate of that which A has to H [v Def 10]

But, as A is to H so is C to D ,

therefore A also has to B the ratio triplicate of that which C has to D

Q E D

PROPOSITION 13

If there be as many numbers as we please in continued proportion, and each by multiplying itself make some number, the products will be proportional and, if the original numbers by multiplying the products make certain numbers the latter will also be proportional

Let there be as many numbers as we please, A, B, C , in continued proportion so that as A is to B so B is to C ,
let A, B, C by multiplying themselves make D, E, F , and by multiplying D, E, F let them make G, H, K

I say that D, E, F and G, H, K are in continued proportion

| | |
|-----------|-----------|
| A _____ | G _____ |
| B _____ | H _____ |
| C _____ | K _____ |
| D _____ | |
| E _____ | M _____ |
| F _____ | N _____ |
| L _____ | P _____ |
| O _____ | Q _____ |

For let A by multiplying B make L

and let the numbers A, B by multiplying L make M, N respectively

And again let B by multiplying C make O ,

and let the numbers B, C by multiplying O make P, Q respectively

Then in manner similar to the foregoing we can prove that

D, L, E and G, M, N, H are continuously proportional in the ratio of A to B and further E, O, F and H, P, Q, K are continuously proportional in the ratio of B to C

Now, as A is to B , so is B to C ,

therefore D, L, E are also in the same ratio with E, O, F ,

and further G, M, N, H in the same ratio with H, P, Q, K

And the multitude of D, L, E is equal to the multitude of E, O, F and that of G, M, N, H to that of H, P, Q, K ,

therefore *ex aequali*

as D is to E so is E to F

and as G is to H , so is H to K [VII 14]

Q, E, D

PROPOSITION 14

If a square measure a square the side will also measure the side and if the side measure the side the square will also measure the square

Let A, B be square numbers let C, D be their sides and let A measure B

I say that C also measures D

A _____

B _____

C _____ D _____

E _____

For let C by multiplying D make E ,

therefore A, E, B are continuously proportional in the ratio of C to D [VIII 11]

And since A, E, B are continuously proportional and A measures B

therefore A also measures E

[III 7]

therefore A has to B the ratio duplicate of that which A has to E [v Def 9]

But, as A is to E , so is C to D

Therefore A has to B the ratio duplicate of that which the side C has to D
Q E D

PROPOSITION 12

Between two cube numbers there are two mean proportional numbers, and the cube has to the cube the ratio triplicate of that which the side has to the side

Let A, B be cube numbers,

and let C be the side of A , and D of B ,

I say that between A, B there are two mean proportional numbers and A has to B the ratio triplicate of that which C has to D

For let C by multiplying

itself make E and by multi

plying D let it make F ,

let D by multiplying itself

make G

and let the numbers C, D by

multiplying F make H, K respectively



Now, since A is a cube and C its side

and C by multiplying itself has made E ,

therefore C by multiplying itself has made E and by multiplying E has made A

For the same reason also

D by multiplying itself has made G and by multiplying G has made B

And since C by multiplying the numbers C, D has made E, F respectively

therefore, as C is to D so is E to F [vii 17]

For the same reason also

as C is to D so is F to G [vii 18]

Again since C by multiplying the numbers E, F has made A, H respectively

therefore as E is to F , so is A to H [vii 17]

But as E is to F , so is C to D

Therefore also as C is to D , so is A to H

Again since the numbers C, D by multiplying F have made H, K respectively

therefore, as C is to D , so is H to K [vii 18]

Again since D by multiplying each of the numbers F, G has made K, B respectively

therefore, as F is to G , so is K to B [vii 17]

But as F is to G so is C to D ,

therefore also as C is to D so is A to H, H to K , and K to B

Therefore H, K are two mean proportionals between A, B

I say next that A also has to B the ratio triplicate of that which C has to D

For since A, H, K, B are four numbers in proportion

therefore A has to B the ratio triplicate of that which A has to H [v Def 10]

But, as A is to H so is C to D

therefore A also has to B the ratio triplicate of that which C has to D
Q E D

PROPOSITION 13

If there be as many numbers as we please in continued proportion, and each by multiplying itself make some number, the products will be proportional and if the original numbers by multiplying the products make certain numbers the latter will also be proportional.

Let there be as many numbers as we please, A, B, C , in continued proportion, so that as A is to B , so is B to C , let A, B, C by multiplying themselves make D, E, F , and by multiplying D, E, F let them make G, H, K .

I say that D, E, F and G, H, K are in continued proportion

| | |
|---------|---------|
| A _____ | G _____ |
| B _____ | H _____ |
| C _____ | K _____ |
| D _____ | M _____ |
| E _____ | N _____ |
| F _____ | P _____ |
| L _____ | Q _____ |
| O _____ | |

For let A by multiplying B make L

and let the numbers A, B by multiplying L make M, N respectively

And again let B by multiplying C make O ,

and let the numbers B, C by multiplying O make P, Q respectively

Then in manner similar to the foregoing we can prove that

D, L, E and G, M, N, H are continuously proportional in the ratio of A to B and further E, O, F and H, P, Q, K are continuously proportional in the ratio of B to C .

Now, as A is to B , so is B to C ,

therefore D, L, E are also in the same ratio with E, O, F ,

and further G, M, N, H in the same ratio with H, P, Q, K .

And the multitude of D, L, E is equal to the multitude of E, O, F and that of G, M, N, H to that of H, P, Q, K ,

therefore *ex aequali*

as D is to E so is E to F

as G is to H so is H to K

and,

[VIII 14]

Q.E.D.

PROPOSITION 14

If a square measure a square the side will also measure the side and if the side measure the side the square will also measure the square

Let A, B be square numbers, let C, D be their sides, and let A measure B ,

I say that C also measures D

A _____

B _____

C _____ D _____

E _____

For let C by multiplying D make E

therefore A, E, B are continuously proportional in the ratio of C to D [VIII 11]

And since A, E, B are continuously proportional and A measures B ,

therefore A also measures E

[VIII 7]

And as A is to E , so is C to D ,
therefore also C measures D [VII Def 20]

Again, let C measure D ,
I say that A also measures B

For, with the same construction we can in a similar manner prove that A , E B are continuously proportional in the ratio of C to D

And since as C is to D , so is A to E ,
and C measures D
therefore A also measures E [VII Def 20]

And A , E B are continuously proportional,
therefore A also measures B

Therefore etc Q E D

PROPOSITION 15

If a cube number measure a cube number the side will also measure the side and, if the side measure the side the cube will also measure the cube

For let the cube number A measure the cube B ,
and let C be the side of A and D of B ,

I say that C measures D

For let C by multiplying itself make E
and let D by multiplying itself make G
further let C by multiplying D make F
and let C , D by multiplying F make H , K respectively

Now it is manifest that E , F
 G and A H K , B are continuously proportional in the ratio of C to D [VIII 11 12]

And since A , H K , B are continuously proportional,
and A measures B ,
therefore it also measures H

[VIII 7]

And as A is to H so is C to D
therefore C also measures D [VII Def 20]

Next let C measure D ,
I say that A will also measure B

For with the same construction, we can prove in a similar manner that A , H , K , B are continuously proportional in the ratio of C to D

And since C measures D
and as C is to D so is A to H ,
therefore A also measures H [VII Def 20]
so that A measures B also Q E D

PROPOSITION 16

If a square number do not measure a square number, neither will the side measure the side and if the side do not measure the side, neither will the square measure the square

Let A , B be square numbers and let C D be their sides and let A not measure B ,

A _____ I say that neither does C measure D
 B _____ For if C measures D A will also measure B [VIII 14]
 C _____ But A does not measure B
 D _____ therefore neither will C measure D
 Again let C not measure D ,
 I say that neither will A measure B
 For if A measures B C will also measure D [VIII 14]
 But C does not measure D
 therefore neither will A measure B Q E D

PROPOSITION 17

If a cube number do not measure a cube number neither will the side measure the side and if the side do not measure the side, neither will the cube measure the cube

For let the cube number A not measure the cube number B ,

and let C be the side of A and D of B ,

A _____ I say that C will not measure D
 B _____ For if C measures D , A will also measure B [VIII 15]
 C _____ But A does not measure B ,
 D _____ therefore neither does C measure D

Again let C not measure D ,

I say that neither will A measure B

For if A measures B C will also measure D [VIII 15]

But C does not measure D

therefore neither will A measure B

Q E D

PROPOSITION 18

Between two similar plane numbers there is one mean proportional number and the plane number has to the plane number the ratio duplicate of that which the corresponding side has to the corresponding side

Let A , B be two similar plane numbers and let the numbers C , D be the sides of A and E , F of B

| | |
|---------|---------|
| A _____ | C _____ |
| B _____ | D _____ |
| E _____ | F _____ |

Now since similar plane numbers are those which have their sides proportional [VII Def 21]

therefore as C is to D so is E to F

I say then that between A , B there is one mean proportional number and A has to B the ratio duplicate of that which C has to E or D to F , that is of that which the corresponding side has to the corresponding side

Now since as C is to D so is E to F

therefore alternately as C is to E so is D to F

[VII 13]

And since A is plane and C , D are its sides

therefore D by multiplying C has made A

For the same reason also

E by multiplying *F* has made *B*

Now let *D* by multiplying *L* make *G*

Then since *D* by multiplying *C* has made *A*, and by multiplying *E* has made *G*

therefore, as *C* is to *E*, so is *A* to *G* [vii 17]

But, as *C* is to *E*, so is *D* to *F*,

therefore also as *D* is to *F*, so is *A* to *G*

Again since *E* by multiplying *D* has made *G*, and by multiplying *F* has made *B*,

therefore as *D* is to *F* so is *G* to *B* [vii 17]

But it was also proved that

as *D* is to *F*, so is *A* to *G*,

therefore also as *A* is to *G* so is *G* to *B*

Therefore *A*, *G*, *B* are in continued proportion

Therefore between *A*, *B* there is one mean proportional number

I say next that *A* also has to *B* the ratio duplicate of that which the corresponding side has to the corresponding side that is of that which *C* has to *E* or *D* to *F*

For since *A*, *G*, *B* are in continued proportion

A has to *B* the ratio duplicate of that which it has to *G* [v Def 9]

And as *A* is to *G* so is *C* to *E* and so is *D* to *F*

Therefore *A* also has to *B* the ratio duplicate of that which *C* has to *E* or *D* to *F* Q E D

PROPOSITION 19

Between two similar solid numbers there fall two mean proportional numbers and the solid number has to the similar solid number the ratio triplicate of that which the corresponding side has to the corresponding side

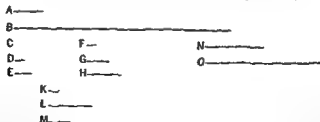
Let *A*, *B* be two similar solid numbers and let *C*, *D*, *E* be the sides of *A* and *F*, *G*, *H* of *B*

Now since similar solid numbers are those which have their sides proportional [vii Def 21]

therefore as *C* is to *D* so is *F* to *G*

and as *D* is to *E* so is *G* to *H*

I say that between *A*, *B* there fall two mean proportional numbers, and *A* has to *B* the ratio triplicate of that which *C* has to *F*, *D* to *G*, and also *E* to *H*



For let *C* by multiplying *D* make *K* and let *F* by multiplying *G* make *L*

Now since *C*, *D* are in the same ratio with *F*, *G*

and *K* is the product of *C*, *D*, and *L* the product of *F*, *G*, *A*, *L* are similar plane numbers, [vii Def 21]

therefore between A, L there is one mean proportional number [VIII 18]

Let it be M

Therefore M is the product of D, F , as was proved in the theorem preceding this [VIII 18]

Now, since D by multiplying C has made K , and by multiplying F has made M ,

therefore as C is to F so is K to M [VII 17]

But, as K is to M so is M to L

Therefore K, M, L are continuously proportional in the ratio of C to F

And since as C is to D so is F to G

alternately therefore, as C is to F , so is D to G [VII 13]

For the same reason also

as D is to G so is E to H

Therefore K, M, L are continuously proportional in the ratio of C to F in the ratio of D to G and also in the ratio of E to H

Next let E, H by multiplying M make N, O respectively

Now, since A is a solid number and C, D, E are its sides

therefore E by multiplying the product of C, D has made A

But the product of C, D is K ,

therefore E by multiplying K has made A

For the same reason also

H by multiplying L has made B

Now since E by multiplying K has made A and further also by multiplying M has made N ,

therefore as K is to M , so is A to N [VII 17]

But, as K is to M so is C to F, D to G and also E to H ,

therefore also as C is to F, D to G , and E to H , so is A to N

Again, since E, H by multiplying M have made N, O respectively

therefore as E is to H , so is N to O [VII 18]

But, as E is to H so is C to F and D to G ,

therefore also, as C is to F, D to G and E to H so is A to N and N to O

Again since H by multiplying M has made O and further also by multiplying L has made B ,

therefore as M is to L so is O to B [VII 17]

But as M is to L so is C to F, D to G and E to H

Therefore also as C is to F, D to G and E to H so not only is O to B but also A to N and N to O

Therefore A, N, O, B are continuously proportional in the aforesaid ratios of the sides

I say that A also has to B the ratio triplicate of that which the corresponding side has to the corresponding side that is of the ratio which the number C has to F or D to G and also L to H

For since A, N, O, B are four numbers in continued proportion therefore A has to B the ratio triplicate of that which A has to N [v Def 10]

But as A is to N so it was proved that C is to F, D to G and also E to H

Therefore A also has to B the ratio triplicate of that which the corresponding side has to the corresponding side that is of the ratio which the number C has to F, D to G and also E to H

PROPOSITION 20

If one mean proportional number fall between two numbers the numbers will be similar plane numbers

For let one mean proportional number C fall between the two numbers A, B ,

I say that A, B are similar plane numbers

Let D, E , the least numbers of those which have the same ratio with A, C be taken, [VII 33]
therefore D measures A the same number of times that E measures C [VII 20]

Now, as many times as D measures A so many units let there be in F ,
therefore F by multiplying D has made A
so that A is plane and D, F are its sides

Again since D, E are the least of the numbers which have the same ratio with C, B

therefore D measures C the same number of times that E measures B [VII 20]



As many times then as E measures B so many units let there be in G

therefore E measures B according to the units in G ,

therefore G by multiplying E has made B

Therefore B is plane and E, G are its sides

Therefore A, B are plane numbers

I say next that they are also similar

For since F by multiplying D has made A and by multiplying E has made C

therefore as D is to E so is A to C that is C to B [VII 17]

Again since E by multiplying F, G has made C, B respectively,

therefore as F is to G , so is C to B [VII 17]

But as C is to B so is D to E

therefore also as D is to E so is F to G

And alternately as D is to F so is E to G [VII 13]

Therefore A, B are similar plane numbers for their sides are proportional
Q E D

PROPOSITION 21

If two mean proportional numbers fall between two numbers the numbers are similar solid numbers

For let two mean proportional numbers C, D fall between the two numbers A, B ,

I say that A, B are similar solid numbers

For let three numbers E, F, G the least of those which have the same ratio with A, C, D be taken [VII 33 or VIII 2]

therefore the extremes of them E, G are prime to one another [VIII 3]

Now, since one mean proportional number F has fallen between E, G ,
therefore E, G are similar plane numbers [VIII 20]

Let then H, K be the sides of E and L, M of G

Therefore it is manifest from the theorem before this that E, F, G are continuously proportional in the ratio of H to L and that of K to M



Now since E, F, G are the least of the numbers which have the same ratio with A, C, D ,
and the multitude of the numbers E, F, G is equal to the multitude of the numbers A, C, D ,

therefore *ex aequali* as E is to G so is A to D [VII 14]

But E, G are prime

primes are also least [VII 21]

and the least measure those which have the same ratio with them the same number of times the greater the greater and the less the less that is, the antecedent the antecedent and the consequent the consequent [VII 20]

therefore E measures A the same number of times that G measures D

Now, as many times as E measures A , so many units let there be in N

Therefore N by multiplying E has made A

But E is the product of H, K

therefore N by multiplying the product of H, K has made A

Therefore A is solid and H, K, N are its sides

Again since E, F, G are the least of the numbers which have the same ratio as C, D, B

therefore E measures C the same number of times that G measures B

Now as many times as E measures C so many units let there be in O

Therefore G measures B according to the units in O

therefore O by multiplying G has made B

But G is the product of L, M

therefore O by multiplying the product of L, M has made B

Therefore B is solid and L, M, O are its sides

therefore A, B are solid

I say that they are also similar

For since N, O by multiplying E have made A, C

therefore as N is to O so is A to C that is E to F [VII 18]

But as E is to F so is H to L and K to M

therefore also as H is to L so is K to M and N to O

And H, K, N are the sides of A and O, L, M the sides of B

Therefore A, B are similar solid numbers

Q E D

PROPOSITION 22

If three numbers be in continued proportion, and the first be square, the third will also be square

Let A, B, C be three numbers in continued proportion, and let A the first be square

I say that C the third is also square

For since between A, C there is one mean proportional number, B

therefore A, C are similar plane numbers

[VIII 20]

But A is square,

therefore C is also square

Q E D

PROPOSITION 23

If four numbers be in continued proportion and the first be cube the fourth will also be cube

Let A, B, C, D be four numbers in continued proportion and let A be cube,

I say that D is also cube

For, since between A, D there are two mean proportional numbers B, C

therefore A, D are similar solid numbers

[VIII 21]

But A is cube

therefore D is also cube

Q E D

PROPOSITION 24

If two numbers have to one another the ratio which a square number has to a square number and the first be square the second will also be square

For let the two numbers A, B have to one another the ratio which the square number C has to the square number D and let A be square

I say that B is also square

For since C, D are square

C, D are similar plane numbers

Therefore one mean proportional number falls between C, D

[VIII 18]

And as C is to D , so is A to B

therefore one mean proportional number falls between A, B also

[VIII 8]

And A is square

therefore B is also square

[VIII 27]
Q E D

PROPOSITION 25

If two numbers have to one another the ratio which a cube number has to a cube number and the first be cube the second will also be cube

For let the two numbers A, B have to one another the ratio which the cube number C has to the cube number D and let A be cube

I say that B is also cube

For since C, D are cube

C, D are similar solid numbers

Therefore two mean proportional numbers fall between C, D [VIII 19]

And, as many numbers as fall between C, D in continued proportion, so many will also fall between those which have the same ratio with them, [VIII 8] so that two mean proportional numbers fall between A, B also

A ———
B ———
C ———
D ———

E ———
F ———

Let E, F so fall

Since, then, the four numbers A, E, F, B are in continued proportion and A is cube

therefore B is also cube

[VIII 23]

Q E D

PROPOSITION 26

Similar plane numbers have to one another the ratio which a square number has to a square number

Let A, B be similar plane numbers,

I say that A has to B the ratio which a square number has to a square number

A ——— B ———
C ———
D ——— E ——— F ———

For since A, B are similar plane numbers

therefore one mean proportional number falls between A, B [VIII 18]

Let it so fall and let it be C ,

and let D, E, F , the least numbers of those which have the same ratio with A, B , be taken [VII 33 or VIII 2]

therefore the extremes of them D, F are square [VIII 11 Por]

And since as D is to F , so is A to B ,

and D, F are square

therefore A has to B the ratio which a square number has to a square number

Q E D

PROPOSITION 27

Similar solid numbers have to one another the ratio which a cube number has to a cube number

Let A, B be similar solid numbers

I say that A has to B the ratio which a cube number has to a cube number

A ——— C ———
B ——— D ———
E ——— F ——— G ——— H ———

For since A, B are similar solid numbers

therefore two mean proportional numbers fall between A, B

[VIII 19]

Let C D so fall
 and let E F G H the least numbers of those which have the same ratio with
 A , C , D , B , and equal with them in multitude be taken, [VII 33 or VIII 2]
 therefore the extremes of them E , H are cube [VIII 2, Por]

And as E is to H , so is A to B ,
 therefore A also has to B the ratio which a cube number has to a cube number

—

— Q E D

BOOK NINE

PROPOSITION 1

If two similar plane numbers by multiplying one another make some number, the product will be square

Let A B be two similar plane numbers and let A by multiplying B make C ,
I say that C is square

A _____ For let A by multiplying itself make D

B _____ Therefore D is square

C _____ Since then A by multiplying itself has made

D _____ D and by multiplying B has made C

therefore as A is to B , so is D to C (VII 17)

And since A B are similar plane numbers

therefore one mean proportional number falls between A B (VIII 18)

But if numbers fall between two numbers in continued proportion as many as fall between them so many also fall between those which have the same ratio, (VIII 8)

so that one mean proportional number falls between D C also

And D is square

therefore C is also square (VIII 22)

Q E D

PROPOSITION 2

If two numbers by multiplying one another make a square number they are similar plane numbers

Let A , B be two numbers and let A by multiplying B make the square number C

A _____ I say that A B are similar plane numbers

B _____ For let A by multiplying itself make D ,

C _____ therefore D is square

D _____ Now since A by multiplying itself has made D and by multiplying B has made C

therefore as A is to B , so is D to C (VII 17)

And since D is square and C is so also

therefore D C are similar plane numbers

Therefore one mean proportional number falls between D C (VIII 18)

And as D is to C so is A to B

therefore one mean proportional number falls between A B also (VIII 8)

But if one mean proportional number fall between two numbers they are similar plane numbers (VIII 20)

therefore A B are similar plane numbers

Q E D

PROPOSITION 3

If a cube number by multiplying itself make some number, the product will be cube

For let the cube number A by multiplying itself make B ,

I say that B is cube

For let C , the side of A , be taken and let C by multiplying itself make D

It is then manifest that C by multiplying D has made A

Now since C by multiplying itself has made D ,

therefore C measures D according to the units in itself

But further the unit also measures C according to the units in it

A —
 B —————
 C — D —

therefore as the unit is to C so is C to D [VII Def 20]

Again, since C by multiplying D has made A ,

therefore D measures A according to the units in C

But the unit also measures C according to the units in it,

therefore as the unit is to C so is D to A

But as the unit is to C so is C to D ,

therefore also as the unit is to C so is C to D and D to A

Therefore between the unit and the number A two mean proportional numbers C , D have fallen in continued proportion

Again since 1 by multiplying itself has made B

therefore A measures B according to the units in itself

But the unit also measures A according to the units in it,

therefore as the unit is to A , so is A to B [VII Def 20]

But between the unit and A two mean proportional numbers have fallen, therefore two mean proportional numbers will also fall between A B [VIII 8]

But if two mean proportional numbers fall between two numbers and the first be cube the second will also be cube [VIII 23]

And A is cube

therefore B is also cube

Q E D

PROPOSITION 4

If a cube number by multiplying a cube number make some number, the product will be cube

For let the cube number A by multiplying the cube number B make C ,

I say that C is cube

For let A by multiplying itself make D ,

therefore D is cube [IX 3]

And since A by multiplying itself has made D and by multiplying B has made C

A —————
 B —————
 C —————
 D —————

therefore as A is to B so is D to C

[VII 17]

And, since A B are cube numbers

A B are similar solid numbers

Therefore two mean proportional numbers fall between A B [VIII 19]
 so that two mean proportional numbers will fall between D C also [VIII 8]

And D is cube

therefore C is also cube

[VIII 23]
 Q E D

PROPOSITION 5

If a cube number by multiplying any number make a cube number, the multiplied number will also be cube

For let the cube number A by multiplying any number B make the cube number C ,

I say that B is cube

A _____ For let A by multiplying itself make D ,
 B _____ therefore D is cube [ix 3]

C _____ Now since A by multiplying itself has
 D _____ made D and by multiplying B has made
 C ,

therefore as A is to B so is D to C [VII 17]

And since D , C are cube,

they are similar solid numbers

Therefore two mean proportional numbers fall between D , C [VIII 19]

And, as D is to C , so is A to B ,

therefore two mean proportional numbers fall between A , B also [VIII 8]

And A is cube,

therefore B is also cube [VIII 23]

PROPOSITION 6

If a number by multiplying itself make a cube number, it will itself also be cube

For let the number A by multiplying itself make the cube number B ,

I say that A is also cube

A _____ For let A by multiplying B make C

B _____ Since then, A by multiplying itself has made B , and
 C _____ by multiplying B has made C

therefore C is cube

And since A by multiplying itself has made B ,

therefore A measures B according to the units in itself

But the unit also measures A according to the units in it

Therefore as the unit is to A , so is A to B [VII Def 20]

And since A by multiplying B has made C ,

therefore B measures C according to the units in A

But the unit also measures A according to the units in it

Therefore as the unit is to A , so is B to C [VII Def 20]

But as the unit is to A so is A to B ,

therefore also as A is to B , so is B to C

And since B , C are cube

they are similar solid numbers

Therefore there are two mean proportional numbers between B , C [VIII 19]

And as B is to C , so is A to B

Therefore there are two mean proportional numbers between A , B also [VIII 8]

And B is cube

therefore A is also cube

[cf VIII 23]

Q E D

PROPOSITION 7

If a composite number by multiplying any number make some number, the product will be solid

For, let the composite number A by multiplying any number B make C ,

I say that C is solid

For since A is composite, it will be measured by some number

[VII Def 13]

Let it be measured by D
and as many times as D measures A ,
so many units let there be in E

A _____
 B _____
 C _____
 D _____ E _____

Since then D measures A according to the units in E ,
therefore E by multiplying D has made A [VII Def 13]

And since A by multiplying B has made C ,
and A is the product of D E ,

therefore the product of D E by multiplying B has made C

Therefore C is solid and D , E , B are its sides Q E D

PROPOSITION 8

If as many numbers as we please beginning from an unit be in continued proportion the third from the unit will be square as will also those which successively leave out one the fourth will be cube as will also all those which leave out two and the sixth will be at once cube and square as will also those which leave out five

Let there be as many numbers as we please, A B , C D , E , F , beginning from an unit and in continued proportion

I say that B the third from the unit is square as are also all those which leave out one C the fourth is cube as are also all those which leave out two and F , the seventh is at once cube and square as are also all those which leave out five

A _____
 B _____
 C _____
 D _____
 E _____
 F _____

For since as the unit is to A , so is A to B ,
therefore the unit measures the number A the same number of times that A measures B [VII Def 20]

But the unit measures the number A according to the units in it,

therefore A also measures B according to the units in A

Therefore A by multiplying itself has made B ,

therefore B is square

And since B C D are in continued proportion and B is square

therefore D is also square [VIII 22]

For the same reason

F is also square

Similarly we can prove that all those which leave out one are square

I say next that C the fourth from the unit is cube as are also all those which leave out two

For since as the unit is to A so is B to C

therefore the unit measures the number A the same number of times that B measures C

But the unit measures the number 1 according to the units in 1,

therefore B also measures C according to the units in A

Therefore A by multiplying B has made C

Since then A by multiplying itself has made B and by multiplying B has made C ,

therefore C is cube

And since C, D, E, F are in continued proportion and C is cube,

therefore F is also cube [VIII 23]

But it was also proved square,

therefore the seventh from the unit is both cube and square

Similarly we can prove that all the numbers which leave out five are also both cube and square Q E D

PROPOSITION 9

If as many numbers as we please beginning from an unit be in continued proportion and the number after the unit be square, all the rest will also be square And if the number after the unit be cube, all the rest will also be cube

Let there be as many numbers as we please, A, B, C, D, E, F , beginning from an unit and in continued proportion and let A , the number after the unit be square,

A _____

B _____

C _____

D _____

E _____

F _____

I say that all the rest will also be square

Now it has been proved that B , the third from the unit is square as are also all those which leave out one

[IX 8]

I say that all the rest are also square

For since A, B, C are in continued proportion,

and A is square

therefore C is also square [VIII 22]

Again since B, C, D are in continued proportion

and B is square

D is also square [VIII 22]

Similarly we can prove that all the rest are also square

Next, let A be cube

I say that all the rest are also cube

Now it has been proved that C , the fourth from the unit is cube as also are all those which leave out two, [IX 8]

I say that all the rest are also cube

For, since as the unit is to A so is A to B ,

therefore the unit measures A the same number of times as A measures B

But the unit measures A according to the units in it

therefore A also measures B according to the units in itself,

therefore A by multiplying itself has made B

And A is cube

But if a cube number by multiplying itself make some number the product is cube [IX 3]

Therefore B is also cube

And since the four numbers A, B, C, D are in continued proportion

and A is cube

D also is cube

[VIII 23]

For the same reason

E is also cube and similarly all the rest are cube Q E D

PROPOSITION 10

If as many numbers as we please beginning from an unit be in continued proportion and the number after the unit be not square neither will any other be square except the third from the unit and all those which leave out one And if the number after the unit be not cube neither will any other be cube except the fourth from the unit and all those which leave out two

Let there be as many numbers as we please $A B C, D E F$, beginning from an unit and in continued proportion

and let A the number after the unit not be square,

I say that neither will any other be square except the third from the unit <and those which leave out one>

For if possible let C be square

But B is also square

[IX 8]

{therefore B, C have to one another the ratio which a square number has to a square number}

And as B is to C so is A to B

therefore A, B have to one another the ratio which a square number has to a square number

{so that A, B are similar plane numbers}

[VIII 26 converse]

And B is square,

therefore A is also square

which is contrary to the hypothesis.

Therefore C is not square

Similarly we can prove that neither is any other of the numbers square except the third from the unit and those which leave out one

Next let A not be cube

I say that neither will any other be cube except the fourth from the unit and those which leave out two

For if possible let D be cube

Now C is also cube for it is fourth from the unit

[IX 8]

And as C is to D so is B to C

therefore B also has to C the ratio which a cube has to a cube

And C is cube

therefore B is also cube

[VIII 23]

And since as the unit is to A so is A to B

and the unit measures A according to the units in it

therefore A also measures B according to the units in itself

therefore A by multiplying itself has made the cube number B

But if a number by multiplying itself make a cube number it is also itself cube

[IX 6]

Therefore A is also cube

which is contrary to the hypothesis

Therefore D is not cube

Similarly we can prove that neither is any other of the numbers cube except the fourth from the unit and those which leave out two Q E D

PROPOSITION 11

If as many numbers as we please beginning from an unit be in continued proportion the less measures the greater according to some one of the numbers which have place among the proportional numbers

Let there be as many numbers as we please, $B C D E$ beginning from the unit A and in continued proportion

A ————— I say that B , the least of the numbers $B C D E$
measures E according to some one of the numbers $C D$

B ————— For since as the unit A is to B , so is D to E
 C ————— therefore the unit A measures the number B the same
 D ————— number of times as D measures E ,

E ————— therefore alternately, the unit A measures D the same
number of times as B measures E [VII 15]

But the unit A measures D according to the units in it,

therefore B also measures E according to the units in D

so that B the less measures E the greater according to some number of those which have place among the proportional numbers —

PORISM And it is manifest that whatever place the measuring number has reckoned from the unit the same place also has the number according to which it measures reckoned from the number measured in the direction of the number before it —

Q E D

PROPOSITION 12

If as many numbers as we please beginning from an unit be in continued proportion by however many prime numbers the last is measured the next to the unit will also be measured by the same

Let there be as many numbers as we please $A B C D$ beginning from an unit and in continued proportion,

I say that by however many prime numbers D is measured A will also be measured by the same

For let D be measured by any prime number E

| | | |
|-----------|-----------|---------------------------------|
| A ————— | F ————— | I say that E measures A |
| B ————— | G ————— | For suppose it does not |
| C ————— | H ————— | now E is prime and any prime |
| D ————— | | number is prime to any which it |
| E ————— | | does not measure [VII 29] |

therefore $E A$ are prime to one another

And since E measures D let it measure it according to F

therefore E by multiplying F has made D

Again since 1 measures D according to the units in C [IX 11 and Por]

therefore A by multiplying C has made D

But further E has also by multiplying F made D

therefore the product of $A C$ is equal to the product of $E F$

Therefore as A is to E so is F to C [VII 19]

But A, E are prime

primes are also least [VII 21]

and the least measure those which have the same ratio the same number of times the antecedent the antecedent and the consequent the consequent

[VII 20]

therefore E measures C

Let it measure it according to G ,

therefore E by multiplying G has made C

But, further, by the theorem before this

A has also by multiplying B made C [IX 11 and Por]

Therefore the product of A, B is equal to the product of E, G

Therefore as $A \equiv$ to E , so is G to B [VII 19]

But A, E are prime,

primes are also least, [VII 21]

and the least numbers measure those which have the same ratio with them the same number of times, the antecedent the antecedent and the consequent the consequent [VII 20]

therefore E measures B

Let it measure it according to H ,

therefore E by multiplying H has made B

But further, A has also by multiplying itself made B , [IX 8]

therefore the product of E, H is equal to the square on A

Therefore as E is to A , so is A to H [VII 19]

But A, E are prime,

primes are also least [VII 21]

and the least measure those which have the same ratio the same number of times the antecedent the antecedent and the consequent the consequent, [VII 20]

therefore E measures A as antecedent antecedent

But again it also does not measure it

which is impossible

Therefore E, A are not prime to one another

Therefore they are composite to one another

But numbers composite to one another are measured by some number [VII Def 14]

And since $E \equiv$ by hypothesis prime

and the prime is not measured by any number other than itself,

therefore E measures A, E ,

so that E measures A

[But it also measures D

therefore E measures A, D]

Similarly we can prove that by however many prime numbers D is measured A will also be measured by the same $Q \ E \ D$

PROPOSITION 13

If as many numbers as we please beginning from an unit be in continued proportion and the number after the unit be prime the greatest will not be measured by any except those which have a place among the proportional numbers

Let there be as many numbers as we please A, B, C, D beginning from an unit and in continued proportion and let A the number after the unit be prime

I say that D the greatest of them will not be measured by any other number except A, B, C

For, if possible let it be measured by E and let E not be the same with any of the numbers A, B, C

It is then manifest that E is not prime

For, if E is prime and measures D

it will also measure A [ix 12], which is prime, though it is not the same with it

| | | |
|---------|---------|----------------------------|
| A ————— | E ————— | which is impossible |
| B ————— | F ————— | Therefore E is not prime |
| C ————— | G ————— | Therefore it is composite |
| D ————— | H ————— | But any composite number |
| | | is measured by some |
| | | prime number, [VII 31] |

therefore E is measured by some prime number

I say next that it will not be measured by any other prime except A

For, if E is measured by another

and E measures D

that other will also measure D

so that it will also measure A [ix 12] which is prime though it is not the same with it

which is impossible

Therefore A measures E

And since E measures D , let it measure it according to F

I say that F is not the same with any of the numbers A, B, C

For if F is the same with one of the numbers A, B, C

and measures D according to E

therefore one of the numbers A, B, C also measures D according to E

But one of the numbers A, B, C measures D according to some one of the numbers A, B, C , [ix 11]

therefore E is also the same with one of the numbers A, B, C

which is contrary to the hypothesis

Therefore F is not the same as any one of the numbers A, B, C

Similarly we can prove that F is measured by A , by proving again that F is not prime

For if it is and measures D

it will also measure A [ix 12] which is prime though it is not the same with it

which is impossible

therefore F is not prime

Therefore it is composite

But any composite number is measured by some prime number [VII 31]

therefore F is measured by some prime number

I say next that it will not be measured by any other prime except A

For if any other prime number measures F

and F measures D

that other will also measure D

so that it will also measure A [ix 12] which is prime though it is not the same with it

which is impossible

Therefore A measures F

And since E measures D according to F

therefore E by multiplying F has made D

But further A has also by multiplying C made D

therefore the product of A, C is equal to the product of E, F [ix 11]

Therefore proportionally, as A is to E , so is F to C

[VII 19]

But A measures E ,

therefore F also measures C

Let it measure it according to G

Similarly then we can prove that G is not the same with any of the numbers

A , B , and that it is measured by A

And since F measures C according to G

therefore F by multiplying G has made C

But further A has also by multiplying B made C

[IX 11]

therefore the product of A , B is equal to the product of F , G

Therefore proportionally, as A is to F so is G to B

[VII 19]

But A measures F

therefore G also measures B

Let it measure it according to H

Similarly then we can prove that H is not the same with A

And since G measures B according to H

therefore G by multiplying H has made B

But further A has also by multiplying itself made B

[IX 8]

therefore the product of H , G is equal to the square on A

Therefore as H is to A , so is A to G

[VII 19]

But A measures G

therefore H also measures A which is prime though it is not the same with it
which is absurd

Therefore D the greatest will not be measured by any other number except
 A , B , C Q E D

PROPOSITION 14

If a number be the least that is measured by prime numbers it will not be measured by any other prime number except those originally measuring it

For let the number 1 be the least that is measured by the prime numbers
 B , C , D

I say that A will not be measured by any other prime number except B , C , D

For if possible let it be measured by the prime number E and let E not

be the same with any one of the numbers B , C , D

Now since E measures A let it measure it according to F ,

therefore E by multiplying F has made A

And A is measured by the prime numbers B , C , D

But if two numbers by multiplying one another make some number and any prime number measure the product it will also measure one of the original numbers

[VII 30]

therefore B , C , D will measure one of the numbers E , F

Now they will not measure E

for E is prime and not the same with any one of the numbers B , C , D

Therefore they will measure F which is less than A

which is impossible for A is by hypothesis the least number measured by B , C , D

Therefore no prime number will measure A except B , C , D Q E D

PROPOSITION 15

If three numbers in continued proportion be the least of those which have the same ratio with them any two whatever added together will be prime to the remaining number

Let A, B, C , three numbers in continued proportion be the least of those which have the same ratio with them

A ————— B —————
 C —————
 D — E — F

I say that any two of the numbers A, B, C whatever added together are prime to the remaining number namely A, B to C, B, C to A , and further, A, C to B

For let two numbers DE, EF , the least of those which have the same ratio with A, B, C , be taken [VIII 2]

It is then manifest that DE by multiplying itself has made A , and by multiplying EF has made B , and further EF by multiplying itself has made C [VIII 2]

Now since DE, EF are least, they are prime to one another [VI 22]

But, if two numbers be prime to one another their sum is also prime to each [VII 28]

therefore DF is also prime to each of the numbers DE, EF

But further, DE is also prime to EF , therefore DF, DE are prime to EF

But, if two numbers be prime to any number their product is also prime to the other [VI 24]

so that the product of FD, DE is prime to EF

hence the product of FD, DE is also prime to the square on EF [VI 25]

But the product of FD, DE is the square on DE together with the product of DE, EF [II 3]

therefore the square on DE together with the product of DE, EF is prime to the square on EF

And the square on DE is A

the product of DE, EF is B ,

and the square on EF is C ,

therefore A, B added together are prime to C

Similarly we can prove that B, C added together are prime to A

I say next that A, C added together are also prime to B

For since DF is prime to each of the numbers DE, EF

the square on DF is also prime to the product of DE, EF [VII 24 25]

But the squares on DE, EF together with twice the product of DE, EF are equal to the square on DF [II 4]

therefore the squares on DE, EF together with twice the product of DE, EF are prime to the product of DE, EF

Separando the squares on DE, EF together with once the product of DE, EF are prime to the product of DE, EF

Therefore *separando* again the squares on DE, EF are prime to the product of DE, EF

And the square on DE is A

the product of DE, EF is B

and the square on EF is C

Therefore A, C added together are prime to B

Q E D

PROPOSITION 16

If two numbers be prime to one another, the second will not be to any other number as the first is to the second

For let the two numbers A, B be prime to one another

I say that B is not to any other number as A is to B

For if possible as A is to B , so let B be to C

Now A, B are prime

primes are also least

[VII 21]

and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent,

therefore A measures B as antecedent antecedent

But it also measures itself

therefore A measures A, B which are prime to one another
which is absurd

Therefore B will not be to C , as A is to B

Q E D

PROPOSITION 17

If there be as many numbers as we please in continued proportion and the extremes of them be prime to one another, the last will not be to any other number as the first to the second

For let there be as many numbers as we please A, B, C, D in continued proportion

and let the extremes of them, A, D , be prime to one another

I say that D is not to any other number as A is to B

A — B —

C —

D —

E —

For if possible as A is to B so let D be to E

therefore alternately as A is to D so is B to E

[VII 13]

But A, D are prime

primes are also least

[VII 21]

and the least numbers measure those which have the same ratio the same number of times the antecedent the antecedent and the consequent the consequent

[VII 20]

Therefore A measures B

And as A is to B so is B to C

Therefore B also measures C

so that A also measures C

And since as B is to C so is C to D

and B measures C

therefore C also measures D

But A measured C

so that A also measures D

But it also measures itself,

therefore A measures A, D which are prime to one another
which is impossible

Therefore D will not be to any other number as A is to B

Q E D

PROPOSITION 18

Given two numbers, to investigate whether it is possible to find a third proportional to them

Let A, B be the given two numbers, and let it be required to investigate whether it is possible to find a third proportional to them

Now A, B are either prime to one another or not

And if they are prime to one another, it has been proved that it is impossible to find a third proportional to them [IX 16]

Next let A, B not be prime to one another,

and let B by multiplying itself make C

Then A either measures C or does not measure it

First let it measure it according to D ,

therefore A by multiplying D has made C

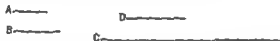
But further, B has also by multiplying itself made C

therefore the product of A, D is equal to the square on B

Therefore as A is to B so is B to D [VII 19]

therefore a third proportional number D has been found to A, B

Next let A not measure C ,



I say that it is impossible to find a third proportional number to A, B

For if possible let D , such third proportional have been found

Therefore the product of A, D is equal to the square on B

But the square on B is C

therefore the product of A, D is equal to C

Hence A by multiplying D has made C

therefore A measures C according to D

But by hypothesis it also does not measure it

which is absurd

Therefore it is not possible to find a third proportional number to A, B when A does not measure C Q E D

PROPOSITION 19

Given three numbers to investigate when it is possible to find a fourth proportional to them

A _____

B _____

C _____

Let A, B, C be the given three numbers and let it be required to investigate when it is possible to find a fourth proportional to them

[The Greek text of this proposition is corrupt. However analogously to Proposition 18 the condition that a fourth proportional to A, B, C exists is that A measure the product of B and C]

PROPOSITION 20

Prime numbers are more than any assigned multitude of prime numbers

Let A, B, C be the assigned prime numbers

I say that there are more prime numbers than A, B, C

For let the least number measured by $A B C$ be taken,
and let it be DE

let the unit DF be added to DE

Then EF is either prime or not

First let it be prime

then the prime numbers $A B, C EF$ have
been found which are more than $A B C$

Next, let EF not be prime

therefore it is measured by some prime number

Let it be measured by the prime number G

I say that G is not the same with any of the numbers A, B, C

For if possible let it be so

Now $A B C$ measure DE

therefore G also will measure DE

But it also measures EF

Therefore G being a number will measure the remainder the unit DF
which is absurd

Therefore G is not the same with any one of the numbers $A B, C$

And by hypothesis it is prime

Therefore the prime numbers $A B C G$ have been found which are more
than the assigned multitude of $A B C$

PROPOSITION 21

If as many even numbers as we please be added together the whole is even

For let as many even numbers as we please $AB, BC CD DE$ be added to-
gether

I say that the whole AE is even

For since each of the numbers AB

$BC CD DE$ is even it has a half part

[xi Def 6]

so that the whole AE also has a half part

But an even number is that which is divisible into two equal parts [id]

therefore AE is even

PROPOSITION 22

If as many odd numbers as we please be added together and their multitude be even,
the whole will be even

For let as many odd numbers as we please $AB BC CD DE$ even in multi-
tude be added together,

I say that the whole AE is even

For since each of the numbers

$AB BC CD DE$ is odd if an unit

be subtracted from each each of

the remainders will be even

so that the sum of them will be even

But the multitude of the units is also even

Therefore the whole AE is also even

PROPOSITION 23

If as many odd numbers as we please be added together, and their multitude be odd the whole will also be odd

For let \equiv many odd numbers as we please, AB, BC, CD , the multitude of which is odd be added together,

$A \quad \equiv \quad C \quad E \quad D$

I say that the whole AD is also odd

Let the unit DE be subtracted from CD ,

therefore the remainder $CE \equiv$ even [VII Def 7]

But $CA \equiv$ also even, [IX 22]

therefore the whole AE is also even [IX 21]

And $DE \equiv$ an unit

Therefore AD is odd [VII Def 7]

Q E D

PROPOSITION 24

If from an even number an even number be subtracted, the remainder will be even

For from the even number AB let the even number BC be subtracted

$A \quad \quad C \quad B$

I say that the remainder CA is even

For since AB is even it has a half part [VII Def 6]

For the same reason BC also has a half part

so that the remainder CA also has a half part and $AC \equiv$ therefore even

Q E D

PROPOSITION 25

If from an even number an odd number be subtracted, the remainder will be odd

For from the even number AB let the odd number BC be subtracted

$A \quad \quad C \quad D \quad B$

I say that the remainder CA is odd

For let the unit CD be subtracted from BC ,

therefore DB is even [VII Def 7]

But AB is also even,

therefore the remainder AD is also even [IX 24]

And $CD \equiv$ an unit

therefore CA is odd [VII Def 7]

Q E D

PROPOSITION 26

If from an odd number an odd number be subtracted the remainder will be even

For from the odd number AB let the odd number BC be subtracted

$A \quad \quad C \quad D \quad B$

I say that the remainder $CA \equiv$ even

For since AB is odd let the unit BD be subtracted

therefore the remainder AD is even [VII Def 7]

For the same reason CD is also even [VII Def 7]

so that the remainder CA is also even [IX 24]

Q E D

PROPOSITION 27

If from an odd number an even number be subtracted the remainder will be odd

For from the odd number AB let the even number BC be subtracted

I say that the remainder CA is odd

Let the unit AD be subtracted,

therefore DB is even [VII Def 7]

$A \quad D \quad C \quad B$

But BC is also even,

therefore the remainder CD is even

[IX 24]

Therefore CA is odd

[VII Def 7]

Q E D

PROPOSITION 28

If an odd number by multiplying an even number make some number, the product will be even

For let the odd number A by multiplying the even number B make C ,

I say that C is even

For since A by multiplying B has made C ,

therefore C is made up of as many numbers equal to B

as there are units in A

[VII Def 15]

And B is even

therefore C is made up of even numbers

But if as many even numbers as we please be added together, the whole is even

[IX 21]

Therefore C is even

Q E D

PROPOSITION 29

If an odd number by multiplying an odd number make some number, the product will be odd

For let the odd number A by multiplying the odd number B make C ,

I say that C is odd

For since A by multiplying B has made C

therefore C is made up of as many numbers equal to

B as there are units in A

[VII Def 15]

And each of the numbers $A \quad B$ is odd

therefore C is made up of odd numbers the multitude of which is odd

Thus C is odd

[IX 25]

Q E D

PROPOSITION 30

If an odd number measure an even number, it will also measure the half of it

For let the odd number A measure the even number B ,

I say that it will also measure the half of it

For since 1 measures B

let it measure it according to C

I say that C is not odd

For if possible let it be so

Then since A measures B according to C

therefore A by multiplying C has made B

Therefore B is made up of odd numbers the multitude of which is odd

Therefore B is odd

[IX 23]

which is absurd for by hypothesis it is even

Therefore C is not odd

therefore C is even

Thus A measures B an even number of times

For this reason then it also measures the half of it

Q E D

PROPOSITION 31

If an odd number be prime to any number, it will also be prime to the double of it

For let the odd number A be prime to any number B ,

and let C be double of B ,

A ———

B ———

C ———

D ———

I say that A is prime to C

For if they are not prime to one another
some number will measure them

Let a number measure them and let it be D

Now A is odd,

therefore D is also odd

And since D which is odd measures C

and C is even

therefore $\{D\}$ will measure the half of C also

[IX 30]

But B is half of C ,

therefore D measures B

But it also measures A

therefore D measures A B which are prime to one another
which is impossible

Therefore A cannot but be prime to C

Therefore A C are prime to one another

Q E D

PROPOSITION 32

Each of the numbers which are continually doubled beginning from a dyad is even times even only

For let as many numbers as we please B C D have been continually doubled beginning from the dyad A ,

A ———

B ———

C ———

D ———

I say that B C D are even times even only

Non that each of the numbers B C D is
even times even is manifest for it is
doubled from a dyad

I say that it is also even times even only

For let an unit be set out

Since then as many numbers as we please beginning from an unit are in continued proportion

and the number A after the unit is prime

therefore D the greatest of the numbers A B C D will not be measured by
any other number except A B C

[IX 13]

And each of the numbers A B C is even

therefore D is even times even only

[VII Def 8]

Similarly we can prove that each of the numbers B C is even times even
only

Q E D

PROPOSITION 33

If a number have its half odd it is even times odd only

For let the number A have its half odd

I say that A is even times odd only

Now that it is even times odd is manifest for the half of it being odd measures it an even number of times [vii Def 9]

I say next that it is also even times odd only

For if A is even times even also

it will be measured by an even number according to an even number [vii Def 8]

so that the half of it will also be measured by an even number though it is odd which is absurd

Therefore A is even times odd only Q E D

PROPOSITION 34

If a number neither be one of those which are continually doubled from a dyad, nor have its half odd it is both even-times even and even-times odd

For let the number A neither be one of those doubled from a dyad nor have its half odd

I say that A is both even times even and even times odd A

Now that A is even times even is manifest

for it has not its half odd [vii Def 8]

I say next that it is also even times odd

For if we bisect A then bisect its half and do this continually we shall come upon some odd number which will measure A according to an even number

For if not we shall come upon a dyad

and A will be among those which are doubled from a dyad

which is contrary to the hypothesis

Thus A is even times odd

But it was also proved even times even

Therefore A is both even times even and even times odd Q E D

PROPOSITION 35

If as many numbers as we please be in continued proportion and there be subtracted from the second and the last numbers equal to the first then as the excess of the second is to the first so will the excess of the last be to all those before it

Let there be as many numbers as we please in continued proportion A, BC, D, EF beginning from A as least

and let there be subtracted from BC and EF the numbers BG, FH each equal to A

I say that as GC is to A so is EH to A, BC, D

For let FA be made equal to BC and FL equal to D

Then since FA is equal to BC

and of these the part FH is equal to the part BG

therefore the remainder HK is equal to the remainder GC

And since as EF is to D so is D to BC and BC to A

while D is equal to FL BC to FA and A to FH

therefore as EF is to FL so is LF to FA and FA to FH

Separando as EL is to LF so is LH to FA , and KH to FH [vii 11 13]



Therefore also as one of the antecedents is to one of the consequents so are all the antecedents to all the consequents [VII 12]

therefore as AH is to FH so are EL, LA, KH to LF, FK, HF

But AH is equal to CG, FH to A and LF, FA, HF to $D, BC, A,$

therefore as CG is to A , so is EH to D, BC, A

Therefore as the excess of the second is to the first so is the excess of the last to all those before it: Q E D

PROPOSITION 36

If as many numbers as we please beginning from an unit be set out continuously in double proportion until the sum of all becomes prime and if the sum multiplied into the last make some number the product will be perfect

For let as many numbers as we please A, B, C, D beginning from an unit be set out in double proportion until the sum of all becomes prime

let E be equal to the sum and let E by multiplying D make $FG,$

I say that FG is perfect

For however many A, B, C, D are in multitude let so many E, HA, L, M be taken in double proportion beginning from $E,$

therefore *ex aequali* as A is to D so is E to M [VII 14]

Therefore the product of F, D is equal to the product of A, M [VII 19]

And the product of E, D is FG

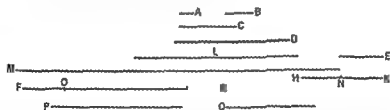
therefore the product of A, M is also FG

Therefore A by multiplying M has made FG

therefore M measures FG according to the units in A

And A is a dyad

therefore FG is double of M



But M, L, HK, E are continuously double of each other
therefore E, HA, L, M, FG are continuously proportional in double proportion

Now let there be subtracted from the second HA and the last FG the numbers HN, FO each equal to the first E

therefore as the excess of the second is to the first so is the excess of the last to all those before it [IX 35]

Therefore as NK is to E so is OG to M, L, HA, E

And NK is equal to E

therefore OG is also equal to M, L, HA, E

But FO is also equal to E

and E is equal to A, B, C, D and the unit

Therefore the whole FG is equal to E, HA, L, M and A, B, C, D and the unit

and it is measured by them

I say also that FG will not be measured by any other number except $4 B C, D, E H K, L M$ and the unit

For if possible let some number P measure FG

and let P not be the same with any of the numbers $A, B, C D, E, H K, L M$

And as many times as P measures FG so many units let there be in Q
therefore Q by multiplying P has made FG

But further, E has also by multiplying D made FG

therefore as E is to Q so is P to D [VII 19]

And since $A B C D$ are continuously proportional beginning from an unit
therefore D will not be measured by any other number except $A B C$ [IX 13]

And by hypothesis P is not the same with any of the numbers $A, B C$,
therefore P will not measure D

But as P is to D so is E to Q

therefore neither does E measure Q [VII Def 20]

And E is prime

and any prime number is prime to any number which it does not measure

[VII 29]

Therefore $E Q$ are prime to one another

But primes are also least

[VII 21]

and the least numbers measure those which have the same ratio the same number of times the antecedent the antecedent and the consequent the consequent

[VII 20]

and as E is to Q so is P to D

therefore E measures P the same number of times that Q measures D

But D is not measured by any other number except $A B, C$

therefore Q is the same with one of the numbers A, B, C

Let it be the same with B

And however many $B C D$ are in multitude let so many $E H K, L$ be taken beginning from E

Now $E H K, L$ are in the same ratio with $B C D$

therefore *ex aequali* as B is to D , so is E to L [VII 14]

Therefore the product of $B L$ is equal to the product of $D E$ [VII 19]

But the product of $D E$ is equal to the product of $Q P$

therefore the product of $Q P$ is also equal to the product of $B L$

Therefore as Q is to B so is L to P [VII 19]

And Q is the same with B

therefore L is also the same with P

which is impossible for by hypothesis P is not the same with any of the numbers set out

Therefore no number will measure FG except $4, B C, D E H K, L M$ and the unit

And FG was proved equal to $4 B C D E H K L M$ and the unit

and a perfect number is that which is equal to its own parts [VII Def 2^o]

therefore FG is perfect

$Q E D$

BOOK TEN

DEFINITIONS I

1 Those magnitudes are said to be *commensurable* which are measured by the same measure and those *incommensurable* which cannot have any common measure

2 Straight lines are *commensurable in square* when the squares on them are measured by the same area, and *incommensurable in square* when the squares on them cannot possibly have any area as a common measure

3 With these hypotheses it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively some in length only and others in square also with an assigned straight line Let then the assigned straight line be called *rational* and those straight lines which are commensurable with it whether in length and in square or in square only, *rational* but those which are incommensurable with it *irrational*

4 And let the square on the assigned straight line be called *rational* and those areas which are commensurable with it *rational* but those which are incommensurable with it *irrational* and the straight lines which produce them *irrational* that is in case the areas are squares the sides themselves but in case they are any other rectilineal figures the straight lines on which are described squares equal to them

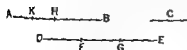
BOOK X PROPOSITIONS

PROPOSITION 1

Two unequal magnitudes being set out if from the greater there be subtracted a magnitude greater than its half and from that which is left a magnitude greater than its half and if this process be repeated continually there will be left some magnitude which will be less than the lesser magnitude set out

Let AB C be two unequal magnitudes of which AB is the greater

I say that if from AB there be subtracted a magnitude greater than its half and from that which is left a magnitude greater than its half and if this process be repeated continually there will be left some magnitude which will be less than the magnitude C



For C if multiplied will sometime be greater than AB [cf v Def 4]

Let it be multiplied and let DE be a multiple of C and greater than AB

let DE be divided into the parts DF FG GE equal to C

from AB let there be subtracted BH greater than its half

and from AH AK greater than its half

and let this process be repeated continually until the divisions in AB are equal in multitude with the divisions in DE

Let then AK KH , HB be divisions which are equal in multitude with DF , FG GE

Now since DE is greater than AB

and from DE there has been subtracted EG less than its half,

and from AB BH greater than its half

therefore the remainder GD is greater than the remainder HA

And since GD is greater than HA ,

and there has been subtracted, from GD the half GF ,

and from HA , HK greater than its half

therefore the remainder DF is greater than the remainder AK

But DF is equal to C

therefore C is also greater than AK

Therefore AK is less than C

Therefore there is left of the magnitude AB the magnitude AK which is less than the lesser magnitude set out namely C Q E D

And the theorem can be similarly proved even if the parts subtracted be halves

PROPOSITION 2

If when the less of two unequal magnitudes is continually subtracted in turn from the greater that which is left never measures the one before it the magnitudes will be incommensurable

For there being two unequal magnitudes AB CD and AB being the less when the less is continually subtracted in turn from the greater, let that which is left over never measure the one before it

I say that the magnitudes AB CD are incommensurable

For if they are commensurable some magnitude will measure them

Let a magnitude measure them if possible and let it be E ,

let AB measuring FD leave CF less than itself,

let CF measuring BG leave AG less than itself

and let this process be repeated continually until there is left some magnitude which is less than E

Suppose this done and let there be left AG less than E

Then since E measures AB

while AB measures DF

therefore E will also measure FD

But it measures the whole CD also

therefore it will also measure the remainder CF

But CF measures BG

therefore E also measures BG

But it measures the whole AB also

therefore it will also measure the remainder AG the greater the less which is impossible



Therefore no magnitude will measure the magnitudes AB , CD ,
 therefore the magnitudes AB , CD are incommensurable [x Def 1]
 Therefore etc. Q E D

PROPOSITION 3

Given two commensurable magnitudes, to find their greatest common measure

Let the two given commensurable magnitudes be AB , CD of which AB is the less

thus it is required to find the greatest common measure of AB , CD

Now the magnitude AB either measures CD or it does not

If then it measures it—and it measures itself also— AB is a common measure of AB , CD

And it is manifest that it is also the greatest,

for a greater magnitude than the magnitude AB will not measure AB

Next, let AB not measure CD

Then if the less be continually subtracted in turn from the greater, that which is left over will sometime measure the one before it, because AB , CD are not incommensurable,

[cf x 2]



let AB , measuring ED leave EC less than itself
 let EC measuring FB leave AF less than itself
 and let AF measure CE

Since, then AF measures CE ,

while CE measures FB ,

therefore AF will also measure FB

But it measures itself also,

therefore AF will also measure the whole AB

But AB measures DE

therefore AF will also measure ED

But it measures CE also,

therefore it also measures the whole CD

Therefore AF is a common measure of AB , CD

I say next that it is also the greatest

For if not there will be some magnitude greater than AF which will measure AB , CD

Let it be G

Since then G measures AB

while AB measures ED

therefore G will also measure ED

But it measures the whole CD also

therefore G will also measure the remainder CE

But CE measures FB

therefore G will also measure FB

But it measures the whole AB also

and it will therefore measure the remainder AF the greater the less which is impossible

Therefore no magnitude greater than AF will measure AB CD
 therefore AF is the greatest common measure of AB CD

Therefore the greatest common measure of the two given commensurable magnitudes AB , CD has been found Q E D

PORISM From this it is manifest that if a magnitude measure two magnitudes it will also measure their greatest common measure

PROPOSITION 4

Given three commensurable magnitudes, to find their greatest common measure

Let A B C be the three given commensurable magnitudes,
 thus it is required to find the greatest common measure of A , B , C

Let the greatest common measure of the two magnitudes A B be taken and let it be D [\propto 3]
 then D either measures C or does not measure it

A —————

B —————

C —————

D ——— E — F —

First let it measure it

Since then D measures C

while it also measures A B

therefore D is a common measure of A B C

And it is manifest that it is also the greatest
 for a greater magnitude than the magnitude D does not measure A B

Next let D not measure C

I say first that C D are commensurable

For since 1 B C are commensurable

some magnitude will measure them

and this will of course measure A B also

so that it will also measure the greatest common measure of A B namely D
[\propto 3 Por]

But it also measures C

so that the said magnitude will measure C D

therefore C D are commensurable

Now let their greatest common measure be taken and let it be E [\propto 3]

Since then E measures D

while D measures A B

therefore E will also measure A B

But it measures C also

therefore E measures A B C

therefore E is a common measure of A , B C

I say next that it is also the greatest

For if possible let there be some magnitude F greater than E and let it measure A B C

Now since F measures A B C

it will also measure A B

and will measure the greatest common measure of A , B [\propto 3 Por]

But the greatest common measure of A B is D

therefore F measures D

But it measures C also

therefore F measures C D

therefore F will also measure the greatest common measure of C D
[\propto 3 Por]

But that is E ,

therefore F will measure E , the greater the less
which is impossible

Therefore no magnitude greater than the magnitude E will measure A, B, C ,
therefore E is the greatest common measure of A, B, C if D do not measure C ,
and if it measure it D is itself the greatest common measure

Therefore the greatest common measure of the three given commensurable
magnitudes has been found

PORISM From this it is manifest that if a magnitude measure three magni-
tudes it will also measure their greatest common measure

Similarly too, with more magnitudes the greatest common measure can be
found and the porism can be extended Q E D

PROPOSITION 5

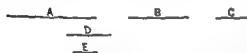
*Commensurable magnitudes have to one another the ratio which a number has to a
number*

Let A, B be commensurable magnitudes,

I say that A has to B the ratio which a number has to a number

For, since A, B are commensurable some magnitude will measure them

Let it measure them and let it be C



And as many times as C meas-
ures A , so many units let there be
in D ,

and as many times as C meas-
ures B , so many units let there
be in E

Since then C measures A according to the units in D

while the unit also measures D according to the units in it

therefore the unit measures the number D the same number of times as the
magnitude C measures A

therefore as C , is to A so is the unit to D , [VII Def 20]

therefore, inversely as A is to C , so is D to the unit [cf v 7 Por]

Again since C measures B according to the units in E

while the unit also measures E according to the units in it

therefore the unit measures E the same number of times as C measures B ,

therefore as C is to B so is the unit to E

But it was also proved that

as A is to C , so is D to the unit

therefore *ex aequali*

as A is to B so is the number D to E [v 22]

Therefore the commensurable magnitudes A, B have to one another the
ratio which the number D has to the number E Q E D

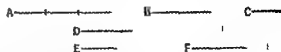
PROPOSITION 6

*If two magnitudes have to one another the ratio which a number has to a number,
the magnitudes will be commensurable*

For let the two magnitudes A, B have to one another the ratio which the
number D has to the number E

I say that the magnitudes A, B are commensurable

For let A be divided into as many equal parts as there are units in D
 and let C be equal to one of them,
 and let F be made up of as
 many magnitudes equal to C
 as there are units in E



Since then there are in A
 as many magnitudes equal to
 C as there are units in D

whatever part the unit is of D , the same part is C of A also
 therefore as C is to A so is the unit to D [vi Def 20]

But the unit measures the number D ,
 therefore C also measures A

And since as C is to A , so is the unit to D
 therefore inversely as A is to C , so is the number D to the unit
 [cf v 7 Por]

Again since there are in F as many magnitudes equal to C as there are units
 in E

therefore as C is to F so is the unit to E [vii Def 20]

But it was also proved that

as A is to C so is D to the unit

therefore *ex aequali* as A is to F so is D to E [v 27]

But as D is to E so is A to B

therefore also as A is to B so is it to F also [v 11]

Therefore A has the same ratio to each of the magnitudes B , F ,

therefore B is equal to F [v 9]

But C measures F

therefore it measures B also

Further it measures A also

therefore C measures A B

Therefore A is commensurable with B

Therefore etc

PROPOSITION 7 From this it is manifest that if there be two numbers as D E and
 a straight line as A it is possible to make a straight line [F] such that the
 given straight line is to it as the number D is to the number E

And if a mean proportional be also taken between A F as B ,
 as A is to F so will the square on A be to the square on B that is as the first
 is to the third so is the figure on the first to that which is similar and similarly
 described on the second [vi 19 Por]

But as A is to F so is the number D to the number E
 therefore it has been contrived that as the number D is to the number E so
 also is the figure on the straight line A to the figure on the straight line B

Q E D

PROPOSITION 7

*Incommensurable magnitudes have not to one another the ratio which a number has
 to a number*

Let A , B be incommensurable magnitudes

I say that A has not to B the ratio which a number has to a number

For, if A has to B the ratio which a number has to a number, A will be com

measurable with B

[x 6]

But it is not,

$\frac{A}{B}$ therefore A has not to B the ratio which a number has to a number

Therefore etc

Q E D

PROPOSITION 8

If two magnitudes have not to one another the ratio which a number has to a number, the magnitudes will be incommensurable

For let the two magnitudes A, B not have to one another the ratio which a number has to a number,

I say that the magnitudes A, B are incommensurable

$\frac{A}{B}$ For if they are commensurable A will have to B the ratio which a number has to a number

[x 5]

But it has not,

therefore the magnitudes A, B are incommensurable

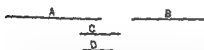
Therefore etc

Q E D

PROPOSITION 9

The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number and squares which have to one another the ratio which a square number has to a square number will also have their sides commensurable in length. But the squares on straight lines incommensurable in length have not to one another the ratio which a square number has to a square number, and squares which have not to one another the ratio which a square number has to a square number will not have their sides commensurable in length either

For let A, B be commensurable in length



I say that the square on A has to the square on B the ratio which a square number has to a square number

For since A is commensurable in length with B ,

therefore A has to B the ratio which a number has to a number [x 5]

Let it have to it the ratio which C has to D

Since then, as A is to B , so is C to D

while the ratio of the square on A to the square on B is duplicate of the ratio of A to B ,

for similar figures are in the duplicate ratio of their corresponding sides

{vi 20 Por }

and the ratio of the square on C to the square on D is duplicate of the ratio of C to D

for between two square numbers there is one mean proportional number and the square number has to the square number the ratio duplicate of that which the side has to the side

{viii 11}

therefore also as the square on A is to the square on B so is the square on C to the square on D

Next as the square on A is to the square on B , so let the square on C be to the square on D ,

I say that A is commensurable in length with B

For since as the square on A is to the square on B , so is the square on C to the square on D ,
while the ratio of the square on A to the square on B is duplicate of the ratio of A to B ,
and the ratio of the square on C to the square on D is duplicate of the ratio of C to D ,

therefore also as A is to B , so is C to D

Therefore A has to B the ratio which the number C has to the number D
therefore A is commensurable in length with B [x 6]

Next let A be incommensurable in length with B ,

I say that the square on A has not to the square on B the ratio which a square number has to a square number

For if the square on A has to the square on B the ratio which a square number has to a square number, A will be commensurable with B

But it is not

therefore the square on A has not to the square on B the ratio which a square number has to a square number

Again let the square on A not have to the square on B the ratio which a square number has to a square number,

I say that A is incommensurable in length with B

For, if A is commensurable with B the square on A will have to the square on B the ratio which a square number has to a square number

But it has not,

therefore A is not commensurable in length with B

Therefore etc

PORISM And it is manifest from what has been proved that straight lines commensurable in length are always commensurable in square also, but those commensurable in square are not always commensurable in length also

[LEMMA It has been proved in the arithmetical books that similar plane numbers have to one another the ratio which a square number has to a square number [VIII 26]

and that if two numbers have to one another the ratio which a square number has to a square number they are similar plane numbers {Converse of VIII 26}

And it is manifest from these propositions that numbers which are not similar plane numbers that is those which have not their sides proportional have not to one another the ratio which a square number has to a square number

For if they have they will be similar plane numbers which is contrary to the hypothesis

Therefore numbers which are not similar plane numbers have not to one another the ratio which a square number has to a square number]

PROPOSITION 10

To find two straight lines incommensurable the one in length only and the other in square also with an assigned straight line

Let A be the assigned straight line,
thus it is required to find two straight lines incommensurable the one in length only and the other in square also with A

Let two numbers B C be set out which have not to one another the ratio

which \equiv square number has to a square number that is which are not similar plane numbers,

and let it be contrived that,

A _____ as B is to C , so is the square on A to the square on D
 D _____ —for we have learnt how to do thus— [x 6 For]
 E _____ therefore the square on A is commensurable with the
 B _____ square on D [x 6]
 C _____

And since B has not to C the ratio which \equiv square number has to a square number,
 therefore neither has the square on A to the square on D the ratio which a square number has to a square number

therefore A is incommensurable in length with D [x 9]

Let E be taken a mean proportional between A D ,
 therefore as A is to D , so is the square on A to the square on E [v Def 9]

But A is incommensurable in length with D ,
 therefore the square on A is also incommensurable with the square on E [x 11]

therefore A is incommensurable in square with E

Therefore two straight lines D E have been found incommensurable D in length only and E in square and of course in length also with the assigned straight line A Q E D

PROPOSITION 11

If four magnitudes be proportional, and the first be commensurable with the second the third will also be commensurable with the fourth and if the first be incommensurable with the second the third will also be incommensurable with the fourth

Let A , B , C D be four magnitudes in proportion so that as A is to B so is C to D

and let A be commensurable with B ,
 A _____ B _____ I say that C will also be commensurable
 C _____ D _____ with D

For since A is commensurable with B
 therefore A has to B the ratio which a number has to a number [x 5]

And as A is to B so is C to D ,

therefore C also has to D the ratio which a number has to a number,
 therefore C is commensurable with D [x 6]

Next let A be incommensurable with B

I say that C will also be incommensurable with D

For since A is incommensurable with B

therefore A has not to B the ratio which a number has to a number [x 7]

And as A is to B so is C to D

therefore neither has C to D the ratio which a number has to a number
 therefore C is incommensurable with D [x 8]

Therefore etc Q E D

PROPOSITION 12

Magnitudes commensurable with the same magnitude are commensurable with one another also

For let each of the magnitudes A B be commensurable with C

I say that A is also commensurable with B

For since A is commensurable with C ,
therefore A has to C the ratio
which a number has to a
number [v 5]

Let it have the ratio which
 D has to E

Again since C is commen-
surable with B ,

therefore C has to B the ratio which a number has to a number [x 5]

Let it have the ratio which F has to G

And given any number of ratios we please namely the ratio which D has to
 E and that which F has to G

let the numbers H, K, L be taken continuously in the given ratios, [cf viii 4]

so that as D is to E , so is H to K ,

and, as F is to G , so is K to L

Since then, as A is to C , so is D to E ,

while as D is to E so is H to K ,

therefore also, as A is to C so is H to K [v 11]

Again since as C is to B so is F to G ,

while as F is to G so is K to L

therefore also as C is to B so is K to L [v 11]

But also as A is to C , so is H to K

therefore *ex aequali* as A is to B , so is H to L [v 27]

Therefore A has to B the ratio which a number has to a number,

therefore A is commensurable with B [x 6]

Therefore etc

Q E D

PROPOSITION 13

If two magnitudes be commensurable and the one of them be incommensurable with
any magnitude the remaining one will also be incommensurable with the same

Let A, B be two commensurable magnitudes and let one of them A be in
commensurable with any other magnitude C

I say that the remaining one B will also be incommensurable with C

For if B is commensurable with C

while A is also commensurable with B

A is also commensurable with C

[x 12]

But it is also incommensurable with it

which is impossible

Therefore B is not commensurable with C

therefore it is incommensurable with it

Therefore etc

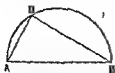
Q E D

LEMMA

Given two unequal straight lines to find by what square the square on the greater is
greater than the square on the less

Let AB, C be the given two unequal straight lines and let AB be the greater
of them,

thus it is required to find by what square the square on AB is greater than the square on C



Let the semicircle ADB be described on AB
and let AD be fitted into it equal to C , [iv 11]
let DB be joined

It is then manifest that the angle ADB is right [iii 31]

and that the square on AB is greater than the square on AD , that is, C by the square on DB [i 47]

Similarly also if two straight lines be given, the straight line the square on which is equal to the sum of the squares on them is found in this manner

Let AD DB be the given two straight lines, and let it be required to find the straight line the square on which is equal to the sum of the squares on them

Let them be placed so as to contain a right angle that formed by AD DB , and let AB be joined

It is again manifest that the straight line the square on which is equal to the sum of the squares on AD , DB is AB [i 47]

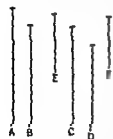
Q E D

PROPOSITION 14

If four straight lines be proportional and the square on the first be greater than the square on the second by the square on a straight line commensurable with the first the square on the third will also be greater than the square on the fourth by the square on a straight line commensurable with the third

And if the square on the first be greater than the square on the second by the square on a straight line incommensurable with the first, the square on the third will also be greater than the square on the fourth by the square on a straight line incommensurable with the third

Let A B C D be four straight lines in proportion so that as A is to B so is C to D



and let the square on A be greater than the square on B by the square on E

and let the square on C be greater than the square on D by the square on F ,

I say that if A is commensurable with E , C is also commensurable with F

and, if A is incommensurable with E C is also incommensurable with F

For since as A is to B so is C to D

therefore also as the square on A is to the square on B so is the square on C to the square on D [vi 22]

But the squares on E B are equal to the square on A

and the squares on D F are equal to the square on C

Therefore as the squares on E B are to the square on B so are the squares on D F to the square on D

therefore *separando* as the square on E is to the square on B so is the square on F to the square on D , [v 17]

therefore also as E is to B so is F to D [vi 22]

therefore *inversely* as B is to E so is D to F

But as A is to B , so also is C to D ,

therefore *ex aequali*, as A is to E , so is C to F [v 22]

Therefore if A is commensurable with E , C is also commensurable with F ,
and if A is incommensurable with E , C is also incommensurable with F [x 11]

Therefore etc

Q E D

PROPOSITION 15

If two commensurable magnitudes be added together, the whole will also be commensurable with each of them and, if the whole be commensurable with one of them the original magnitudes will also be commensurable

For let the two commensurable magnitudes AB , BC be added together

I say that the whole AC is also commensurable with each of the magnitudes AB
 BC



For since AB , BC are commensurable,
some magnitude will measure them

Let it measure them and let it be D

Since then D measures AB , BC , it will also measure the whole AC

But it measures AB , BC also,

therefore D measures AB , BC , AC

therefore AC is commensurable with each of the magnitudes AB , BC

[x Def 1]

Next, let AC be commensurable with AB ,

I say that AB , BC are also commensurable

For since AC , AB are commensurable some magnitude will measure them

Let it measure them and let it be D

Since then D measures CA , AB it will also measure the remainder BC

But it measures AB also

therefore D will measure AB , BC ,

therefore AB , BC are commensurable

[x Def 1]

Therefore etc

Q E D

PROPOSITION 16

If two incommensurable magnitudes be added together the whole will also be incommensurable with each of them and, if the whole be incommensurable with one of them the original magnitudes will also be incommensurable

For let the two incommensurable magnitudes AB , BC be added together I say that the whole AC is also incommensurable with each of the magnitudes AB , BC

For if CA , AB are not incommensurable some magnitude will measure them

Let it measure them if possible and let it be D

Since then D measures CA , AB

therefore it will also measure the remainder BC

But it measures AB also

therefore D measures AB , BC

Therefore AB , BC are commensurable

but they were also by hypothesis incommensurable
which is impossible



Therefore no magnitude will measure CA, AB ,
 therefore CA, AB are incommensurable [X Def 1]
 Similarly we can prove that AC, CB are also incommensurable
 Therefore AC is incommensurable with each of the magnitudes AB, BC
 Next let AC be incommensurable with one of the magnitudes AB, BC
 First let it be incommensurable with AB ,
 I say that AB, BC are also incommensurable
 For, if they are commensurable some magnitude will measure them
 Let it measure them, and let it be D
 Since then, D measures AB, BC ,
 therefore it will also measure the whole AC
 But it measures AB also
 therefore D measures CA, AB
 Therefore CA, AB are commensurable,
 but they were also by hypothesis incommensurable
 which is impossible
 Therefore no magnitude will measure AB, BC
 therefore AB, BC are incommensurable [X Def 1]
 Therefore etc Q E D

LEMMA

If to any straight line there be applied a parallelogram deficient by a square figure the applied parallelogram is equal to the rectangle contained by the segments of the straight line resulting from the application

For let there be applied to the straight line AB the parallelogram AD deficient by the square figure DB ,
 I say that AD is equal to the rectangle contained by AC, CB



This is indeed at once manifest

for since DB is a square
 DC is equal to CB

and AD is the rectangle AC, CD , that is the rectangle AC, CB

Therefore etc

Q E D

PROPOSITION 17

If there be two unequal straight lines and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure and if it divide it into parts which are commensurable in length then the square on the greater will be greater than the square on the less by the square on a straight line commensurable with the greater

And if the square on the greater be greater than the square on the less by the square on a straight line commensurable with the greater and if there be applied to the greater a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, it will divide it into parts which are commensurable in length

Let A, BC be two unequal straight lines of which BC is the greater and let there be applied to BC a parallelogram equal to the fourth part of the square on the less A that is equal to the square on the half of A and deficient by a square figure Let this be the rectangle BD, DC [cf Lemma]

and let BD be commensurable in length with DC ,

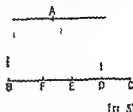
I say that the square on BC is greater than the square on A by the square on a straight line commensurable with BC

For let BC be bisected at the point E

and let EF be made equal to DE

Therefore the remainder DC is equal to BF

And since the straight line BC has been cut into equal parts at E and into unequal parts at D therefore the rectangle contained by BD, DC , together with the square on ED , is equal to the square on EC ,



And the same is true of their quadruples, therefore four times the rectangle BD, DC , together with four times the square on DE is equal to four times the square on EC

But the square on A is equal to four times the rectangle BD, DC and the square on DF is equal to four times the square on DE , for DF is double of DE

And the square on BC is equal to four times the square on EC , for again BC is double of CE

Therefore the squares on A, DF are equal to the square on BC , so that the square on BC is greater than the square on A by the square on DF

It is to be proved that BC is also commensurable with DF

Since BD is commensurable in length with DC

therefore BC is also commensurable in length with CD [x 15]

But CD is commensurable in length with CD, BF for CD is equal to BF [x 6]

Therefore BC is also commensurable in length with BF, CD [x 12]

so that BC is also commensurable in length with the remainder FD [x 15] therefore the square on BC is greater than the square on A by the square on a straight line commensurable with BC

Next let the square on BC be greater than the square on A by the square on a straight line commensurable with BC

let a parallelogram be applied to BC equal to the fourth part of the square on A and deficient by a square figure and let it be the rectangle BD, DC

It is to be proved that BD is commensurable in length with DC

With the same construction we can prove similarly that the square on BC is greater than the square on A by the square on FD

But the square on BC is greater than the square on A by the square on a straight line commensurable with BC

Therefore BC is commensurable in length with FD

so that BC is also commensurable in length with the remainder the sum of BF, DC [x 15]

But the sum of BF, DC is commensurable with DC [x 6]

so that BC is also commensurable in length with CD [x 12]

and therefore *separando* BD is commensurable in length with DC [x 15]

Therefore etc. Q E D

PROPOSITION 18

If there be two unequal straight lines and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square

figure, and if it divide it into parts which are incommensurable the square on the greater will be greater than the square on the less by the square on a straight line incommensurable with the greater

And if the square on the greater be greater than the square on the less by the square on a straight line incommensurable with the greater, and if there be applied to the greater a parallelogram equal to the fourth part of the square on the less and deficient by a square figure it divides it into parts which are incommensurable

Let A, BC be two unequal straight lines of which BC is the greater, and to BC let there be applied a parallelogram equal to the fourth part of the square on the less A , and deficient by a square figure. Let this be the rectangle BD, DC , [cf Lemma before $\propto 17$]

and let BD be incommensurable in length with DC ,
I say that the square on BC is greater than the square on A by the square on a straight line incommensurable with BC

For with the same construction as before, we can prove similarly that the square on BC is greater than the square on A by the square on FD

It is to be proved that BC is incommensurable in length with DF

Since BD is incommensurable in length with DC ,
therefore BC is also incommensurable in length with CD [$\propto 16$]

But DC is commensurable with the sum of BF, DC , [$\propto 6$]

therefore BC is also incommensurable with the sum of BF, DC [$\propto 13$]

so that BC is also incommensurable in length with the remainder FD [$\propto 16$]

And the square on BC is greater than the square on A by the square on FD ,
therefore the square on BC is greater than the square on A by the square on a straight line incommensurable with BC

Again, let the square on BC be greater than the square on A by the square on a straight line incommensurable with BC and let there be applied to BC a parallelogram equal to the fourth part of the square on A and deficient by a square figure. Let this be the rectangle BD, DC

It is to be proved that BD is incommensurable in length with DC

For with the same construction, we can prove similarly that the square on BC is greater than the square on A by the square on FD

But the square on BC is greater than the square on A by the square on a straight line incommensurable with BC ,

therefore BC is incommensurable in length with FD ,

so that BC is also incommensurable with the remainder the sum of BF, DC

[$\propto 16$]

But the sum of BF, DC is commensurable in length with DC , [$\propto 6$]

therefore BC is also incommensurable in length with DC [$\propto 13$]

so that, *separando* BD is also incommensurable in length with DC [$\propto 16$]

Therefore etc Q E D

LEMMA

Since it has been proved that straight lines commensurable in length are always commensurable in square also while those commensurable in square are not always commensurable in length also but can of course be either commensurable or incommensurable in length it is manifest that if any straight line be commensurable in length with a given rational straight line, it is called

rational and commensurable with the other not only in length but in square also, since straight lines commensurable in length are always commensurable in square also

But, if any straight line be commensurable in square with a given rational straight line then, if it is also commensurable in length with it it is called in this case also rational and commensurable with it both in length and in square but if again any straight line being commensurable in square with a given rational straight line be incommensurable in length with it, it is called in this case also rational but commensurable in square only

PROPOSITION 19

The rectangle contained by rational straight lines commensurable in length is rational

For let the rectangle AC be contained by the rational straight lines AB , BC commensurable in length,

I say that AC is rational

For on AB let the square AD be described,

therefore AD is rational [x Def 4]

And since AB is commensurable in length with BC ,

while AB is equal to BD ,

therefore BD is commensurable in length with BC

And as BD is to BC , so is DA to AC [vi 1]

Therefore DA is commensurable with AC [x 11]

But DA is rational

therefore AC is also rational

[x Def 4]

Therefore etc

Q E D

PROPOSITION 20

If a rational area be applied to a rational straight line it produces as breadth a straight line rational and commensurable in length with the straight line to which it is applied

For let the rational area AC be applied to AB , a straight line once more rational in any of the aforesaid ways producing BC as breadth,

I say that BC is rational and commensurable in length with BA

For on AB let the square AD be described

therefore AD is rational [x Def 4]

But AC is also rational

therefore DA is commensurable with AC

And as DA is to AC so is DB to BC [vi 1]

Therefore DB is also commensurable with BC , [x 11]

and DB is equal to BA ,

therefore AB is also commensurable with BC

But AB is rational

therefore BC is also rational and commensurable in length with AB

Therefore etc

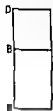
Q E D

PROPOSITION 21

The rectangle contained by rational straight lines commensurable in square only is irrational and the side of the square equal to it is irrational Let the latter be called medial

For let the rectangle AC be contained by the rational straight lines AB , BC commensurable in square only,

I say that AC is irrational, and the side of the square equal to it is irrational, and let the latter be called *medial*



For on AB let the square AD be described,

therefore AD is rational

[x Def 4]

And since AB is incommensurable in length with BC

for by hypothesis they are commensurable in square only,

while $4B$ is equal to BD

therefore DB is also incommensurable in length with BC

And, as DB is to BC , so is AD to AC

[vi 1]

therefore DA is incommensurable with AC

[x 11]

But DA is rational,

therefore AC is irrational

so that the side of the square equal to AC is also irrational

[x Def 4]

And let the latter be called *medial*

q e d

LEMMA

If there be two straight lines then, as the first is to the second so is the square on the first to the rectangle contained by the two straight lines

Let FE , EG be two straight lines

I say that, as FE is to EG so is the square on FE to the rectangle FE , EG

For on FE let the square FD be described

and let GD be completed



Since then, as FE is to EG so is FD to DG ,

[vi 1]

and FD is the square on FE

and DG the rectangle DE , EG that is the rectangle FE , EG

therefore as FE is to EG so is the square on FE to the rectangle FE , EG

Similarly also as the rectangle GE , EF is to the square on EF that is as GD is to FD , so is GE to EF

q e d

PROPOSITION 22

The square on a medial straight line if applied to a rational straight line produces as breadth a straight line rational and incommensurable in length with that to which it is applied



Let A be medial and CB rational

and let a rectangular area BD equal to the square on A be applied to BC producing CD as breadth

I say that CD is rational and incommensurable in length with CB

For since A is medial the square on it is equal to a rectangular area contained by rational straight lines commensurable in square only

[x 21]

Let the square on it be equal to GF

But the square on it is also equal to BD

therefore BD is equal to GF



But it is also equiangular with it,
and in equal and equiangular parallelograms the sides about the equal angles
are reciprocally proportional, [vi 14]

therefore, proportionally, as BC is to EG , so is EF to CD

Therefore also as the square on BC is to the square on EG , so is the square
on EF to the square on CD [vi 22]

But the square on CB is commensurable with the square on EG , for each of
these straight lines is rational

therefore the square on EF is also commensurable with the square on CD [x 11]

But the square on EF is rational,

therefore the square on CD is also rational, [x Def 4]

therefore CD is rational

And since EF is incommensurable in length with EG ,

for they are commensurable in square only,

and as EF is to EG so is the square on EF to the rectangle FE, EG , [Lemma]
therefore the square on EF is incommensurable with the rectangle FE, EG [x 11]

But the square on CD is commensurable with the square on EF , for the
straight lines are rational in square

and the rectangle DC, CB is commensurable with the rectangle FE, EG for
they are equal to the square on A ,

therefore the square on CD is also incommensurable with the rectangle DC
 CB [x 13]

But as the square on CD is to the rectangle DC, CB , so is DC to CB , [Lemma]

therefore DC is incommensurable in length with CB [x 11]

Therefore CD is rational and incommensurable in length with CB

■ x p

PROPOSITION 23

A straight line commensurable with a medial straight line is medial

Let A be medial and let B be commensurable with A ,

I say that B is also medial

For let a rational straight line CD be set out
and to CD let the rectangular area CE equal to the
square on A be applied producing ED as breadth,
therefore ED is rational and incommensurable in
length with CD [x 22]

And let the rectangular area CF equal to the square
on B be applied to CD producing DF as breadth

Since, then A is commensurable with B
the square on A is also commensurable with the
square on B

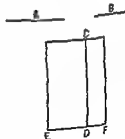
But EC is equal to the square on A

and CF is equal to the square on B ,

therefore EC is commensurable with CF

And as EC is to CF so is ED to DF

therefore ED is commensurable in length with DF



[vi 11]

[x 11]

But ED is rational and incommensurable in length with DC
 therefore DF is also rational [x Def 3] and incommensurable in length with DC [x 13]

Therefore CD, DF are rational and commensurable in square only

But the straight line the square on which is equal to the rectangle contained by rational straight lines commensurable in square only is medial, [x 21]
 therefore the side of the square equal to the rectangle CD, DF is medial

And B is the side of the square equal to the rectangle CD, DF ,

therefore B is medial Q E D

PROPOSITION 23 From this it is manifest that an area commensurable with a medial area is medial

[And in the same way as was explained in the case of rationals [Lemma following x 18] it follows as regards medials, that a straight line commensurable in length with a medial straight line is called *medial* and commensurable with it not only in length but in square also since in general, straight lines commensurable in length are always commensurable in square also

But if any straight line be commensurable in square with a medial straight line then, if it is also commensurable in length with it, the straight lines are called in this case too medial and commensurable in length and in square but if in square only, they are called medial straight lines commensurable in square only]

PROPOSITION 24

The rectangle contained by medial straight lines commensurable in length is medial

For let the rectangle AC be contained by the medial straight lines AB, BC which are commensurable in length

I say that AC is medial

For on AB let the square AD be described

therefore AD is medial

And, since AB is commensurable in length with BC

while AB is equal to BD ,

therefore DB is also commensurable in length with BC ,

so that DA is also commensurable with AC [vi 1 x 11]

But DA is medial

therefore AC is also medial

[x 23 Por]

Q E D

PROPOSITION 25

The rectangle contained by medial straight lines commensurable in square only is either rational or medial

For let the rectangle AC be contained by the medial straight lines AB, BC which are commensurable in square only

I say that AC is either rational or medial

For on AB, BC let the squares AD, BE be described

therefore each of the squares AD, BE is medial

Let a rational straight line FG be set out

to FG let there be applied the rectangular parallelogram GH equal to AD producing FH as breadth

to HM let there be applied the rectangular parallelogram MK equal to AC , producing HK as breadth

and further to AN let there be similarly applied AL equal to BE producing KL as breadth,

therefore FH HA AL are in a straight line

Since then each of the squares AD , BE
is medial

and AD is equal to GH , and BE to NL
therefore each of the rectangles GH , NL is
also medial

And they are applied to the rational
straight line FG

therefore each of the straight lines FH KL
is rational and incommensurable in length
with FG [x 22]

And since AD is commensurable with BE

therefore GH is also commensurable with NL

And as GH is to NL so is GH to KL

therefore FH is commensurable in length with KL

Therefore FH KL are rational straight lines commensurable in length, [vi 1]

therefore the rectangle FH KL is rational [x 11]

And since DB is equal to BA and OB to BC , [x 19]

therefore as DB is to BC so is AB to BO

But as DB is to BC so is DA to AC [vi 1]

and as AB is to BO so is AC to CO , [vd]

therefore as DA is to AC so is AC to CO

But AD is equal to GH AC to MA and CO to AL ,

therefore as GH is to MA , so is MA to AL

therefore also as FH is to HA so is HA to AL [vi 1, v 11]

therefore the rectangle FH KL is equal to the square on HA [vi 17]

But the rectangle FH AL is rational

therefore the square on HA is also rational

Therefore HA is rational

And if it is commensurable in length with FG

HN is rational [x 19]

but if it is incommensurable in length with FG

HA HM are rational straight lines commensurable in square only and there-
fore HN is medial [x 21]

Therefore HN is either rational or medial

But HA is equal to AC

therefore AC is either rational or medial

Therefore etc

Q E D

PROPOSITION 26

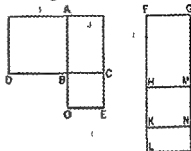
A medial area does not exceed a medial area by a rational area

For if possible let the medial area AB exceed the medial area AC by the
rational area DB

and let a rational straight line EF be set out

to EF let there be applied the rectangular parallelogram FH equal to AB pro-
ducing EH as breadth

and let the rectangle FG equal to AC be subtracted



therefore the remainder BD is equal to the remainder KH

But DB is rational

therefore KH is also rational

Since then each of the rectangles AB , AC is medial

and AB is equal to FH and AC to FG

therefore each of the rectangles FH , FG is also medial

And they are applied to the rational straight line EF ,

therefore each of the straight lines HE EG is rational and incommensurable in length with EF [x 22]

And, since [DB is rational and is equal to KH ,

therefore] KH is [also] rational,

and it is applied to the rational straight line EF

therefore GH is rational and commensurable in length with EF [x 20]

But EG is also rational and is incommensurable in length with EF

therefore EG is incommensurable in length with GH [x 13]

And as EG is to GH so is the square on EG to the rectangle EG , GH , therefore the square on EG is incommensurable with the rectangle EG , GH [x 11]

But the squares on EG , GH are commensurable with the square on EG for both are rational

and twice the rectangle EG , GH is commensurable with the rectangle EG , GH for it is double of it, [x 6]

therefore the squares on EG , GH are incommensurable with twice the rectangle EG , GH [x 13]

therefore also the sum of the squares on EG , GH and twice the rectangle EG , GH , that is the square on EH [II 4] is incommensurable with the squares on EG , GH [x 16]

But the squares on EG , GH are rational

therefore the square on EH is irrational [x Def 4]

Therefore EH is irrational

But it is also rational

which is impossible

Therefore etc

Q E D

PROPOSITION 27

To find medial straight lines commensurable in square only which contain a rational rectangle

Let two rational straight lines A , B commensurable in square only be set out,

let C be taken a mean proportional between A , B [VI 13]

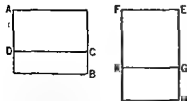
and let it be contrived that

as A is to B , so is C to D [VI 12]

Then since A , B are rational and commensurable in square only

the rectangle A , B that is the square on C

[VI 17] is medial [x 21]



Therefore C is medial [x 21]
 And since as A is to B , so is C to D ,
 and A, B are commensurable in square only — —
 therefore C, D are also commensurable in square only [x 11]
 And C is medial,
 therefore D is also medial [x 23, addition]
 Therefore C, D are medial and commensurable in square only
 I say that they also contain a rational rectangle
 For since as A is to B so is C to D
 therefore alternately as A is to C so is B to D [v 16]
 But as A is to C so is C to B ,
 therefore also as C is to B so is B to D
 therefore the rectangle $C D$ is equal to the square on B
 But the square on B is rational
 therefore the rectangle $C D$ is also rational
 Therefore medial straight lines commensurable in square only have been
 found which contain a rational rectangle Q E D

PROPOSITION 28

To find medial straight lines commensurable in square only which contain a medial rectangle

Let the rational straight lines A, B, C commensurable in square only be set out

let D be taken a mean proportional between A, B , [v 13]
 and let it be contrived that
 as B is to C so is D to E [vi 12]

Since A, B are rational straight
 lines commensurable in square only
 therefore the rectangle $A B$ that
 is the square on D [vi 17] is medi
 al [x 21]

A —————
 B ————— D —————
 C ————— E —————

Therefore D is medial [x 21]
 And since B, C are commensurable in square only
 and as B is to C so is D to E
 therefore D, E are also commensurable in square only [x 11]
 But D is medial

therefore E is also medial [x 23 addition]

Therefore D, E are medial straight lines commensurable in square only

I say next that they also contain a medial rectangle

For since as B is to C so is D to E

therefore alternately as B is to D so is C to E [v 16]

But as B is to D so is D to A

therefore also as D is to A so is C to E

therefore the rectangle $A C$ is equal to the rectangle $D E$ [vi 16]

But the rectangle $A C$ is medial

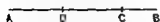
therefore the rectangle $D E$ is also medial

Therefore medial straight lines commensurable in square only have been
 found which contain a medial rectangle Q E D

LEMMA 1

To find two square numbers such that their sum is also square

Let two numbers AB , BC be set out, and let them be either both even or both odd



Then since whether an even number is subtracted from an even number, or an odd number from an odd number the remainder is even [ix 24 26]

therefore the remainder AC is even

Let AC be bisected at D

Let AB , BC also be either similar plane numbers or square numbers which are themselves also similar plane numbers

Now the product of AB , BC together with the square on CD is equal to the square on BD [ii 6]

And the product of AB , BC is square inasmuch as it was proved that if two similar plane numbers by multiplying one another make some number the product is square [ix 1]

Therefore two square numbers, the product of AB , BC , and the square on CD have been found which when added together make the square on BD

And it is manifest that two square numbers, the square on BD and the square on CD , have again been found such that their difference, the product of AB , BC is a square, whenever AB , BC are similar plane numbers

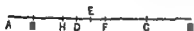
But when they are not similar plane numbers two square numbers the square on BD and the square on DC , have been found such that their difference the product of AB , BC is not square Q E D

LEMMA 2

To find two square numbers such that their sum is not square

For let the product of AB , BC as we said be square and CA even

and let CA be bisected by D



It is then manifest that the square product of AB , BC together with the square on CD is equal to the square on BD

[See Lemma 1]

Let the unit DE be subtracted, therefore the product of AB , BC together with the square on CE is less than the square on BD

I say then that the square product of AB , BC together with the square on CE will not be square

For if it is square it is either equal to the square on BE or less than the square on BE but cannot any more be greater lest the unit be divided

First if possible, let the product of AB , BC together with the square on CE be equal to the square on BE

and let GA be double of the unit DE

Since then the whole AC is double of the whole CD

and in them AG is double of DE

therefore the remainder GC is also double of the remainder EC , therefore GC is bisected by E

Therefore the product of GB , BC together with the square on CE is equal to the square on BE [11 6]

But the product of AB , BC together with the square on CE is also, by hypothesis equal to the square on BE ,
therefore the product of GB , BC together with the square on CE is equal to the product of AB , BC together with the square on CE

And, if the common square on CE be subtracted,
it follows that AB is equal to GB
which is absurd

Therefore the product of AB , BC together with the square on CE is not equal to the square on BE

I say next that neither is it less than the square on BE

For, if possible, let it be equal to the square on BF
and let HA be double of DF

Now it will again follow that HC is double of CF
so that CH has also been bisected at F

and for this reason the product of HB , BC together with the square on FC is equal to the square on BF [11 6]

But, by hypothesis the product of AB , BC together with the square on CE is also equal to the square on BF

Thus the product of HB , BC together with the square on CF will also be equal to the product of AB , BC together with the square on CE
which is absurd

Therefore the product of AB , BC together with the square on CE is not less than the square on BE

And it was proved that neither is it equal to the square on BE

Therefore the product of AB , BC together with the square on CE is not square Q E D

PROPOSITION 29

To find two rational straight lines commensurable in square only and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater

For let there be set out any rational straight line AB and two square numbers CD , DE such that their difference CE is not square, [Lemma 1]

let there be described on AB the semicircle AFB

and let it be contrived that

as DC is to CE so is the square on BA to the square on AF [x 6 Por]

Let FB be joined

Since as the square on BA is to the square on AF ,
so is DC to CE

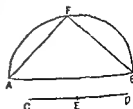
therefore the square on BA has to the square on AF
the ratio which the number DC has to the number CE ,

therefore the square on BA is commensurable with the square on AF [x 6]

But the square on AB is rational [x Def 4]

therefore the square on AF is also rational

therefore AF is also rational [1d]



And since DC has not to CE the ratio which a square number has to a square number
neither has the square on BA to the square on AF the ratio which a square number has to a square number

therefore AB is incommensurable in length with AF [τ 9]

Therefore BA, AF are rational straight lines commensurable in square only
And since as DC is to CE , so is the square on BA to the square on AF ,
therefore, *convertendo* as CD is to DE , so is the square on AB to the square on BF [ν 19 Por, III 31 i 47]

But CD has to DE the ratio which a square number has to a square number
therefore also the square on AB has to the square on BF the ratio which a square number has to a square number,

therefore AB is commensurable in length with BF [κ 9]

And the square on AB is equal to the squares on AF, FB
therefore the square on AB = greater than the square on AF by the square on BF commensurable with AB

Therefore there have been found two rational straight lines BA, AF commensurable in square only and such that the square on the greater AB is greater than the square on the less AF by the square on BF commensurable in length with AB Q E D

PROPOSITION 30

To find two rational straight lines commensurable in square only and such that the square on the greater is greater than the square on the less by the square on a straight line incommensurable in length with the greater

Let there be set out a rational straight line AB ,

and two square numbers CE, BD such that their sum CD is not square [Lemma 2]

let there be described on AB the semicircle AFB ,

let it be contrived that

as DC is to CE , so = the square on BA to the square on AF [τ 6, Por]

and let FB be joined

Then in a similar manner to the preceding we can prove that BA, AF are rational straight lines commensurable in square only

And since as DC is to CE , so is the square on BA to the square on AF
therefore *convertendo* as CD is to DE so is the square on AB to the square on BF [ν 19 Por III 31 i 47]

But CD has not to DE the ratio which a square number has to a square number

therefore neither has the square on AB to the square on BF the ratio which a square number has to a square number,

therefore AB is incommensurable in length with BF [κ 9]

And the square on AB is greater than the square on AF by the square on FB incommensurable with AB

Therefore AB, AF are rational straight lines commensurable in square only and the square on AB is greater than the square on AF by the square on FB incommensurable in length with AB Q E D



PROPOSITION 31

To find two medial straight lines commensurable in square only, containing a rational rectangle and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater

Let there be set out two rational straight lines A B commensurable in square only and such that the square on A being the greater is greater than the square on B the less by the square on a straight line commensurable in length with A [x 20]

And let the square on C be equal to the rectangle A B

Now the rectangle A B is medial [x 21]

therefore the square on C is also medial,

therefore C is also medial [x 21]

Let the rectangle C D be equal to the square on B

Now the square on B is rational

therefore the rectangle C D is also rational

And since as A is to B so is the rectangle A B to the square on B

while the square on C is equal to the rectangle A B ,

and the rectangle C D is equal to the square on B ,

therefore as A is to B so is the square on C to the rectangle C D ,

But as the square on C is to the rectangle C D so is C to D ,

therefore also as A is to B so is C to D

But A is commensurable with B in square only

therefore C is also commensurable with D in square only [x 11]

And C is medial

therefore D is also medial

[x 23 addition]

And since as A is to B so is C to D

and the square on A is greater than the square on B by the square on a straight line commensurable with A ,

therefore also the square on C is greater than the square on D by the square on a straight line commensurable with C [x 14]

Therefore two medial straight lines C D commensurable in square only and containing a rational rectangle have been found and the square on C is greater than the square on D by the square on a straight line commensurable in length with C

Similarly also it can be proved that the square on C exceeds the square on D by the square on a straight line incommensurable with C when the square on A is greater than the square on B by the square on a straight line incommensurable with A [x 30]

PROPOSITION 32

To find two medial straight lines commensurable in square only containing a medial rectangle and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater

Let there be set out three rational straight lines A B C commensurable in square only and such that the square on A is greater than the square on C by the square on a straight line commensurable with A [x 20]

and let the square on D be equal to the rectangle A B

Therefore the square on D is medial

therefore D is also medial

[x 21]

Let the rectangle D, E be equal to the rectangle B, C

A _____

B _____

C _____

D _____

E _____

Then since, as the rectangle A, B is to the rectangle B, C , so $\equiv A$ to C ,

while the square on D is equal to the rectangle A, B ,

and the rectangle D, E is equal to the rectangle B, C ,

therefore, as $A \equiv$ to C , so is the square on D to the rectangle D, E

But as the square on D is to the rectangle D, E , so $\equiv D$ to E ,

therefore also as A is to C , so is D to E

But $A \equiv$ commensurable with C in square only,

therefore D is also commensurable with E in square only [x 11]

But $D \equiv$ medial,

therefore E is also medial

[x 23 addition]

And, since, as A is to C , so is D to E ,

while the square on A is greater than the square on C by the square on \equiv straight line commensurable with A ,

therefore also the square on D will be greater than the square on E by the square on a straight line commensurable with D [x 14]

I say next that the rectangle D, E is also medial

For, since the rectangle B, C is equal to the rectangle D, E , while the rectangle B, C is medial [x 21]

therefore the rectangle D, E is also medial

Therefore two medial straight lines D, E , commensurable in square only and containing a medial rectangle have been found such that the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater

Similarly again it can be proved that the square on D is greater than the square on E by the square on a straight line incommensurable with D , when the square on A is greater than the square on C by the square on a straight line incommensurable with A [x 30]

LEMMA

Let ABC be a right angled triangle having the angle A right and let the perpendicular AD be drawn,



I say that the rectangle CB, BD is equal to the square on BA ,

the rectangle BC, CD equal to the square on CA ,

the rectangle BD, DC equal to the square on AD

and further, the rectangle BC, AD equal to the rectangle BA, AC

And first that the rectangle CB, BD is equal to the square on BA

For since in a right angled triangle AD has been drawn from the right angle perpendicular to the base therefore the triangles ABD, ADC are similar both to the whole ABC and to one another [vi 8]

And since the triangle ABC is similar to the triangle ABD , therefore as CB is to BA so is BA to BD , [vi 4]

therefore the rectangle CB BD is equal to the square on AB [VI 14]

For the same reason the rectangle BC CD is also equal to the square on AC

And since if in a right angled triangle a perpendicular be drawn from the right angle to the base the perpendicular so drawn is a mean proportional between the segments of the base [VI 8 Por]

therefore as BD is to DA , so is AD to DC ,

therefore the rectangle BD DC is equal to the square on AD [VI 17]

I say that the rectangle BC AD is also equal to the rectangle BA AC

For since as we said ABC is similar to ABD ,

therefore as BC is to CA so is BA to AD [VI 4]

Therefore the rectangle BC AD is equal to the rectangle BA AC [VI 16]

Q E D

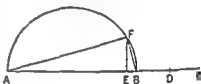
PROPOSITION 33

To find two straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial

Let there be set out two rational straight lines AB BC commensurable in square only and such that the square on the greater AB is greater than the square on the less BC by the square on a straight line incommensurable with AB [X 30]

let BC be bisected at D

let there be applied to AB a parallelogram equal to the square on either of the straight lines BD DC and deficient by a square figure and let it be the rectangle AE EB , [VI 28]



let the semicircle AFB be described on AB ,

let EF be drawn at right angles to AB ,

and let AF FB be joined

Then since AB BC are unequal straight lines and the square on AB is greater than the square on BC by the square on a straight line incommensurable with AB while there has been applied to AB a parallelogram equal to the fourth part of the square on BC that is to the square on half of it and deficient by a square figure making the rectangle AE EB ,

therefore AE is incommensurable with EB [X 18]

And as AE is to EB so is the rectangle BA AE to the rectangle AB BE

while the rectangle BA AE is equal to the square on AF

and the rectangle AB BE to the square on BF

therefore the square on AF is incommensurable with the square on FB ,

therefore AF FB are incommensurable in square

And since AB is rational

therefore the square on AB is also rational,

so that the sum of the squares on AF FB is also rational [I 4]

And since again the rectangle AE EB is equal to the square on EF

and by hypothesis the rectangle AE FB is also equal to the square on BD

therefore FE is equal to BD

therefore BC is double of FE

so that the rectangle AB BC is also commensurable with the rectangle AB EF

But the rectangle AB BC is medial, [x 21]
 therefore the rectangle AB , EF is also medial [x 23 Por]
 But the rectangle AB , EF is equal to the rectangle AF FB , [Lemma]
 therefore the rectangle AF FB is also medial

But it was also proved that the sum of the squares on these straight lines is rational

Therefore two straight lines AF , FB incommensurable in square have been found which make the sum of the squares on them rational but the rectangle contained by them medial Q E D

PROPOSITION 34

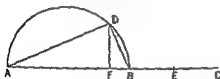
To find two straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational

Let there be set out two medial straight lines AB BC , commensurable in square only such that the rectangle which they contain is rational and the square on AB \equiv greater than the square on BC by the square on a straight line incommensurable with AB , [x 31, ad fin]

let the semicircle ADB be described on AB ,

let BC be bisected at E

let there be applied to AB a parallelogram equal to the square on BE and deficient by a square figure, namely the rectangle AF FB [v 1 23]



therefore AF is incommensurable in length with FB [v 18]

Let FD be drawn from F at right angles to AB ,
 and let AD DB be joined

Since AF is incommensurable in length with FB
 therefore the rectangle BA , AF is also incommensurable with the rectangle AB , BF [x 11]

But the rectangle BA AF \equiv equal to the square on AD and the rectangle AB , BF to the square on DB ,

therefore the square on AD is also incommensurable with the square on DB

And since the square on AB is medial

therefore the sum of the squares on AD DB is also medial [III 31 i 47]

And since BC is double of DF ,

therefore the rectangle AB BC is also double of the rectangle AB FD

But the rectangle AB BC is rational

therefore the rectangle AB FD is also rational [x 6]

But the rectangle AB FD is equal to the rectangle AD DB [Lemma]

so that the rectangle AD DB is also rational

Therefore two straight lines AD DB incommensurable in square have been found which make the sum of the squares on them medial but the rectangle contained by them rational Q E D

PROPOSITION 35

To find two straight lines incommensurable in square which make the sum of the squares on them medial and the rectangle contained by them medial and moreover incommensurable with the sum of the squares on them

Let there be set out two medial straight lines AB , BC commensurable in square only containing a medial rectangle and such that the square on AB is greater than the square on BC by the square on a straight line incommensurable with AB , [x 32, *ad fin*]

let the semicircle ADB be described on AB ,
and let the rest of the construction be as above

Then, since AP is incommensurable in length with FB [x 18]
 AD is also incommensurable in square with DB [x 11]

And since the square on AB is medial
therefore the sum of the squares on AD , DB is also medial [III 31 i 47]

And since the rectangle AF , FB is equal to the square on each of the straight lines BE , DF ,

therefore BE is equal to DF , [ii]
therefore BC is double of FD

so that the rectangle AB , BC is also double of the rectangle AB , FD

But the rectangle AB , BC is medial,

therefore the rectangle AB , FD is also medial [x 32 *Por*]

And it is equal to the rectangle AD , DB , [Lemma after x 37]

therefore the rectangle AD , DB is also medial

And since AB is incommensurable in length with BC ,
while CB is commensurable with BE ,

therefore AB is also incommensurable in length with BE [x 13]

so that the square on AB is also incommensurable with the rectangle AB , BE [x 11]

But the squares on AD , DB are equal to the square on AB , [i 47]
and the rectangle AB , FD , that is the rectangle AD , DB , is equal to the rectangle AB , BE

therefore the sum of the squares on AD , DB is incommensurable with the rectangle AD , DB

Therefore two straight lines AD , DB incommensurable in square have been found which make the sum of the squares on them medial and the rectangle contained by them medial and moreover incommensurable with the sum of the squares on them Q E D

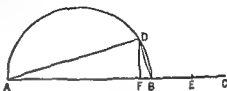
PROPOSITION 36

If two rational straight lines commensurable in square only be added together, the whole is irrational and let it be called binomial

For let two rational straight lines AB , BC commensurable in square only be added together

I say that the whole AC is irrational

For since AB is incommensurable in length with BC —for they are commensurable in square only—
and as AB is to BC , so is the rectangle AB , BC to the square on BC ,
therefore the rectangle AB , BC is incommensurable with the square on BC [x 11]



But twice the rectangle AB, BC is commensurable with the rectangle AB, BC [x 6], and the squares on AB, BC are commensurable with the square on BC —for AB, BC are rational straight lines commensurable in square only—

[x 15]

therefore twice the rectangle AB, BC is incommensurable with the squares on AB, BC

[x 13]

And, *componendo* twice the rectangle AB, BC together with the squares on AB, BC , that is, the square on AC [II 4] is incommensurable with the sum of the squares on AB, BC

[x 16]

But the sum of the squares on AB, BC is rational

therefore the square on AC is irrational

so that AC is also irrational

[x Def 4]

And let it be called *binomial*

Q E D

PROPOSITION 37

If two medial straight lines commensurable in square only and containing a rational rectangle be added together the whole is irrational, and let it be called a first binomial straight line

For let two medial straight lines AB, BC commensurable in square only and containing a rational rectangle be added together

I say that the whole AC is irrational



For since AB is incommensurable in length with BC

therefore the squares on AB, BC are also incommensurable with twice the rectangle AB, BC ,

[cf x 36, II 9-20]

and *componendo* the squares on AB, BC together with twice the rectangle AB, BC , that is the square on AC [II 4] is incommensurable with the rectangle AB, BC

[x 16]

But the rectangle AB, BC is rational for by hypothesis AB, BC are straight lines containing a rational rectangle

therefore the square on AC is irrational,

therefore AC is irrational

[x Def 4]

And let it be called a first binomial straight line

Q E D

PROPOSITION 38

If two medial straight lines commensurable in square only and containing a medial rectangle be added together the whole is irrational and let it be called a second binomial straight line

For let two medial straight lines AB, BC commensurable in square only and containing a medial rectangle be added together



I say that AC is irrational

For let a rational straight line DE be set out and let the parallelogram DF equal to the square on AC be applied to DE producing DG as breadth

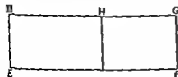
[I 44]

Then since the square on AC is equal to

the squares on AB, BC and twice the rectangle AB, BC

[II 4]

let EH equal to the squares on AB, BC be applied to DE



therefore the remainder HF is equal to twice the rectangle AB, BC

And, since each of the straight lines AB, BC is medial

therefore the squares on AB, BC are also medial

But by hypothesis, twice the rectangle AB, BC is also medial

And EH is equal to the squares on AB, BC ,

while FH is equal to twice the rectangle AB, BC ,

therefore each of the rectangles EH, HF is medial

And they are applied to the rational straight line DE ,

therefore each of the straight lines DH, HG is rational and incommensurable in length with DE [x 2]

Since then AB is incommensurable in length with BC ,

and as AB is to BC so is the square on AB to the rectangle AB, BC ,

therefore the square on AB is incommensurable with the rectangle AB, BC [x 11]

But the sum of the squares on AB, BC is commensurable with the square on AB [x 15]

and twice the rectangle AB, BC is commensurable with the rectangle AB, BC [x 6]

Therefore the sum of the squares on AB, BC is incommensurable with twice the rectangle AB, BC [x 13]

But EH is equal to the squares on AB, BC ,

and HF is equal to twice the rectangle AB, BC

Therefore EH is incommensurable with HF

so that DH is also incommensurable in length with HG [vi 1, x 11]

Therefore DH, HG are rational straight lines commensurable in square only, so that DG is irrational [x 36]

But DE is rational,

and the rectangle contained by an irrational and a rational straight line is irrational [cf x 20]

therefore the area DF is irrational

and the side of the square equal to it is irrational [x Def 4]

But AC is the side of the square equal to DF

therefore AC is irrational

And let it be called a second binomial straight line Q E D

PROPOSITION 39

If two straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial, be added together, the whole straight line is irrational and let it be called major

I or let two straight lines AB, PC incommensurable in square and fulfilling the given conditions [x 33] be added together

I say that AC is irrational

For, since the rectangle AB, BC is medial

twice the rectangle AB, BC is also medial [x 6 and 23 For]

But the sum of the squares on AB, BC is rational

therefore twice the rectangle AB, BC is incommensurable with the sum of the squares on AB, BC

so that the squares on AB, BC together with twice the rectangle AB, BC that



is, the square on AC , is also incommensurable with the sum of the squares on AB, BC , [x 16]

therefore the square on AC is irrational

so that AC is also irrational

[x Def 4]

And let it be called *major*

Q E D

PROPOSITION 40

If two straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational be added together, the whole straight line is irrational and let it be called the side of a rational plus a medial area

For let two straight lines AB, BC incommensurable in square and fulfilling the given conditions [x 34] be added together, I say that AC is irrational

For, since the sum of the squares on AB, BC is medial while twice the rectangle AB, BC is rational

therefore the sum of the squares on AB, BC is incommensurable with twice the rectangle AB, BC ,

so that the square on AC is also incommensurable with twice the rectangle AB, BC [x 16]

But twice the rectangle AB, BC = rational,

therefore the square on AC is irrational

Therefore AC is irrational

[x Def 4]

And let it be called the side of a rational plus a medial area

Q E D

PROPOSITION 41

If two straight lines incommensurable in square which make the sum of the squares on them medial and the rectangle contained by them medial and also incommensurable with the sum of the squares on them be added together, the whole straight line is irrational and let it be called the side of the sum of two medial areas

For let two straight lines AB, BC incommensurable in square and satisfying the given conditions [x 35] be added together

I say that AC is irrational

Let a rational straight line DE be set out and let there be applied to DE the rectangle DF equal to the squares on AB, BC , and the rectangle GH equal to twice the rectangle AB, BC

therefore the whole DH is equal to the square on AC [II 4]

Now since the sum of the squares on AB, BC is medial, and is equal to DF ,

therefore DF is also medial

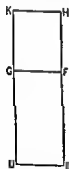
And it is applied to the rational straight line DE therefore DG is rational and incommensurable in length with DE [x 22]

For the same reason GK is also rational and incommensurable in length with GF that is DE

A B C

And since the squares on AB, BC are incommensurable with twice the rectangle AB, BC

DF is incommensurable with GH



so that DG is also incommensurable with GK [vi 1, x 11]

And they are rational

therefore DG GK are rational straight lines commensurable in square only,
therefore DK is irrational and what is called binomial [x 36]

But DE is rational

therefore DH is irrational and the side of the square which is equal to it is irrational [x Def 4]

But AC is the side of the square equal to HD ,
therefore AC is irrational

And let it be called the *side of the sum of two medial areas* Q E D

LEMMA

And that the aforesaid irrational straight lines are divided only in one way into the straight lines of which they are the sum and which produce the types in question we will now prove after premising the following lemma

Let the straight line AB be set out let the whole be cut into unequal parts at each of the points C D

and let AC be supposed greater than DB

I say that the squares on AC CB are greater than the squares on AD DB

For let AB be bisected at E

Then since AC is greater than DB ,

let DC be subtracted from each

therefore the remainder AD is greater than the remainder CB

But AE is equal to EB

therefore DE is less than EC

therefore the points C D are not equidistant from the point of bisection

And since the rectangle AC , CB together with the square on EC is equal to the square on EB [ii 5]

and further the rectangle AD DB together with the square on DE is equal to the square on EB [ii 4]

therefore the rectangle AC CB together with the square on EC is equal to the rectangle AD DB together with the square on DE

And of these the square on DE is less than the square on EC ,
therefore the remainder the rectangle AC CB is also less than the rectangle AD DB

so that twice the rectangle AC CB is also less than twice the rectangle AD , DB

Therefore also the remainder the sum of the squares on AC , CB , is greater than the sum of the squares on AD DB Q E D

PROPOSITION 42

A binomial straight line is divided into its terms at one point only

Let AB be a binomial straight line divided into its terms at C ,
therefore AC CB are rational straight lines
commensurable in square only

I say that AB is not divided at another point into two rational straight lines commensurable in square only

For if possible let it be divided at D also so that AD DB are also rational straight lines commensurable in square only

It is then manifest that AC is not the same with DB

For, if possible let it be so

Then AD will also be the same as CB ,

and as AC is to CB , so will BD be to DA ,

thus AB will be divided at D also in the same way as by the division at C which is contrary to the hypothesis

Therefore AC is not the same with DB

For this reason also the points C , D are not equidistant from the point of bisection

Therefore that by which the squares on AC , CB differ from the squares on AD DB is also that by which twice the rectangle AD DB differs from twice the rectangle AC , CB ,

because both the squares on AC CB together with twice the rectangle AC , CB and the squares on AD DB together with twice the rectangle AD , DB are equal to the square on AB [II 4]

But the squares on AC , CB differ from the squares on AD , DB by a rational area

for both are rational

therefore twice the rectangle AD , DB also differs from twice the rectangle AC , CB by a rational area though they are medial [X 21]

which is absurd for a medial area does not exceed a medial by a rational area [X 20]

Therefore a binomial straight line is not divided at different points

therefore it is divided at one point only Q E D

PROPOSITION 43

A first binomial straight line is divided at one point only

Let AB be a first binomial straight line divided at C , so that AC CB are medial straight lines commensurable in square only and containing a rational rectangle; [X 37]

I say that AB is not so divided at another point

For if possible let it be divided at D also, so that AD , DB are also medial straight lines commensurable in square only and containing a rational rectangle

Since then that by which twice the rectangle AD DB differs from twice the rectangle AC CB is that by which the squares on AC CB differ from the squares on AD DB

while twice the rectangle AD DB differs from twice the rectangle AC , CB by a rational area—for both are rational—

therefore the squares on AC CB also differ from the squares on AD DB by a rational area though they are medial

which is absurd

[X 20]

Therefore a first binomial straight line is not divided into its terms at different points

therefore it is so divided at one point only

Q E D

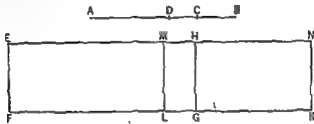
PROPOSITION 44

A second bimedial straight line is divided at one point only

Let AB be a second bimedial straight line divided at C , so that AC , CB are medial straight lines commensurable in square only and containing a medial rectangle, [x 38]

it is then manifest that C is not at the point of bisection, because the segments are not commensurable in length

I say that AB is not so divided at another point



For if possible let it be divided at D also so that AC is not the same with DB , but AC is supposed greater

it is then clear that the squares on AD , DB are also as we proved above [Lemma] less than the squares on AC , CB

and suppose that AD , DB are medial straight lines commensurable in square only and containing a medial rectangle

Now let a rational straight line LF be set out

let there be applied to EF the rectangular parallelogram EA equal to the square on AB ,

and let EG , equal to the squares on AC , CB , be subtracted therefore the remainder HK is equal to twice the rectangle AC , CB [ii 4]

Again let there be subtracted EL equal to the squares on AD , DB , which were proved less than the squares on AC , CB [Lemma]

therefore the remainder MA is also equal to twice the rectangle AD , DB

Now since the squares on AC , CB are medial

therefore EG is medial

And it is applied to the rational straight line EF ,

therefore EH is rational and incommensurable in length with EF [x 27]

For the same reason

HN is also rational and incommensurable in length with EF

And since AC , CB are medial straight lines commensurable in square only

therefore AC is incommensurable in length with CB

But as AC is to CB so is the square on AC to the rectangle AC , CB , therefore the square on AC is incommensurable with the rectangle AC , CB [x 11]

But the squares on AC , CB are commensurable with the square on AC , for AC , CB are commensurable in square [x 15]

And twice the rectangle AC , CB is commensurable with the rectangle AC , CB [x 6]

Therefore the squares on AC , CB are also incommensurable with twice the rectangle AC , CB [x 13]

But EG is equal to the squares on AC CB ,
 and HA is equal to twice the rectangle AC CB ,
 therefore EG is incommensurable with HA ,

so that EH is also incommensurable in length with HN [VI 1, x 11]

And they are rational

therefore EH HN are rational straight lines commensurable in square only

But if two rational straight lines commensurable in square only be added together the whole is the irrational which is called binomial [x 36]

Therefore EN is a binomial straight line divided at H [x 36]

In the same way EM , MN will also be proved to be rational straight lines commensurable in square only,

and EN will be a binomial straight line divided at different points H and M

And EH is not the same with MN

For the squares on AC CB are greater than the squares on AD , DB

But the squares on AD , DB are greater than twice the rectangle AD DB
 therefore also the squares on AC CB , that is, EG are much greater than twice the rectangle AD , DB that is MA

so that EH is also greater than MN

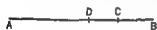
Therefore EH is not the same with MN

Q E D

PROPOSITION 45

A major straight line is divided at one and the same point only

Let AB be a major straight line divided at C , so that AC CB are incommensurable in square and make the sum of the squares on AC CB rational but the rectangle AC CB medial,



I say that AB is not so divided at another point

For if possible let it be divided at D also so that AD DB are also incommensurable in square and make the sum of the squares on AD DB rational but the rectangle contained by them medial

Then since that by which the squares on AC CB differ from the squares on AD DB is also that by which twice the rectangle AD DB differs from twice the rectangle AC CB

while the squares on AC CB exceed the squares on AD DB by a rational area—for both are rational—

therefore twice the rectangle AD DB also exceeds twice the rectangle AC CB by a rational area though they are medial

which is impossible

[x 26]

Therefore a major straight line is not divided at different points

therefore it is only divided at one and the same point Q E D

PROPOSITION 46

The side of a rational plus a medial area is divided at one point only

Let AB be the side of a rational plus a medial area divided at C so that AC , CB are incommensurable in square and make the sum of the squares on AC CB medial but twice the rectangle AC CB rational [x 40]



I say that AB is not so divided at another point

For if possible let it be divided at D also so that AD DB are also incommensurable in square and make the sum of the squares on AD DB medial but twice the rectangle AD DB rational

measurable in square and make the sum of the squares on AD DB medial but twice the rectangle AD , DB rational

Since then that by which twice the rectangle AC CB differs from twice the squares on AD DB is also that by which the squares on AD DB differ from the squares on AC CB

while twice the rectangle AC CB exceeds twice the rectangle AD DB by a rational area,

therefore the squares on AD , DB also exceed the squares on AC , CB by a rational area though they are medial

which is impossible

[x 26]

Therefore the side of a rational plus a medial area is not divided at different points,

therefore it is divided at one point only

Q E D

PROPOSITION 47

The side of the sum of two medial areas is divided at one point only

Let AB be divided at C so that AC CB are incommensurable in square and make the sum of the squares on AC CB medial and the rectangle AC , CB medial and also incommensurable with the sum of the squares on them

I say that AB is not divided at another point so as to fulfil the given conditions

For if possible let it be divided at D so that again AC is of course not the same as BD but AC is supposed greater

let a rational straight line EF be set out

and let there be applied to EF the rectangle EG equal to the squares on AC CB

and the rectangle HA equal to twice the rectangle AC CB

therefore the whole EA is equal to the square on AB

[ii 4]

Again let EL equal to the squares on AD DB be applied to EF , therefore the remainder twice the rectangle AD DB is equal to the remainder MA

And since by hypothesis the sum of the squares on AC CB is medial therefore EG is also medial

And it is applied to the rational straight line EF

therefore HE is rational and incommensurable in length with EF

[x 27]

For the same reason

HA is also rational and incommensurable in length with EF

And since the sum of the squares on AC CB is incommensurable with twice the rectangle AC CB

therefore EG is also incommensurable with GN

so that EH is also incommensurable with HN

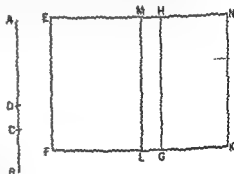
[xi 1 & 11]

And they are rational

therefore EH HN are rational straight lines commensurable in square only,

therefore EN is a binomial straight line divided at H

[x 36]



Similarly we can prove that it is also divided at M

And EH is not the same with MN ,

therefore a binomial has been divided at different points

which is absurd

[x 42]

Therefore a side of the sum of two medial areas is not divided at different points

therefore it is divided at one point only

Q E D

DEFINITIONS II

1 Given a rational straight line and a binomial divided into its terms such that the square on the greater term is greater than the square on the lesser by the square on a straight line commensurable in length with the greater, then if the greater term be commensurable in length with the rational straight line set out, let the whole be called a *first binomial* straight line

2 but if the lesser term be commensurable in length with the rational straight line set out let the whole be called a *second binomial*

3 and if neither of the terms be commensurable in length with the rational straight line set out let the whole be called a *third binomial*

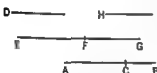
4 Again if the square on the greater term be greater than the square on the lesser by the square on a straight line incommensurable in length with the greater then if the greater term be commensurable in length with the rational straight line set out let the whole be called a *fourth binomial*,

5 if the lesser a *fifth binomial*,

6 and if neither a *sixth binomial*

PROPOSITION 48

To find the first binomial straight line

Let two numbers AC CB be set out such that the sum of them AB has to BC the ratio which a square number has to a square number but has not to CA the ratio which a square number has to a square number, [Lemma 1 after x 28]

 let any rational straight line D be set out and let EF be commensurable in length with D

Therefore EF is also rational

Let it be contrived that

as the number BA is to AC so is the square on EF to the square on FG

[x 6 Por]

But AB has to AC the ratio which a number has to a number therefore the square on EF also has to the square on FG the ratio which a number has to a number

so that the square on EF is commensurable with the square on FG [x 6]

And EF is rational

therefore FG is also rational

And since BA has not to AC the ratio which a square number has to a square number

neither therefore has the square on EF to the square on FG the ratio which a square number has to a square number

therefore EF is incommensurable in length with FG [x 9]

Therefore EF , FG are rational straight lines commensurable in square only,
therefore EG is binomial [x 36]

I say that it is also a first binomial straight line

For since as the number BA is to AC , so is the square on EF to the square on FG ,

while BA is greater than AC ,

therefore the square on EF is also greater than the square on FG

Let then the squares on FG , H be equal to the square on EF

Now since, as BA is to AC , so is the square on EF to the square on FG ,
therefore, *convertendo*

as AB is to BC , so is the square on EF to the square on H [x 10 Por]

But AB has to BC the ratio which a square number has to a square number
therefore the square on EF also has to the square on H the ratio which a square number has to a square number

Therefore EF is commensurable in length with H [x 9]
therefore the square on EF is greater than the square on FG by the square on a straight line commensurable with EF

And EF , FG are rational and EF is commensurable in length with D

Therefore EF is a first binomial straight line Q.E.D.

PROPOSITION 49

To find the second binomial straight line

Let two numbers AC , CB be set out such that the sum of them AB has to BC the ratio which a square number has to a square number but has not to AC the ratio which a square number has to a square number,
let a rational straight line D be set out and let EF be commensurable in length with D

therefore EF is rational

Let it be contrived then that

as the number CA is to AB so also is the square on EF to the square on FG [x 6 Por]

therefore the square on EF is commensurable with the square on FG [x 6]

Therefore FG is also rational

Now since the number CA has not to AB the ratio which a square number has to a square number neither has the square on EF to the square on FG the ratio which a square number has to a square number

Therefore EF is incommensurable in length with FG , [x 9]
therefore EF , FG are rational straight lines commensurable in square only

therefore EG is binomial [x 36]

It is next to be proved that it is also a second binomial straight line

For since inversely as the number BA is to AC , so is the square on GF to the square on FE ,

while BA is greater than AC

therefore the square on GF is greater than the square on FE

Let the squares on EF , H be equal to the square on GF ,
therefore *convertendo* as AB is to BC so is the square on FG to the square on H [x 19 Por]

But AB has to BC the ratio which a square number has to a square number,



therefore the square on FG also has to the square on H the ratio which a square number has to a square number

Therefore FG is commensurable in length with H , [x 9]
so that the square on FG is greater than the square on FE by the square on a straight line commensurable with FG

And FG FE are rational straight lines commensurable in square only and EF , the lesser term is commensurable in length with the rational straight line D set out

Therefore EG is a second binomial straight line

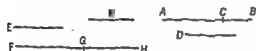
Q E D

PROPOSITION 50

To find the third binomial straight line

Let two numbers AC , CB be set out such that the sum of them AB has to BC the ratio which a square number has to a square number but has not to AC the ratio which a square number has to a square number

Let any other number D , not square be set out also, and let it not have to either of the numbers BA , AC the ratio which a square number has to a square number



Let any rational straight line E be set out and let it be contrived that as D is to AB so is the square on E to the square on FG , [x 6, Por]

therefore the square on E is commensurable with the square on FG [x 6]

And E is rational,

therefore FG is also rational

And since D has not to AB the ratio which a square number has to a square number

neither has the square on E to the square on FG the ratio which a square number has to a square number

therefore E is incommensurable in length with FG [x 9]

Next let it be contrived that as the number BA is to AC so is the square on FG to the square on GH [x 6 Por]

therefore the square on FG is commensurable with the square on GH [x 6]

But FG is rational

therefore GH is also rational

And since BA has not to $1C$ the ratio which a square number has to a square number

neither has the square on FG to the square on HQ the ratio which a square number has to a square number

therefore FG is incommensurable in length with GH [x 9]

Therefore FG GH are rational straight lines commensurable in square only

therefore FH is binomial [x 36]

I say next that it is also a third binomial straight line

For since as D is to AB so is the square on E to the square on FG

and as BA is to AC so is the square on FG to the square on GH

therefore ex aequali, as D is to AC so is the square on E to the square on GH

[x 22]

But D has not to AC the ratio which a square number has to a square number,
therefore neither has the square on E to the square on GH the ratio which a square number has to a square number,

therefore E is incommensurable in length with GH [x 9]

And since, as BA is to AC , so is the square on FG to the square on GH ,

therefore the square on FG is greater than the square on GH

Let then the squares on GH , A be equal to the square on FG ,

therefore *convertendo* as AB is to BC , so is the square on FG to the square on K [v 19, Por]

But AB has to BC the ratio which a square number has to a square number,
therefore the square on FG also has to the square on K the ratio which a square number has to a square number,

therefore FG is commensurable in length with A [x 9]

Therefore the square on FG is greater than the square on GH by the square on a straight line commensurable with FG

And FG , GH are rational straight lines commensurable in square only, and neither of them is commensurable in length with E

Therefore FH is a third binomial straight line

Q E D

PROPOSITION 51

To find the fourth binomial straight line

Let two numbers AC , CB be set out such that AB neither has to BC , nor yet to AC the ratio which a square number has to a square number

Let a rational straight line D be set out

and let EF be commensurable in length with D ,

therefore EF is also rational

Let it be contrived that, as the number BA is to AC so is the square on EF to the square on FG [x 6, Por]

therefore the square on EF is commensurable with the square on FG , [x 6]

therefore FG is also rational

Now since BA has not to AC the ratio which a square number has to a square number

neither has the square on EF to the square on FG the ratio which a square number has to a square number

therefore EF is incommensurable in length with FG [x 9]

Therefore EF , FG are rational straight lines commensurable in square only so that EG is binomial

I say next that it is also a fourth binomial straight line

For since as BA is to AC so is the square on EF to the square on FG

therefore the square on EF is greater than the square on FG

Let then the squares on FG , H be equal to the square on EF ,
therefore *convertendo* as the number AB is to BC , so is the square on EF to the square on H [v 19 Por]

But AB has not to BC the ratio which a square number has to a square number

therefore neither has the square on EF to the square on H the ratio which a square number has to a square number

Therefore EF is incommensurable in length with H , [x 9]
 therefore the square on EF is greater than the square on GF by the square on
 a straight line incommensurable with EF

And EF , FG are rational straight lines commensurable in square only, and
 EF is commensurable in length with D

Therefore EG is a fourth binomial straight line Q E D

PROPOSITION 52

To find the fifth binomial straight line

Let two numbers AC , CB be set out such that AB has not to either of them
 the ratio which a square number has to a square number,

let any rational straight line D be set out

and let EF be commensurable with D ,

therefore EF is rational

Let it be contrived that, as CA is to AB , so is the square on EF
 to the square on FG [x 11, Por.]

But CA has not to AB the ratio which a square number
 has to a square number,

therefore neither has the square on EF to the square on FG the
 ratio which a square number has to a square number

Therefore EF , FG are rational straight lines commensurable
 in square only, [x 9]

therefore EG is binomial [x 36]

I say next that it is also a fifth binomial straight line

For since, as CA is to AB so is the square on EF to the square on FG
 inversely, as BA is to AC so is the square on FG to the square on FE ,

therefore the square on GF is greater than the square on FE

Let then the squares on EF , H be equal to the square on GF ,
 therefore, *convertendo* as the number AB is to BC , so is the square on GF to
 the square on H [v 19 Por.]

But AB has not to BC the ratio which a square number has to a square
 number

therefore neither has the square on FG to the square on H the ratio which a
 square number has to a square number

Therefore FG is incommensurable in length with H [x 9]
 so that the square on FG is greater than the square on FE by the square on a
 straight line incommensurable with FG

And GF , FE are rational straight lines commensurable in square only and
 the lesser term EF is commensurable in length with the rational straight line
 D set out

Therefore EG is a fifth binomial straight line Q E D

PROPOSITION 53

To find the sixth binomial straight line

Let two numbers AC , CB be set out such that AB has not to either of them
 the ratio which a square number has to a square number

and let there also be another number D which is not square and which has not
 to either of the numbers BA , AC the ratio which a square number has to a
 square number

Let any rational straight line E be set out,
and let it be contrived that, as D is to AB so is the square on E to the square on FG [x 6 Por]

therefore the square on E is commensurable with the square on FG [x 6]

And E is rational

therefore FG is also rational

Now since D has not to AB the ratio which a square number has to a square number

neither has the square on E to the square on FG the ratio which a square number has to a square number,
therefore E is incommensurable in length with FG [x 9]

Again let it be contrived that, as BA is to AC , so is the square on FG to the square on GH [x 6, Por]

Therefore the square on FG is commensurable with the square on GH [x 6]

Therefore the square on GH is rational

therefore GH is rational

And since BA has not to AC the ratio which a square number has to a square number,

neither has the square on FG to the square on GH the ratio which a square number has to a square number,

therefore FG is incommensurable in length with GH [x 9]

Therefore FG, GH are rational straight lines commensurable in square only

therefore FH is binomial [x 10]

It is next to be proved that it is also a sixth binomial straight line

For since as D is to AB so is the square on E to the square on FG
and also as BA is to AC , so is the square on FG to the square on GH
therefore *ex aequali* as D is to AC so is the square on E to the square on GH [x 22]

But D has not to AC the ratio which a square number has to a square number

therefore neither has the square on E to the square on GH the ratio which a square number has to a square number

therefore E is incommensurable in length with GH [x 9]

But it was also proved incommensurable with FG

therefore each of the straight lines FG, GH is incommensurable in length with E

And since as BA is to AC so is the square on FG to the square on GH
therefore the square on FG is greater than the square on GH

Let then the squares on CH, A be equal to the square on FG

therefore *convertendo* as AB is to BC , so is the square on FG to the square on A [x 19 Por]

But AB has not to BC the ratio which a square number has to a square number

so that neither has the square on FG to the square on A the ratio which a square number has to a square number

Therefore FG is incommensurable in length with A [x 9]

therefore the square on FG is greater than the square on GH by the square on a straight line incommensurable with FG

And FG GH are rational straight lines commensurable in square only and neither of them is commensurable in length with the rational straight line E set out

Therefore FH is a sixth binomial straight line

Q E D

LEMMA

Let there be two squares AB , BC , and let them be placed so that DB is in a straight line with BE ,

therefore FB is also in a straight line with BG

Let the parallelogram AC be completed

I say that AC is a square that DG is a mean proportional between AB , BC and further that DC is a mean proportional between AC , CB

For, since DB is equal to BF and BE to BG ,

therefore the whole DE is equal to the whole FG

But DE is equal to each of the straight lines AH , KC ,

and FG is equal to each of the straight lines AK , HC [I. 34],

therefore each of the straight lines AH , KC is also equal to each of the straight lines AK , HC

Therefore the parallelogram AC is equilateral

And it is also rectangular

therefore AC is a square

And since as FB is to BG so is DB to BE

while as FB is to BG so is AB to DG ,

and as DB is to BE , so is DG to BC

[VI. 1]

therefore also as AB is to DG so is DG to BC

[V. 11]

Therefore DG is a mean proportional between AB BC

I say next that DC is also a mean proportional between AC CB

For since as AD is to DK so is KG to GC —

for they are equal respectively—

and *componendo* as AK is to AD so is AC to CG

[V. 18]

while as AK is to AD so is AC to CD

and as KC is to CG so is DC to CB

[VI. 1]

therefore also as AC is to DC so is DC to CB

[V. 11]

Therefore DC is a mean proportional between AC CB

Being what it was proposed to prove

PROPOSITION 54

If an area be contained by a rational straight line and the first binomial, the side of the area is the irrational straight line which is called binomial

For let the area AC be contained by the rational straight line AB and the first binomial AD ,

I say that the side of the area AC is the irrational straight line which is called binomial

For since AD is a first binomial straight line let it be divided into its terms at E

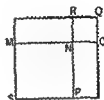
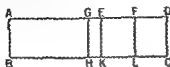
and let AE be the greater term

It is then manifest that AE ED are rational straight lines commensurable in square only

the square on AE is greater than the square on ED by the square on a straight line commensurable with AE ,
 and AE is commensurable in length with the rational straight line AB set out

1 2 11 [x Def 11 1]

Let ED be bisected at the point F



Then since the square on AE is greater than the square on ED by the square on a straight line commensurable with AE

therefore, if there be applied to the greater AE a parallelogram equal to the fourth part of the square on the less that is to the square on EF , and deficient by a square figure it divides it into commensurable parts [x 17]

Let then the rectangle AG GE equal to the square on EF be applied to AE
 therefore AG is commensurable in length with FG

Let GH EK FL be drawn from G E F parallel to either of the straight lines AB CD

let the square SN be constructed equal to the parallelogram AH and the square NQ equal to GA [ii 14]

and let them be placed so that MN is in a straight line with NO ,

therefore RN is also in a straight line with NP

And let the parallelogram SQ be completed

therefore SQ is a square

[Lemma]

Now since the rectangle AG GE is equal to the square on EF

therefore as AG is to EF so is FE to EG ,

[vi 14]

therefore also as AH is to EL so is EL to AG

[vi 11]

therefore EL is a mean proportional between AH GA

But AH is equal to SN and GA to NQ

therefore EL is a mean proportional between SN NQ

But MR is also a mean proportional between the same SN NQ [Lemma]

therefore EL is equal to MR

so that it is also equal to PO

But AH GK are also equal to SN NQ

therefore the whole AC is equal to the whole SQ that is to the square on MO

therefore MO is the side of AC

I say next that MO is binomial

For since AC is commensurable with GE

therefore AF is also commensurable with each of the straight lines AG GE [x 15]

But AE is also by hypothesis commensurable with AB

therefore AG GE are also commensurable with AB [x 12]

And AB is rational

therefore each of the straight lines AG GE is also rational

therefore each of the rectangles AH GA is rational, [x 19]

and AH is commensurable with GK

But AH is equal to SN , and GK to ΛQ ,
therefore SN, NQ that is, the squares on $MN, \Lambda O$ are rational and commensurable

And since AE is incommensurable in length with ED
while AE is commensurable with AG , and DE is commensurable with EF
therefore AG is also incommensurable with EF [x 13]
so that AH is also incommensurable with EL [vi 1, x 11]

But AH is equal to SN , and EL to MR ,
therefore SN is also incommensurable with MR

But, as SN is to MR , so is PN to NR , [vi 1]
therefore PN is incommensurable with NR [x 11]

But PN is equal to MN , and NR to NO ,
therefore MN is incommensurable with NO

And the square on MN is commensurable with the square on NO
and each is rational,

therefore MN, NO are rational straight lines commensurable in square only
Therefore MO is binomial [x 36] and the side of AC Q E D

PROPOSITION 55

If an area be contained by a rational straight line and the second binomial the side of the area is the irrational straight line which is called a first bimedial

For let the area $ABCD$ be contained by the rational straight line AB and the second binomial AD ,

I say that the side of the area AC is a first bimedial straight line

For, since AD is a second binomial straight line let it be divided into its terms at E so that AE is the greater term

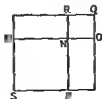
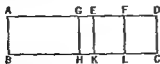
therefore AE, ED are rational straight lines commensurable in square only, the square on AE is greater than the square on ED by the square on a straight line commensurable with AE

and the lesser term ED is commensurable in length with AB [x Def II 2]

Let ED be bisected at F
and let there be applied to AE the rectangle $AG GE$ equal to the square on EF and deficient by a square figure

therefore AG is commensurable in length with GE [x 17]

Through $G E F$ let $GH EK FL$ be drawn parallel to $AB CD$
let the square SN be constructed equal to the parallelogram AH and the square NQ equal to GK



and let them be placed so that MN is in a straight line with NO

therefore RN is also in a straight line with NP

Let the square SQ be completed

It is then manifest from what was proved before that MR is a mean proportional between SN , NQ and is equal to EL , and that MO is the "side" of the area AC

It is now to be proved that MO is a first bimedial straight line

Since AE is commensurable in length with ED

while ED is commensurable with AB

therefore AE is incommensurable with AB [x 13]

And since AG is commensurable with EG ,

AE is also commensurable with each of the straight lines AG , GE [x 15]

But AE is incommensurable in length with AB ,

therefore AG , GE are also incommensurable with AB [x 13]

Therefore BA , AG and BA , GE are pairs of rational straight lines commensurable in square only

so that each of the rectangles AH , GK is medial [x 21]

Hence each of the squares SN , NQ is medial

Therefore MN , NO are also medial

And since AG is commensurable in length with GE

AH is also commensurable with GK ,

[xi 1, x 11]

that is SN is commensurable with NQ

that is the square on MN with the square on NO

And since AE is incommensurable in length with ED ,

while AE is commensurable with AG ,

and ED is commensurable with EF

therefore AG is incommensurable with EF , [x 13]

so that AH is also incommensurable with EL ,

that is SN is incommensurable with MR ,

that is PN with NR ,

[vi 1, x 11]

that is MN is incommensurable in length with NO

But MA , NO were proved to be both medial and commensurable in square

therefore MA , NO are medial straight lines commensurable in square only

I say next that they also contain a rational rectangle

For since DE is by hypothesis commensurable with each of the straight lines AB , EF

therefore EF is also commensurable with EA [x 14]

And each of them is rational,

therefore EL that is MR is rational

[x 19]

and MR is the rectangle MN , NO

But if two medial straight lines commensurable in square only and containing a rational rectangle be added together the whole is irrational and is called a first bimedial straight line [x 31]

Therefore MO is a first bimedial straight line

Q E D

PROPOSITION 56

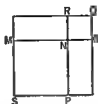
If an area be contained by a rational straight line and the third binomial the "side" of the area is the irrational straight line called a second bimedial

For let the area $ABCD$ be contained by the rational straight line AB and the third binomial AD divided into its terms at E of which terms AE is the greater

I say that the "side" of the area AC is the irrational straight line called a second bimedial

For let the same construction be made as before

Now, since AD is a third binomial straight line,



therefore AE , ED are rational straight lines commensurable in square only, the square on AE is greater than the square on ED by the square on a straight line commensurable with AE , and neither of the terms AE , ED is commensurable in length with AB

[x Def 11 3]

Then in manner similar to the foregoing, we shall prove that MO is the 'side' of the area AC

and MN NO are medial straight lines commensurable in square only, so that MO is bimedial

It is next to be proved that it is also a second bimedial straight line

Since DE is incommensurable in length with AB that is with EK

and DE is commensurable with EF

therefore EF is incommensurable in length with EK [x 13]

And they are rational,

therefore FE EK are rational straight lines commensurable in square only

Therefore EL that is MR is medial

[x 21]

And it is contained by MN , NO

therefore the rectangle MN NO is medial

Therefore MO is a second bimedial straight line

[x 38]

Q E D

PROPOSITION 57

If an area be contained by a rational straight line and the fourth binomial the 'side' of the area is the irrational straight line called major

For let the area AC be contained by the rational straight line AB and the fourth binomial AD divided into its terms at E , of which terms let AE be the greater

I say that the side of the area AC is the irrational straight line called major

For since AD is a fourth binomial straight line

therefore AE ED are rational straight lines commensurable in square only the square on AE is greater than the square on ED by the square on a straight line incommensurable with AE

and AE is commensurable in length with AB [x Def 11 4]

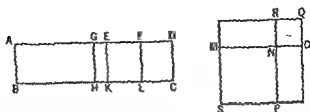
Let DE be bisected at F

and let there be applied to AE a parallelogram the rectangle AG , GE equal to the square on EF

therefore AG is incommensurable in length with GE [x 18]

Let GH EK FL be drawn parallel to AB

and let the rest of the construction be as before
it is then manifest that MO is the 'side' of the area AC



It is next to be proved that MO is the irrational straight line called major
Since AG is incommensurable with EG

AH is also incommensurable with GK that is SN with NQ [v 1, x 11]

therefore MN NO are incommensurable in square

And since AE is commensurable with AB

AH is rational [x 19]

and it is equal to the squares on MN NO

therefore the sum of the squares on MN NO is also rational

And, since DE is incommensurable in length with AB that is with EK

while DE is commensurable with EF ,

therefore EF is incommensurable in length with EK [x 13]

Therefore EA EF are rational straight lines commensurable in square only

therefore LE that is, MR , is medial [x 21]

And it is contained by MN NO ,

therefore the rectangle MN NO is medial

And the [sum] of the squares on MN NO is rational

and MN NO are incommensurable in square

But if two straight lines incommensurable in square and making the sum of the squares on them rational but the rectangle contained by them medial be added together the whole is irrational and is called major [x 39]

Therefore MO is the irrational straight line called major and is the 'side' of the area AC Q E D

PROPOSITION 58

If an area be contained by a rational straight line and the fifth binomial the 'side' of the area is the irrational straight line called the side of a rational plus a medial area

For let the area AC be contained by the rational straight line AB and the fifth binomial AD divided into its terms at E so that AE is the greater term,

I say that the 'side' of the area AC is the irrational straight line called the side of a rational plus a medial area

For let the same construction be made as before shown,

it is then manifest that MO is the 'side' of the area AC

It is then to be proved that MO is the side of a rational plus a medial area

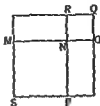
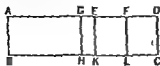
For since AG is incommensurable with GE [x 18]

therefore EH is also commensurable with HE [vi 1, x 11]

that is the square on MN with the square on NO ,

therefore MN NO are incommensurable in square

And since AD is a fifth binomial straight line and ED the lesser segment therefore ED is commensurable in length with AB [X. Def. 11. 5]



But AE is incommensurable with ED ,

therefore AB is also incommensurable in length with AE [X. 13]

Therefore AA that is the sum of the squares on MN NO , is medial [X. 21]

And, since DE is commensurable in length with AB , that is, with FA ,

while DE is commensurable with EF ,

therefore EF is also commensurable with EA [X. 12]

And EK is rational,

therefore EL that is, MR , that is the rectangle MN NO , is also rational [X. 19]

Therefore MN , NO are straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational

Therefore MO is the side of a rational plus a medial area [X. 40] and is the "side of the area AC " Q.E.D.

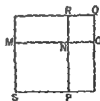
PROPOSITION 50

If an area be contained by a rational straight line and the sixth binomial, the "side" of the area is the irrational straight line called the side of the sum of two medial areas

For let the area $ABCD$ be contained by the rational straight line AB and the sixth binomial AD , divided into its terms at E so that AE is the greater term

I say that the "side" of AC is the side of the sum of two medial areas

Let the same construction be made as before shown



It is then manifest that MO is the "side" of AC and that MA is incommensurable in square with NO

Now since EA is incommensurable in length with AB

therefore EA AB are rational straight lines commensurable in square only therefore AA that is the sum of the squares on MN NO is medial [X. 21]

Again since ED is incommensurable in length with AB ,

therefore FE is also incommensurable with EK , [x 13]
 therefore FE, EK are rational straight lines commensurable in square only,
 therefore EL that is MR , that is the rectangle MN, NO is medial [x 9]
 And since AE is incommensurable with EF

AK is also incommensurable with EL [vi 1 x 11]

But AK is the sum of the squares on MN, NO

and EL is the rectangle MN, NO

therefore the sum of the squares on MN, NO is incommensurable with the rectangle MN, NO

And each of them is medial, and MN, NO are incommensurable in square

Therefore MO is the side of the sum of two medial areas [x 41] and is the side of AC Q E D

LEMMA

If a straight line be cut into unequal parts, the squares on the unequal parts are greater than twice the rectangle contained by the unequal parts

Let AB be a straight line and let it be cut into unequal parts at C and let AC be the greater

I say that the squares on AC, CB are greater than twice the rectangle AC, CB
 For let AB be bisected at D

Since then, a straight line has been cut into equal parts at D , and into unequal parts at C

therefore the rectangle AC, CB together with the square on CD is equal to the square on AD [ii 5]

so that the rectangle AC, CB is less than the square on AD ,
 therefore twice the rectangle AC, CB is less than double of the square on AD

But the squares on AC, CB are double of the squares on AD, DC , [ii 8]
 therefore the squares on AC, CB are greater than twice the rectangle AC, CB Q E D

PROPOSITION 60

The square on the binomial straight line applied to a rational straight line produces as breadth the first binomial

Let AB be a binomial straight line divided into its terms at C so that AC is the greater term

let a rational straight line DF be set out
 and let $DEFG$ equal to the square on AB be applied to DE producing DG as its breadth

I say that DG is a first binomial straight line

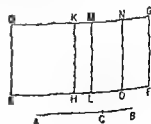
For let there be applied to DE the rectangle DH equal to the square on AC and KL equal to the square on BC

therefore the remainder twice the rectangle AC, CB is equal to MF

Let MG be bisected at N and let NO be drawn parallel [to ML or GF]

Therefore each of the rectangles MO, NF is equal to once the rectangle AC, CB

Now, since AB is a binomial divided into its terms at C



therefore AC , CB are rational straight lines commensurable in square only, [x 36]
 therefore the squares on AC , CB are rational and commensurable with one another

so that the sum of the squares on AC , CB is also rational [x 15]

And it is equal to DL ,

therefore DL is rational

And it is applied to the rational straight line DE ,

therefore DM is rational and commensurable in length with DE [x 20]

Again since AC , CB are rational straight lines commensurable in square only

therefore twice the rectangle AC , CB that is MF is medial [x 21]

And it is applied to the rational straight line ML ,

therefore MG is also rational and incommensurable in length with ML , that is DE [x 23]

But MD is also rational and is commensurable in length with DE ,

therefore DV is incommensurable in length with MG [x 13]

And they are rational

therefore DM , MG are rational straight lines commensurable in square only

therefore DG is binomial [x 30]

It is next to be proved that it is also a first binomial straight line

Since the rectangle AC , CB is a mean proportional between the squares on AC , CB [cf Lemma after x 53]

therefore MO is also a mean proportional between DH , KL

Therefore, as DH is to MO so is MO to KL

that is as DA is to AN so is MA to AK [vi 1]

therefore the rectangle DK , AW is equal to the square on MA [vi 17]

And since the square on AC is commensurable with the square on CB

DH is also commensurable with KL

so that DA is also commensurable with AK [vi 1 x 11]

And, since the squares on AC , CB are greater than twice the rectangle AC , CB , [Lemma]

therefore DL is also greater than MF ,

so that DM is also greater than MG [vi 1]

And the rectangle DA , AK is equal to the square on AN that is to the fourth part of the square on MQ

and DA is commensurable with AK

But if there be two unequal straight lines and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure and if it divide it into commensurable parts the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater [x 17]

therefore the square on DV is greater than the square on MG by the square on a straight line commensurable with DM

And DM , MG are rational

and DV , which is the greater term is commensurable in length with the rational straight line DE set out

Therefore DG is a first binomial straight line

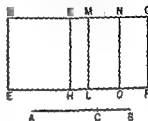
[x Def II 1]

Q E D

PROPOSITION 61

The square on the first binomial straight line applied to a rational straight line produces as breadth the second binomial

Let AB be a first binomial straight line divided into its medials at C , of which medials AC is the greater, let a rational straight line DE be set out, and let there be applied to DE the parallelogram DF equal to the square on AB , producing DG as its breadth



I say that DG is a second binomial straight line
 For let the same construction as before be made
 Then, since AB is a first binomial divided at C ,
 therefore AC CB are medial straight lines commensurable in square only and containing a rational rectangle,
 so that the squares on AC , CB are also medial [x 3d]
 Therefore DL is medial [x 15 and 13, Por.]
 And it has been applied to the rational straight line DE ,
 therefore MD is rational and incommensurable in length with DE [x 27]
 Again, since twice the rectangle AC CB is rational MF is also rational
 And it is applied to the rational straight line ML
 therefore MG is also rational and commensurable in length with ML that is DE [x 20]

therefore DM is incommensurable in length with MG [x 13]

And they are rational

therefore DM MG are rational straight lines commensurable in square only
 therefore DG is binomial [x 3b]

It is next to be proved that it is also a second binomial straight line

For, since the squares on AC CB are greater than twice the rectangle AC , CB

therefore DL is also greater than MF

so that DM is also greater than MG [vi 1]

And, since the square on AC is commensurable with the square on CB ,

DH is also commensurable with KL

so that DK is also commensurable with AM [vi 1 & 11]

And the rectangle DK AM is equal to the square on MN ,
 therefore the square on DM is greater than the square on MG by the square on a straight line commensurable with DM [x 17]

And MG is commensurable in length with DE

Therefore DG is a second binomial straight line [x Def. 11]

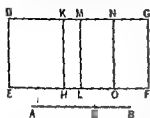
PROPOSITION 62

The square on the second binomial straight line applied to a rational straight line produces as breadth the third binomial

Let AB be a second binomial straight line divided into its medials at C , so that AC is the greater segment
 let DE be any rational straight line
 and to DE let there be applied the parallelogram DF equal to the square on AB and producing DG as its breadth,

I say that DG is a third binomial straight line

Let the same construction be made as before shown



Then, since AB is a second binomial divided at C ,

therefore AC , CB are medial straight lines commensurable in square only and containing a medial rectangle [x 35]

so that the sum of the squares on AC , CB is also medial [x 15 and 23 Por]

And it is equal to DL

therefore DL is also medial

And it is applied to the rational straight line DE

therefore MD is also rational and incommensurable in length with DE [x 22]

For the same reason

MG is also rational and incommensurable in length with ML that is, with DE , therefore each of the straight lines DM , MG is rational and incommensurable in length with DE

And since AC is incommensurable in length with CB ,

and as AC is to CB so is the square on AC to the rectangle AC , CB ,

therefore the square on AC is also incommensurable with the rectangle AC , CB [x 11]

Hence the sum of the squares on AC , CB is incommensurable with twice the rectangle AC , CB , [x 12 13]

that is, DL is incommensurable with MF ,

so that DM is also incommensurable with MG [vi 1, x 11]

And they are rational,

therefore DG is binomial

[x 36]

It is to be proved that it is also a third binomial straight line

In manner similar to the foregoing we may conclude that DM is greater than MG

and that DK is commensurable with KM

And the rectangle DK , KM is equal to the square on MN ,

therefore the square on DM is greater than the square on MG by the square on a straight line commensurable with DM

And neither of the straight lines DM , MG is commensurable in length with DE

Therefore DG is a third binomial straight line

[x Def II 3]

Q E D

PROPOSITION 63

The square on the major straight line applied to a rational straight line produces as breadth the fourth binomial

Let AB be a major straight line divided at C so that AC is greater than CB

let DE be a rational straight line

and to DE let there be applied the parallelogram DF equal to the square on AB and producing DG as its breadth

I say that DG is a fourth binomial straight line

Let the same construction be made as before shown

Then since AB is a major straight line divided at C ,

AC , CB are straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial

[x 39]

Since, then the sum of the squares on AC , CB is rational therefore DL is rational therefore DM is also rational and commensurable in length with DE

[x 20]

Again since twice the rectangle AC , CB , that is MF is medial, and it is applied to the rational straight line ML therefore MG is also rational and incommensurable in length with DE ,

[x 22]

therefore DM is also incommensurable in length with MG

[x 13]

Therefore DM , MG are rational straight lines commensurable in square only

therefore DG is binomial

[x 36]

It is to be proved that it is also a fourth binomial straight line

In manner similar to the foregoing we can prove that DM is greater than MG

and that the rectangle DK , AM is equal to the square on MN

Since then the square on LC is incommensurable with the square on CB , therefore DL is also incommensurable with AL ,

so that DK is also incommensurable with AM

[vi 1 & 11]

But if there be two unequal straight lines and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure and if it divide it into incommensurable parts then the square on the greater will be greater than the square on the less by the square on a straight line incommensurable in length with the greater

[x 13]

therefore the square on DM is greater than the square on MG by the square on a straight line incommensurable with DM

And DM , MG are rational straight lines commensurable in square only and DM is commensurable with the rational straight line DE set out

Therefore DG is a fourth binomial straight line

[x Def II 4]

Q E D

PROPOSITION 64

The square on the side of a rational plus a medial area applied to a rational straight line produces as breadth the fifth binomial

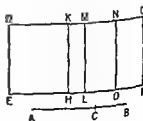
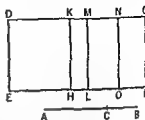
Let AB be the side of a rational plus a medial area divided into its straight lines at C so that AC is the greater, let a rational straight line DE be set out and let there be applied to DE the parallelogram DF equal to the square on AB producing DG as its breadth,

I say that DG is a fifth binomial straight line

Let the same construction as before be made.

Since then AB is the side of a rational plus a medial area divided at C

therefore AC , CB are straight lines incommensurable in square which make



the sum of the squares on them medial but the rectangle contained by them rational [x 40]

Since, then the sum of the squares on AC CB is medial

therefore DL is medial

so that DM is rational and incommensurable in length with DE [x 22]

Again, since twice the rectangle AC , CB , that is MF is rational

therefore MG is rational and commensurable with DE [x 20]

Therefore DM is incommensurable with MG [x 13]

therefore DM , MG are rational straight lines commensurable in square only

therefore DG is binomial [x 36]

I say next that it is also a fifth binomial straight line

For it can be proved similarly that the rectangle DA AM is equal to the square on AN

and that DA is incommensurable in length with AM ,

therefore the square on DM is greater than the square on MG by the square on a straight line incommensurable with DM [x 18]

And DM , MG are commensurable in square only, and the less MG , is commensurable in length with DE

Therefore DG is a fifth binomial

Q E D

PROPOSITION 65

The square on the side of the sum of two medial areas applied to a rational straight line produces as breadth the sixth binomial

Let AB be the side of the sum of two medial areas, divided at C

let DE be a rational straight line

and let there be applied to DE the parallelogram DF equal to the square on AB producing DG as its breadth,

I say that DG is a sixth binomial straight line

For let the same construction be made as before

Then since AB is the side of the sum of two medial areas divided at C

therefore AC CB are straight lines incommensurable in square which make the sum of the squares on them medial the rectangle contained by them medial and moreover the sum of the squares on them incommensurable with the rectangle contained by them [x 41]

so that in accordance with what was before proved each of the rectangles DL MF is medial

And they are applied to the rational straight line DE

therefore each of the straight lines DM MG is rational and incommensurable in length with DE [x 22]

And since the sum of the squares on AC , CB is incommensurable with twice the rectangle AC CB

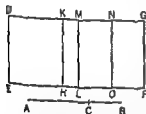
therefore DL is incommensurable with MF

Therefore DM is also incommensurable with MG [vi 1 x 11]

therefore DM MG are rational straight lines commensurable in square only

therefore DG is binomial [x 36]

I say next that it is also a sixth binomial straight line



Similarly again we can prove that the rectangle DA , AM is equal to the square on NN ,

and that DA is incommensurable in length with KM ,
and for the same reason the square on DM is greater than the square on MG
by the square on a straight line incommensurable in length with DM

And neither of the straight lines DM MG is commensurable in length with the rational straight line DE set out

Therefore DG is a sixth binomial straight line

Q E D

PROPOSITION 66

A straight line commensurable in length with a binomial straight line is itself also binomial and the same in order

Let AB be binomial and let CD be commensurable in length with AB ,

I say that CD is binomial and the same

in order with AB

For since AB is binomial

let it be divided into its terms at E

and let AE be the greater term,

therefore AE EB are rational straight lines commensurable in square only [x 36]

Let it be contrived that

as AB is to CD so is AE to CF ,

[vi 17]

therefore also the remainder EB is to the remainder FD as AB is to CD [v 19]

But AB is commensurable in length with CD

therefore AE is also commensurable with CF , and EB with FD [x 11]

And AE EB are rational

therefore CF FD are also rational

And as AE is to CF so is EB to FD

[v 11]

Therefore alternately as AE is to EB so is CF to FD

[v 16]

But AE EB are commensurable in square only

therefore CF FD are also commensurable in square only [x 11]

And they are rational

therefore CD is binomial

[x 36]

I say next that it is the same in order with AB

For the square on AE is greater than the square on EB either by the square on a straight line commensurable with AE or by the square on a straight line incommensurable with it

If then the square on AE is greater than the square on EB by the square on a straight line commensurable with AE

the square on CF will also be greater than the square on FD by the square on a straight line commensurable with CF [x 14]

And if AE is commensurable with the rational straight line set out, CF will also be commensurable with it [x 17]

and for this reason each of the straight lines AB CD is a first binomial that is the same in order [x Def II 1]

But if EB is commensurable with the rational straight line set out FD is also commensurable with it [x 17]

and for this reason again CD will be the same in order with AB

for each of them will be a second binomial [x Def II 2]

But, if neither of the straight lines AE , EB is commensurable with the rational straight line set out neither of the straight lines CF , FD will be commensurable with it [x 13]

and each of the straight lines AB , CD is a third binomial [x Def II 3]

But, if the square on AE is greater than the square on EB by the square on a straight line incommensurable with AE , the square on CF is also greater than the square on FD by the square on a straight line incommensurable with CF [x 14]

And if AE is commensurable with the rational straight line set out, CF is also commensurable with it, and each of the straight lines AB , CD is a fourth binomial [x Def II 4]

But if EB is so commensurable, so is FD also and each of the straight lines AB , CD will be a fifth binomial [x Def II 5]

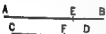
But if neither of the straight lines AE , EB is so commensurable, neither of the straight lines CF , FD is commensurable with the rational straight line set out

and each of the straight lines AB , CD will be a sixth binomial [x Def II 6]

Hence a straight line commensurable in length with a binomial straight line is binomial and the same in order Q E D

PROPOSITION 67

A straight line commensurable in length with a binomial straight line is itself also binomial and the same in order

Let AB be binomial, and let CD be commensurable in length with AB ,
 I say that CD is binomial and the same in order with AB

For, since AB is binomial, let it be divided into its medials at E , therefore AE , EB are medial straight lines commensurable in square only [x 37 38]

And let it be contrived that as AB is to CD so is AE to CF , therefore also the remainder EB is to the remainder FD as AB is to CD [v 19]

But AB is commensurable in length with CD , therefore AE , EB are also commensurable with CF , FD respectively [x 11]

But AE , EB are medial, therefore CF , FD are also medial [x 23]

And since, as AE is to EB so is CF to FD [v 11]

and AE , EB are commensurable in square only, CF , FD are also commensurable in square only [x 11]

But they were also proved medial, therefore CD is binomial

I say next that it is also the same in order with AB

For since as AE is to EB so is CF to FD therefore also as the square on AE is to the rectangle AE , EB , so is the square on CF to the rectangle CF , FD ,

therefore alternately

as the square on AE is to the square on CF so is the rectangle AE , EB to the rectangle CF , FD [v 16]

But the square on AE is commensurable with the square on CF ,

therefore the rectangle AE, EB is also commensurable with the rectangle CF, FD

If therefore the rectangle AE, EB is rational

the rectangle CF, FD is also rational

[and for this reason CD is a first bimedial],

but if medial medial

and each of the straight lines AB, CD is a second bimedial

And for this reason CD will be the same in order with AB

[x 31]

[x 23, Por]

[x 33]

Q E D

PROPOSITION 68

A straight line commensurable with a major straight line is itself also major

Let AB be major and let CD be commensurable with AB ,

I say that CD is major

Let AB be divided at E ,

therefore AE, EB are straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial

[x 39]

Let the same construction be made as before

Then since as AB is to CD so is AE to CF and EB to FD ,

therefore also as AE is to CF , so is EB to FD

[v 11]

But AB is commensurable with CD ,

therefore AE, EB are also commensurable with CF, FD respectively

[x 11]

And since as AE is to CF so is EB to FD ,

alternately also

as AE is to EB , so is CF to FD ,

[v 15]

therefore also *componendo*

as AB is to BE , so is CD to DF ,

[v 15]

therefore also as the square on AB is to the square on BE , so is the square on CD to the square on DF

[vi 20]

Similarly we can prove that as the square on AB is to the square on AE so also is the square on CD to the square on CF

Therefore also as the square on AB is to the squares on AE, EB so is the square on CD to the squares on CF, FD ,

therefore also alternately

as the square on AB is to the square on CD so are the squares on AE, EB to the squares on CF, FD

[v 15]

But the square on AB is commensurable with the square on CD ,

therefore the squares on AE, EB are also commensurable with the squares on CF, FD

And the squares on AE, EB together are rational,

therefore the squares on CF, FD together are rational

Similarly also twice the rectangle AE, EB is commensurable with twice the rectangle CF, FD

And twice the rectangle AE, EB is medial,

therefore twice the rectangle CF, FD is also medial

[x 23 Por]

Therefore CF, FD are straight lines incommensurable in square which make at the same time the sum of the squares on them rational but the rectangle contained by them medial therefore the whole CD is the irrational straight line called major

[x 39]

Therefore a straight line commensurable with the major straight line \equiv major Q E D

PROPOSITION 69

A straight line commensurable with the side of a rational plus a medial area is itself also the side of \equiv rational plus a medial area

Let AB be the side of a rational plus a medial area and let CD be commensurable with AB

it is to be proved that CD is also the side of a rational plus a medial area

Let AB be divided into its straight lines at E ,
therefore AE EB are straight lines incommensurable in square
which make the sum of the squares on them medial but the
rectangle contained by them rational [x 40]

Let the same construction be made as before

We can then prove similarly that

CF FD are incommensurable in square

and the sum of the squares on AE , EB is commensurable with the sum of the squares on CF FD

and the rectangle AE , EB with the rectangle CF , FD ,

\equiv that the sum of the squares on CF FD is also medial, and the rectangle CF FD rational

Therefore CD is the side of a rational plus a medial area

Q E D

PROPOSITION 70

A straight line commensurable with the side of the sum of two medial areas is the side of the sum of two medial areas

Let AB be the side of the sum of two medial areas and CD commensurable with AB

it is to be proved that CD is also the side of the sum of two medial areas

For since AB is the side of the sum of two medial areas
let it be divided into its straight lines at E ,
therefore AE EB are straight lines incommensurable in square
which make the sum of the squares on them medial the rectangle
contained by them medial and furthermore the sum of the
squares on AE , EB incommensurable with the rectangle AE EB [x 41]

Let the same construction be made as before

We can then prove similarly that

CF FD are also incommensurable in square

the sum of the squares on AE EB is commensurable with the sum of the squares on CF FD

and the rectangle AE EB with the rectangle CF FD

so that the sum of the squares on CF FD is also medial

the rectangle CF FD is medial

and moreover the sum of the squares on CF FD is incommensurable with the rectangle CF FD

Therefore CD is the side of the sum of two medial areas

Q E D

PROPOSITION 71

If \square rational and a medial area be added together, four irrational straight lines arise namely a binomial or a first bimedial or a major or a side of a rational plus a medial area

Let AB be rational and CD medial

I say that the 'side' of the area AD is a binomial or a first bimedial or a major or \square side of a rational plus a medial area

For AB is either greater or less than CD

First let it be greater

let a rational straight line EF be set out

let there be applied to EF the rectangle EG equal to AB , producing EH as breadth

and let HI equal to DC be applied to EF producing HK as breadth

Then since AB is rational and is equal to EG

therefore EG is also rational

And it has been applied to EF producing EH as breadth

therefore EH is rational and commensurable in length with EF [\propto 20]

Again since CD is medial and \square equal to HI

therefore HI \square also medial

And it is applied to the rational straight line EF producing HK as breadth, therefore HK is rational and incommensurable in length with EF [\propto 17]

And since CD is medial

while AB is rational

therefore AB is incommensurable with CD ,

so that EG is also incommensurable with HI

But as EG is to HI so is EH to HK [\propto 11]

therefore EH is also incommensurable in length with HK [\propto 11]

And both are rational

therefore EH HK are rational straight lines commensurable in square only

therefore EK is a binomial straight line divided at H [\propto 36]

And since AB is greater than CD

while AB is equal to EG and CD to HI

therefore EG is also greater than HI

therefore EH is also greater than HK

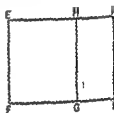
The square then on EH is greater than the square on HK either by the square on a straight line commensurable in length with EH or by the square on a straight line incommensurable with it

First let the square on it be greater by the square on a straight line commensurable with itself

Now the greater straight line HE is commensurable in length with the rational straight line EF set out

therefore EK is a first binomial

[\propto Def. II. 1]



But EF is rational

and if an area be contained by a rational straight line and the first binomial the side of the square equal to the area is binomial [x 54]

Therefore the "side" of EI is binomial

so that the "side" of AD is also binomial

Next let the square on EH be greater than the square on HA by the square on a straight line incommensurable with EH

Now the greater straight line EH is commensurable in length with the rational straight line EF set out

therefore EK is a fourth binomial [x Def II 4]

But EF is rational

and if an area be contained by a rational straight line and the fourth binomial, the 'side' of the area is the irrational straight line called major [x 57]

Therefore the 'side' of the area EI is major,

so that the "side" of the area AD is also major

Next, let AB be less than CD ,

therefore EG is also less than HI ,

so that EH is also less than HA

Now the square on HK is greater than the square on EH either by the square on a straight line commensurable with HA or by the square on a straight line incommensurable with it

First let the square on it be greater by the square on a straight line commensurable in length with itself

Now the lesser straight line EH is commensurable in length with the rational straight line EF set out,

therefore EK is a second binomial [x Def II 2]

But EF is rational

and if an area be contained by a rational straight line and the second binomial the side of the square equal to it is a first binomial, [x 55]

therefore the 'side' of the area EI is a first binomial

so that the 'side' of AD is also a first binomial

Next let the square on HK be greater than the square on HE by the square on a straight line incommensurable with HA

Now the lesser straight line EH is commensurable with the rational straight line EF set out,

therefore EK is a fifth binomial [x Def II 5]

But EF is rational

and, if an area be contained by a rational straight line and the fifth binomial the side of the square equal to the area is a side of a rational plus a medial area [x 58]

Therefore the 'side' of the area EI is a side of a rational plus a medial area

so that the 'side' of the area AD is also a side of a rational plus a medial area

Therefore etc

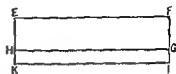
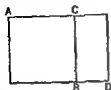
Q E D

PROPOSITION 72

If two medial areas incommensurable with one another be added together, the remaining two irrational straight lines arise namely either a second binomial or a side of the sum of two medial areas

For let two medial areas AB , CD incommensurable with one another be added together,

I say that the "side" of the area AD is either a second binomial or a side of the sum of two medial areas



For AB is either greater or less than CD

First if it so chance let AB be greater than CD

Let the rational straight line EF be set out

and to EF let there be applied the rectangle EG equal to AB and producing EH as breadth and the rectangle HI equal to CD and producing HK as breadth.

Now since each of the areas AB , CD is medial

therefore each of the areas EG , HI is also medial

And they are applied to the rational straight line FE producing EH , HK as breadth

therefore each of the straight lines EH , HK is rational and incommensurable in length with EF [x 11]

And, since AB is incommensurable with CD ,

and AB is equal to EG , and CD to HI ,

therefore EG is also incommensurable with HI

But as EG is to HI , so is EH to HK [vi 1]

therefore EH is incommensurable in length with HK [x 11]

Therefore EH , HK are rational straight lines commensurable in square only,

therefore EK is binomial [x 36]

But the square on EH is greater than the square on HK either by the square on a straight line incommensurable with EH or by the square on a straight line incommensurable with it

First let the square on it be greater by the square on a straight line commensurable in length with itself

Now neither of the straight lines EH , HK is commensurable in length with the rational straight line EF set out

therefore EK is a third binomial [x Def 11 3]

But EF is rational,

and if an area be contained by a rational straight line and the third binomial the "side" of the area is a second binomial [x 56]

therefore the "side" of AD that is of AD is a second binomial

Next let the square on EH be greater than the square on HK by the square on a straight line incommensurable in length with EH

Now each of the straight lines EH , HK is incommensurable in length with EF ,

therefore EK is a sixth binomial [x Def 11 6]

But, if an area be contained by a rational straight line and the sixth bi

nomial the 'side' of the area is the side of the sum of two medial areas [\propto 59] so that the "side" of the area AD is also the side of the sum of two medial areas

Therefore etc

Q E D

The binomial straight line and the irrational straight lines after it are neither the same with the medial nor with one another

For the square on a medial if applied to a rational straight line produces as breadth a straight line rational and incommensurable in length with that to which it is applied [\propto 22]

But the square on the binomial if applied to a rational straight line produces as breadth the first binomial [\propto 60]

The square on the first binomial if applied to a rational straight line produces as breadth the second binomial [\propto 61]

The square on the second binomial if applied to a rational straight line produces as breadth the third binomial [\propto 62]

The square on the major if applied to a rational straight line, produces as breadth the fourth binomial [\propto 63]

The square on the side of a rational plus a medial area, if applied to a rational straight line produces as breadth the fifth binomial [\propto 64]

The square on the side of the sum of two medial areas if applied to a rational straight line produces as breadth the sixth binomial [\propto 65]

And the said breadths differ both from the first and from one another from the first because it is rational and from one another because they are not the same in order

so that the irrational straight lines themselves also differ from one another

PROPOSITION 73

If from a rational straight line there be subtracted a rational straight line commensurable with the whole in square only the remainder is irrational and let it be called an apotome

For from the rational straight line AB let the rational straight line BC commensurable with the whole in square only be subtracted,

I say that the remainder AC is the irrational straight line called apotome

For, since AB is incommensurable in length with BC , and as AB is to BC so is the square on AB to the rectangle $AB \cdot BC$, therefore the square on AB is incommensurable with the rectangle $AB \cdot BC$ [\propto 11]

But the squares on $AB \cdot BC$ are commensurable with the square on AB [\propto 12]

and twice the rectangle $AB \cdot BC$ is commensurable with the rectangle $AB \cdot BC$ [\propto 6]

And inasmuch as the squares on AB, BC are equal to twice the rectangle $AB \cdot BC$ together with the square on CA [II 7]

therefore the squares on $AB \cdot BC$ are also incommensurable with the remainder the square on AC [\propto 13 16]

But the squares on $AB \cdot BC$ are rational

therefore AC is irrational

[\propto Def 4]

And let it be called an apotome

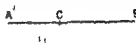
Q E D

PROPOSITION 74

If from a medial straight line there be subtracted a medial straight line which is commensurable with the whole in square only, and which contains with the whole a rational rectangle the remainder is irrational. And let it be called a first apotome of a medial straight line.

For from the medial straight line AB let there be subtracted the medial straight line BC which is commensurable with AB in square only and with AB makes the rectangle AB, BC rational,

I say that the remainder AC is irrational and let it



be called a first apotome of a medial straight line.

For since AB, BC are medial

the squares on AB, BC are also medial

But twice the rectangle AB, BC is rational, therefore the squares on AB, BC are incommensurable with twice the rectangle AB, BC ,

therefore twice the rectangle AB, BC is also incommensurable with the remainder the square on AC , [cf II 7]

since if the whole is incommensurable with one of the magnitudes, the original magnitudes will also be incommensurable [x Ib]

But twice the rectangle AB, BC is rational

therefore the square on AC is irrational,

therefore AC is irrational [x Def 4]

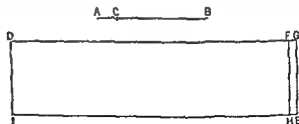
And let it be called a first apotome of a medial straight line Q E D

PROPOSITION 75

If from a medial straight line there be subtracted a medial straight line which is commensurable with the whole in square only and which contains with the whole a medial rectangle the remainder is irrational and let it be called a second apotome of a medial straight line.

For from the medial straight line AB let there be subtracted the medial straight line CB which is commensurable with the whole AB in square only and such that the rectangle AB, BC which it contains with the whole AB , is medial [x 28]

I say that the remainder AC is irrational and let it be called a second apotome of a medial straight line.



For let a rational straight line DI be set out, let DE , equal to the squares on AB, BC , be applied to DI , producing DG as breadth,

and let DH equal to twice the rectangle $AB \cdot BC$ be applied to DI producing DF as breadth,

therefore the remainder FE is equal to the square on AC [II 7]

Now, since the squares on AB, BC are medial and commensurable

therefore DE is also medial [X 15 and 23, Por.]

And it is applied to the rational straight line DI producing DG as breadth, therefore DG is rational and incommensurable in length with DI [X 22]

Again since the rectangle AB, BC is medial,

therefore twice the rectangle AB, BC is also medial [X 23, Por.]

And it is equal to DH ,

therefore DH is also medial

And it has been applied to the rational straight line DI , producing DF as breadth

therefore DF is rational and incommensurable in length with DI [X 22]

And, since AB, BC are commensurable in square only,

therefore AB is incommensurable in length with BC

therefore the square on AB is also incommensurable with the rectangle $AB \cdot BC$ [X 11]

But the squares on AB, BC are commensurable with the square on AB , [X 15]

and twice the rectangle $AB \cdot BC$ is commensurable with the rectangle $AB \cdot BC$, [X 6]

therefore twice the rectangle AB, BC is incommensurable with the squares on AB, BC [X 13]

But DE is equal to the squares on AB, BC ,

and DH to twice the rectangle AB, BC ,

therefore DE is incommensurable with DH

But as DE is to DH so is GD to DF , [XI 1]

therefore GD is incommensurable with DF [X 11]

And both are rational,

therefore GD, DF are rational straight lines commensurable in square only

therefore FG is an apotome [X 73]

But DI is rational

and the rectangle contained by a rational and an irrational straight line is irrational, [deduction from X 20]

and its 'side' is irrational

And AC is the 'side' of FE

therefore AC is irrational

And let it be called a second apotome of a medial straight line Q.E.D.

PROPOSITION 76

If from a straight line there be subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the squares on them added together rational but the rectangle contained by them medial the remainder is irrational and let it be called minor

For from the straight line AB let there be subtracted the straight line BC

which is incommensurable in square with the whole and fulfils the given conditions [X 33]

I say that the remainder AC is the irrational straight line called minor

For, since the sum of the squares on AB , BC is rational while twice the rectangle AB , BC is medial
 therefore the squares on AB , BC are incommensurable with twice the rectangle AB , BC ,
 and *contertendo*, the squares on AB , BC are incommensurable with the remainder the square on AC [II 7 & 16]

But the squares on AB , BC are rational
 therefore the square on AC is irrational,
 therefore AC is irrational

And let it be called *minor*

Q E D

PROPOSITION 77

If from a straight line there be subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the sum of the squares on them medial, but twice the rectangle contained by them rational, the remainder is irrational and let it be called that which produces with a rational area a medial whole

For from the straight line AB let there be subtracted the straight line BC which is incommensurable in square with AB and fulfils the given conditions, [x 34]

I say that the remainder AC is the irrational straight line aforesaid

For, since the sum of the squares on AB , BC is medial

while twice the rectangle AB , BC is rational

therefore the squares on AB , BC are incommensurable with twice the rectangle AB , BC

therefore the remainder also the square on AC is incommensurable with twice the rectangle AB , BC [II 7 & 16]

And twice the rectangle AB , BC is rational

therefore the square on AC is irrational,

therefore AC is irrational

And let it be called that which produces with a rational area a medial whole

Q E D

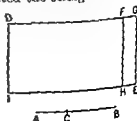
PROPOSITION 78

If from a straight line there be subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the sum of the squares on them medial twice the rectangle contained by them medial and further, the squares on them incommensurable with twice the rectangle contained by them the remainder is irrational and let it be called that which produces with a medial area a medial whole

For from the straight line AB let there be subtracted the straight line BC incommensurable in square with AB and fulfilling the given conditions [x 30]

I say that the remainder AC is the irrational straight line called that which produces with a medial area a medial whole

For let a rational straight line DI be set out to DI let there be applied DE equal to the squares on AB , BC , producing DG as breadth



and let DH equal to twice the rectangle AB, BC be subtracted
Therefore the remainder FE is equal to the square on AC [II 7]
so that AC is the side of FE

Now since the sum of the squares on AB, BC is medial and is equal to DE ,
therefore DE is medial

And it is applied to the rational straight line DI producing DG as breadth,
therefore DG is rational and incommensurable in length with DI [X 22]

Again since twice the rectangle AB, BC is medial and is equal to DH
therefore DH is medial

And it is applied to the rational straight line DI , producing DF as breadth
therefore DF is also rational and incommensurable in length with DI [X 22]

And since the squares on AB, BC are incommensurable with twice the rectangle AB, BC ,

therefore DE is also incommensurable with DH

But as DE is to DH , so also is DG to DF [VI 1]

therefore DG is incommensurable with DF [X 11]

And both are rational,

therefore GD, DF are rational straight lines commensurable in square only

Therefore FG is an apotome [X 73]

And FH is rational,

but the rectangle contained by a rational straight line and an apotome is irrational [deduction from X 20]

and its side is irrational

And AC is the side of FE

therefore AC is irrational

And let it be called that which produces with a medial area a medial whole

Q E D

PROPOSITION 79

To an apotome only one rational straight line can be annexed which is commensurable with the whole in square only

Let AB be an apotome and BC an annex to it,

$A \quad B \quad C \quad D$ therefore AC, CB are rational straight lines
commensurable in square only [X 73]

I say that no other rational straight line can be annexed to AB which is commensurable with the whole in square only

For if possible let BD be so annexed

therefore AD, DB are also rational straight lines commensurable in square only [X 73]

Now since the excess of the squares on AD, DB over twice the rectangle AD, DB is also the excess of the squares on AC, CB over twice the rectangle AC, CB ,

for both exceed by the same the square on AB [II 7]

therefore alternately the excess of the squares on AD, DB over the squares on AC, CB is the excess of twice the rectangle AD, DB over twice the rectangle AC, CB

But the squares on AD, DB exceed the squares on AC, CB by a rational area
for both are rational

therefore twice the rectangle AD, DB also exceeds twice the rectangle AC, CB by a rational area

which is impossible

for both are medial [x 21] and a medial area does not exceed a medial by a rational area [x 26]

Therefore no other rational straight line can be annexed to AB which is commensurable with the whole in square only

Therefore only one rational straight line can be annexed to an apotome which is commensurable with the whole in square only Q E D

PROPOSITION 80

To a first apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a rational rectangle

For let AB be a first apotome of a medial straight line and let BC be an annex to AB , therefore AC CB are medial straight lines commensurable in square only and such that the rectangle AC CB which they contain is rational, [x 74]

I say that no other medial straight line can be annexed to AB which is commensurable with the whole in square only and which contains with the whole a rational area

For if possible let DB also be so annexed therefore AD DB are medial straight lines commensurable in square only and such that the rectangle AD DB which they contain is rational [x 14]

Now since the excess of the squares on AD DB over twice the rectangle AD DB is also the excess of the squares on AC CB over twice the rectangle AC CB ,

for they exceed by the same the square on AB , [ii 7] therefore alternately the excess of the squares on AD DB over the squares on AC CB is also the excess of twice the rectangle AD DB over twice the rectangle AC CB

But twice the rectangle AD DB exceeds twice the rectangle AC CB by a rational area

for both are rational

Therefore the squares on AD DB also exceed the squares on AC CB by a rational area

which is impossible

for both are medial [x 15 and 23 1 or] and a medial area does not exceed a medial by a rational area [x 26]

Therefore etc

Q E D

PROPOSITION 81

To a second apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a medial rectangle

Let AB be a second apotome of a medial straight line and BC an annex to AB , therefore AC CB are medial straight lines commensurable in square only and such that the rectangle AC CB which they contain is medial [x 13]

I say that no other medial straight line can be annexed to AB which is com

measurable with the whole in square only and which contains with the whole a medial rectangle

For, if possible let BD also be so annexed,

therefore AD, DB are also medial straight lines commensurable in square only and such that the rectangle AD, DB which they contain is medial [x 75]

Let a rational straight line EF be set out
let EG equal to the squares on AC, CB be applied to EF , producing EM as breadth
and let HG equal to twice the rectangle AC, CB be subtracted producing HM as breadth

therefore the remainder EL is equal to the square on AB , [x 7]

so that AB is the "side" of EL

Again let EI equal to the squares on AD, DB be applied to EF , producing EN as breadth

But EL is also equal to the square on AB ,
therefore the remainder HI is equal to twice the rectangle AD, DB [x 7]

Now since AC, CB are medial straight lines
therefore the squares on AC, CB are also medial

And they are equal to EG

therefore EG is also medial [x 15 and 23 Por.]

And it is applied to the rational straight line EF , producing EM as breadth
therefore EM is rational and incommensurable in length with EF [x 22]

Again since the rectangle AC, CB is medial

twice the rectangle AC, CB is also medial [x 23 Por.]

And it is equal to HG ,

therefore HG is also medial

And it is applied to the rational straight line EF producing HM as breadth,
therefore HM is also rational and incommensurable in length with EF [x 22]

And since AC, CB are commensurable in square only

therefore AC is incommensurable in length with CB

But as AC is to CB , so is the square on AC to the rectangle AC, CB ,
therefore the square on AC is incommensurable with the rectangle AC, CB [x 11]

But the squares on AC, CB are commensurable with the square on AC ,
while twice the rectangle AC, CB is commensurable with the rectangle AC, CB [x 6]

therefore the squares on AC, CB are incommensurable with twice the rectangle AC, CB [x 13]

And EG is equal to the squares on AC, CB ,

while GH is equal to twice the rectangle AC, CB

therefore EG is incommensurable with HG

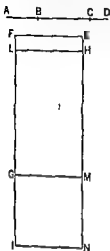
But, as EG is to HG so is EM to HM [vi 1]

therefore EM is incommensurable in length with HM [x 11]

And both are rational

therefore EM, HM are rational straight lines commensurable in square only

therefore EH is an apotome and HM an annex to it [x 73]



Similarly we can prove that HN is also an annex to it,
therefore to an apotome different straight lines are annexed which are com-
mensurable with the wholes in square only
which is impossible [x 19]

Therefore etc Q E D

PROPOSITION 82

To a minor straight line only one straight line can be annexed which is incommensurable in square with the whole and which makes with the whole, the sum of the squares on them rational but twice the rectangle contained by them medial

Let AB be the minor straight line and let BC be an annex to AB ,
therefore AC , CB are straight lines incommensurable in square which make
the sum of the squares on them rational, but twice [x 76]
the rectangle contained by them medial $\overline{A} \quad \overline{B} \quad \overline{C} \quad \overline{D}$

I say that no other straight line can be annexed to AB fulfilling the same conditions

For, if possible let BD be so annexed,
therefore AD , DB are also straight lines incommensurable in square which fulfil the aforesaid conditions [x 16]

Now since the excess of the squares on AD , DB over the squares on AC , CB
is also the excess of twice the rectangle AD , DB over twice the rectangle AC , CB ,

while the squares on AD , DB exceed the squares on AC , CB by a rational area,
for both are rational

therefore twice the rectangle AD , DB also exceeds twice the rectangle AC , CB
by a rational area

which is impossible for both are medial [x 26]

Therefore to a minor straight line only one straight line can be annexed
which is incommensurable in square with the whole and which makes the
squares on them added together rational but twice the rectangle contained by
them medial Q E D

PROPOSITION 83

To a straight line which produces with a rational area a medial whole only one straight line can be annexed which is incommensurable in square with the whole straight line and which with the whole straight line makes the sum of the squares on them medial but twice the rectangle contained by them rational

Let AB be the straight line which produces with a rational area a medial whole

and let BC be an annex to AB ,
therefore AC , CB are straight lines incommensurable in square which fulfil the given conditions [x 17]

I say that no other straight line can be annexed to AB which fulfils the same conditions

For if possible let BD be so annexed
therefore AD , DB are also straight lines incommensurable in square which fulfil the given conditions [x 77]

Since then as in the preceding cases
the excess of the squares on AD , DB over the squares on AC , CB is also the

excess of twice the rectangle AD DB over twice the rectangle AC , CB , while twice the rectangle AD , DB exceeds twice the rectangle AC , CB by a rational area

for both are rational

therefore the squares on AD DB also exceed the squares on AC , CB by a rational area

which is impossible for both are medial [X 26]

Therefore no other straight line can be annexed to AB which is incommensurable in square with the whole and which with the whole fulfils the aforesaid conditions,

therefore only one straight line can be so annexed

Q E D

PROPOSITION 84

To a straight line which produces with a medial area a medial whole only one straight line can be annexed which is incommensurable in square with the whole straight line and which with the whole straight line makes the sum of the squares on them medial and twice the rectangle contained by them both medial and also incommensurable with the sum of the squares on them

Let AB be the straight line which produces with a medial area a medial whole

and BC an annex to it,

therefore AC , CB are straight lines incommensurable in square which fulfil the aforesaid conditions [X 78]

A B C D



I say that no other straight line can be annexed to AB which fulfils the aforesaid conditions

For if possible let BD be so annexed so that AD DB are also straight lines incommensurable in square which make the squares on AD DB added together medial twice the rectangle AD DB medial and also the squares

on AD DB incommensurable with twice the rectangle AD DB [X 78]

Let a rational straight line BF be set out let EG equal to the squares on AC CB be applied to EF producing EM as breadth

and let HG equal to twice the rectangle AC CB be applied to EF producing HM as breadth

therefore the remainder the square on AB [II 7] is equal to EL

therefore AB is the side of EL

Again let EI equal to the squares on AD DB be applied to EI producing EN as breadth

But the square on AB is also equal to EL , therefore the remainder twice the rectangle AD DB [II 7] is equal to HI

Now since the sum of the squares on AC CB is medial and is equal to EG therefore EG is also medial

And it is applied to the rational straight line EF producing EM as breadth therefore EM is rational and incommensurable in length with EF [X 22]

Again, since twice the rectangle AC, CB is medial and is equal to HG
therefore HG is also medial

And it is applied to the rational straight line EF producing HM as breadth
therefore HM is rational and incommensurable in length with EF [v 22]

And, since the squares on AC, CB are incommensurable with twice the rectangle AC, CB

EG is also incommensurable with HG ,

therefore EM is also incommensurable in length with MH [vi 1 & 11]

And both are rational,

therefore EM, MH are rational straight lines commensurable in square only
therefore EH is an apotome and HM an annex to it [x 13]

Similarly we can prove that EH is again an apotome and HN an annex to it

Therefore to an apotome different rational straight lines are annexed which
are commensurable with the wholes in square only

which was proved impossible

[x 19]

Therefore no other straight line can be so annexed to AB

Therefore to AB only one straight line can be annexed which is incommensurable in square with the whole and which with the whole makes the squares on them added together medial twice the rectangle contained by them medial, and also the squares on them incommensurable with twice the rectangle contained by them

Q E D

DEFINITIONS III

1 Given a rational straight line and an apotome if the square on the whole be greater than the square on the annex by the square on a straight line commensurable in length with the whole and the whole be commensurable in length with the rational straight line set out let the apotome be called a *first apotome*

2 But if the annex be commensurable in length with the rational straight line set out and the square on the whole be greater than that on the annex by the square on a straight line commensurable with the whole let the apotome be called a *second apotome*

3 But if neither be commensurable in length with the rational straight line set out and the square on the whole be greater than the square on the annex by the square on a straight line commensurable with the whole, let the apotome be called a *third apotome*

4 Again if the square on the whole be greater than the square on the annex by the square on a straight line incommensurable with the whole then if the whole be commensurable in length with the rational straight line set out let the apotome be called a *fourth apotome*

5 if the annex be so commensurable a *fifth*

6 and if neither a *sixth*

PROPOSITION 85

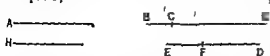
To find the first apotome

Let a rational straight line A be set out

and let BG be commensurable in length with A ,

therefore BG is also rational

Let two square numbers DE , EF be set out, and let their difference FD not be square,



therefore neither has ED to DF the ratio which a square number has to a square number

Let it be contrived that

as ED is to DF , so is the square on BG to the square on GC , [x 6 Por]
therefore the square on BG is commensurable with the square on GC [x 6]

But the square on BG is rational,

therefore the square on GC is also rational,
therefore GC is also rational

And since ED has not to DF the ratio which a square number has to a square number,

therefore neither has the square on BG to the square on GC the ratio which a square number has to a square number,

therefore BG is incommensurable in length with GC [x 9]

And both are rational,

therefore BG , GC are rational straight lines commensurable in square only

therefore BC is an apotome [x 73]

I say next that it is also a first apotome

For let the square on H be that by which the square on BG is greater than the square on GC

Now since as ED is to FD so is the square on BG to the square on GC ,
therefore also, *converendo* [v 19, Por]

as DE is to EF , so is the square on GB to the square on H

But DE has to EF the ratio which a square number has to a square number,
for each is square,

therefore the square on GB also has to the square on H the ratio which a square number has to a square number

therefore BG is commensurable in length with H [x 9]

And the square on BG is greater than the square on GC by the square on H ,
therefore the square on BG is greater than the square on GC by the square on a straight line commensurable in length with BG

And the whole BG is commensurable in length with the rational straight line A set out

Therefore BC is a first apotome [x Def III 1]

Therefore the first apotome BC has been found

(Being) that which it was required to find Q E D

PROPOSITION 86

To find the second apotome

Let a rational straight line A be set out and GC commensurable in length with A ,

therefore GC is rational

Let two square numbers DE , EF be set out and let their difference FD be square

Now let it be contrived that as FD is to DE so is the square on CG to the square on GB

Therefore the square on CG is commensurable with the square on GB

But the square on CG is rational,
 therefore the square on GB is also rational,
 therefore GB is rational

And, since the square on GC has not to the
 square on GB the ratio which a square num-
 ber has to a square number
 CG is incommensurable in length with GB

[x 9]

And both are rational,
 therefore CG , GB are rational straight lines
 commensurable in square only,

therefore BC is an apotome

[x 73]

I say next that it is also a second apotome

For let the square on H be that by which the square on BG is greater than
 the square on GC

Since then as the square on BG is to the square on GC , so is the number ED
 to the number DF

therefore *convertendo*

as the square on BG is to the square on H , so is DE to EF [v 19 Por]

And each of the numbers DE , EF is square
 therefore the square on BG has to the square on H the ratio which a square
 number has to a square number

therefore BG is commensurable in length with H [x 9]

And the square on BG is greater than the square on GC by the square on H ,
 therefore the square on BG is greater than the square on GC by the square on a
 straight line commensurable in length with BG

And CG the annex is commensurable with the rational straight line A set
 out

Therefore BC is a second apotome

[x Def III 2]

Therefore the second apotome BC has been found

Q E D

PROPOSITION 87

To find the third apotome

Let a rational straight line A be set out
 let three numbers E , BC , CD be set out
 which have not to one another the ratio
 which a square number has to a square num-
 ber but let CB have to BD the ratio which a
 square number has to a square number

Let it be contrived that as E is to BC so is
 the square on A to the square on FG
 and as BC is to CD , so is the square on FG
 to the square on GH

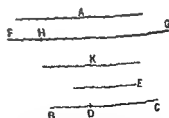
[x 6 Por]

Since then, as E is to BC so is the square on A to the square on FG ,
 therefore the square on A is commensurable with the square on FG [x 6]

But the square on A is rational

therefore the square on FG is also rational

therefore FG is rational



And since E has not to BC the ratio which a square number has to a square number,
therefore neither has the square on A to the square on FG the ratio which a square number has to a square number,

therefore A is incommensurable in length with FG [x 9]

Again, since as BC is to CD so is the square on FG to the square on GH
therefore the square on FG is commensurable with the square on GH [x 6]

But the square on FG is rational,

therefore the square on GH is also rational,

therefore GH is rational

And since BC has not to CD the ratio which a square number has to a square number,

therefore neither has the square on FG to the square on GH the ratio which a square number has to a square number

therefore FG is incommensurable in length with GH [x 9]

And both are rational

therefore FG, GH are rational straight lines commensurable in square only,

therefore FH is an apotome [x 73]

I say next that it is also a third apotome

For since, as E is to BC so is the square on A to the square on FG

and, as BC is to CD so is the square on FG to the square on GH

therefore *ex aequali*, as E is to CD so is the square on A to the square on GH [v 22]

But E has not to CD the ratio which a square number has to a square number
therefore neither has the square on A to the square on GH the ratio which a square number has to a square number,

therefore A is incommensurable in length with GH [x 9]

Therefore neither of the straight lines FG, GH is commensurable in length with the rational straight line A set out

Now let the square on A be that by which the square on FG is greater than the square on GH

Since then as BC is to CD so is the square on FG to the square on GH
therefore *convertendo* as BC is to BD so is the square on FG to the square on A [v 19 Por]

But BC has to BD the ratio which a square number has to a square number,
therefore the square on FG also has to the square on A the ratio which a square number has to a square number

Therefore FG is commensurable in length with A [x 9]
and the square on FG is greater than the square on GH by the square on a straight line commensurable with FG

And neither of the straight lines FG, GH is commensurable in length with the rational straight line A set out

therefore FH is a third apotome [x Def III 3]

Therefore the third apotome FH has been found

Q E D

PROPOSITION 88

To find the fourth apotome

Let a rational straight line A be set out and BG commensurable in length with it

therefore BG is also rational

Let two numbers DF , FE be set out such that the whole DE has not to either of the numbers DF , FE the ratio which a square number has to a square number



Let it be contrived that as DE is to EF , so is the square on BG to the square on GC , [x 6 Por]

therefore the square on BG is commensurable with the square on GC [x 6]

But the square on BG is rational

therefore the square on GC is also rational,

therefore GC is rational

Now since DE has not to EF the ratio which a square number has to a square number,

therefore neither has the square on BG to the square on GC the ratio which a square number has to a square number,

therefore BG is incommensurable in length with GC [x 9]

And both are rational,

therefore BG , GC are rational straight lines commensurable in square only,

therefore BC is an apotome [x 73]

Now let the square on H be that by which the square on BG is greater than the square on GC

Since then as DE is to EF so is the square on BG to the square on GC , therefore also *convertendo* as ED is to DF so is the square on GB to the square on H [v 19 Por]

But ED has not to DF the ratio which a square number has to a square number

therefore neither has the square on GB to the square on H the ratio which a square number has to a square number

therefore BG is incommensurable in length with H [x 9]

And the square on BG is greater than the square on GC by the square on H , therefore the square on BG is greater than the square on GC by the square on a straight line incommensurable with BG

And the whole BG is commensurable in length with the rational straight line A set out

Therefore BC is a fourth apotome [x Def n 4]

Therefore the fourth apotome has been found Q E D

PROPOSITION 89

To find the fifth apotome

Let a rational straight line A be set out and let CG be commensurable in length with A

therefore CG is rational

Let two numbers DF , FE be set out such that DE again has not to either of the numbers DF , FE the ratio which a square number has to a square number,

and let it be contrived that as FE is to ED so is the square on CG to the square on GB

Therefore the square on GB is also rational, [x 6]
therefore BG is also rational

Now since as DE is to EF , so is the square on BG to the square on GC

while DE has not to EF the ratio which a square number has to a square number

therefore neither has the square on BG to the square on GC the ratio which a square number has to a square number

therefore BG is incommensurable in length with GC [x 9]

And both are rational,
therefore BG GC are rational straight lines commensurable in square only,

therefore BC is an apotome [x 73]

I say next that it is also a fifth apotome

For let the square on H be that by which the square on BG is greater than the square on GC

Since then as the square on BG is to the square on GC , so is DE to EF ,
therefore, *convertendo* as ED is to DF so is the square on BG to the square on H [v 10 Por]

But ED has not to DF the ratio which a square number has to a square number,
therefore neither has the square on BG to the square on H the ratio which a square number has to a square number

therefore BG is incommensurable in length with H [x. 9]

And the square on BG is greater than the square on GC by the square on H ,
therefore the square on GB is greater than the square on GC by the square on a straight line incommensurable in length with GB

And the annex CG is commensurable in length with the rational straight line A set out

therefore BC is a fifth apotome [x Def III 5]

Therefore the fifth apotome BC has been found q.e.d.

PROPOSITION 90

To find the sixth apotome

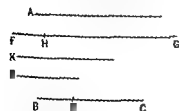
Let a rational straight line A be set out and three numbers E BC , CD not having to one another the ratio which a square number has to a square number and further let CB also not have to BD the ratio which a square number has to a square number

Let it be contrived that as E is to BC , so is the square on A to the square on FG ,
and as BC is to CD so is the square on FG to the square on GH [x 6 Por]

Now since as E is to BC so is the square on A to the square on FG

therefore the square on A is commensurable with the square on FG [x 6]

But the square on A is rational



therefore the square on FG is also rational

therefore FG is also rational

And, since E has not to BC the ratio which a square number has to a square number,

therefore neither has the square on A to the square on FG the ratio which a square number has to a square number

therefore A is incommensurable in length with FG [x 9]

Again since as BC is to CD so is the square on FG to the square on GH therefore the square on FG is commensurable with the square on GH [x 6]

But the square on FG is rational

therefore the square on GH is also rational,

therefore GH is also rational

And since BC has not to CD the ratio which a square number has to a square number,

therefore neither has the square on FG to the square on GH the ratio which a square number has to a square number

therefore FG is incommensurable in length with GH [x 9]

And both are rational

therefore FG, GH are rational straight lines commensurable in square only,

therefore FH is an apotome [x 13]

I say next that it is also a sixth apotome

For since as E is to BC , so is the square on A to the square on FG

and as BC is to CD so is the square on FG to the square on GH ,

therefore *ex aequali* as E is to CD , so is the square on A to the square on GH [v 22]

But E has not to CD the ratio which a square number has to a square number

therefore neither has the square on A to the square on GH the ratio which a square number has to a square number,

therefore A is incommensurable in length with GH , [x 9]

therefore neither of the straight lines FG, GH is commensurable in length with the rational straight line A

Now let the square on A be that by which the square on FG is greater than the square on GH

Since then as BC is to CD so is the square on FG to the square on GH therefore *convertendo* as CB is to BD , so is the square on FG to the square on A [v 19, Por]

But CB has not to BD the ratio which a square number has to a square number,

therefore neither has the square on FG to the square on A the ratio which a square number has to a square number

therefore FG is incommensurable in length with A [x 9]

And the square on FG is greater than the square on GH by the square on A , therefore the square on FG is greater than the square on GH by the square on a straight line incommensurable in length with FG

And neither of the straight lines FG, GH is commensurable with the rational straight line A set out

Therefore FH is a sixth apotome

Therefore the sixth apotome FH has been found

[x Def III 6]
Q E D

PROPOSITION 91

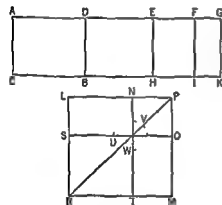
If an area be contained by a rational straight line and a first apotome the "side" of the area is an apotome

For let the area AB be contained by the rational straight line AC and the first apotome AD

I say that the "side" of the area AB is an apotome

For since AD is a first apotome let DG be its annex,
therefore AG GD are rational straight lines commensurable in square only [x 73]

And the whole AG is commensurable with the rational straight line AC set out
and the square on AG is greater than the square on GD by the square on a straight line commensurable in length with AG [x Defs III 1]
if therefore there be applied to AG a parallelogram equal to the fourth part of the square on DG and deficient by a square figure it divides it into commensurable parts [x 17]



Let DG be bisected at E
let there be applied to AG a parallelogram equal to the square on EG and deficient by a square figure
and let it be the rectangle AF FG
therefore AF is commensurable with FG

And through the points E F G let EH FI GK be drawn parallel to AC
Now since AF is commensurable in length with FG
therefore AG is also commensurable in length with each of the straight lines AF FG [x 15]

But AG is commensurable with AC ,
therefore each of the straight lines AF FG is commensurable in length with AC [x 12]

And AC is rational

therefore each of the straight lines AF FG is also rational
so that each of the rectangles AI FK is also rational [x 19]

Now since DE is commensurable in length with EG
therefore DG is also commensurable in length with each of the straight lines DE EG [x 15]

But DG is rational and incommensurable in length with AC
therefore each of the straight lines DE , EG is also rational and incommensurable in length with AC [x 13]

therefore each of the rectangles DH EK is medial [x 21]

Now let the square LM be made equal to AI and let there be subtracted the square NO having a common angle with it the angle LPN , and equal to FK

therefore the squares LM NO are about the same diameter [vi 26]

Let PR be their diameter and let the figure be drawn

Since then the rectangle contained by AF , FG is equal to the square on EG
therefore, as AF is to EG , so is EG to FG [vi 17]

But, as AF is to LG so is AI to EK

and as EG is to FG so is EK to AF , [vi 11]

therefore EK is a mean proportional between AI , AF [v 11]

But MN is also a mean proportional between LM , NO , as was before proved [Lemma after v 53]

and AI is equal to the square LM , and AF to AO ,

therefore MA is also equal to EK

But EK is equal to DH and MN to LO ,

therefore DA is equal to the gnomon UVW and NO

But AA is also equal to the squares LM , NO

therefore the remainder AB is equal to ST

But ST is the square on LN ,

therefore the square on LN is equal to AB ,

therefore LN is the "side" of AB

I say next that LN is an apotome

For since each of the rectangles AI , FA is rational,

and they are equal to LM , NO

therefore each of the squares LM , NO that is, the squares on LP , PN respectively is also rational

therefore each of the straight lines LP , PN is also rational

Again since DH is medial and is equal to LO

therefore LO is also medial

Since then LO is medial

while NO is rational

therefore LO is incommensurable with NO

But as LO is to NO so is LP to PN [vi 11]

therefore LP is incommensurable in length with PN [v 11]

And both are rational

therefore LP , PN are rational straight lines commensurable in square only

therefore LN is an apotome [v 73]

And it is the "side" of the area AB

therefore the "side" of the area AB is an apotome

Therefore etc

Q E D

PROPOSITION 92

If an area be contained by a rational straight line and a second apotome, the "side" of the area is a first apotome of a medial straight line

For let the area AB be contained by the rational straight line AC and the second apotome AD

I say that the "side" of the area AB is a first apotome of a medial straight line

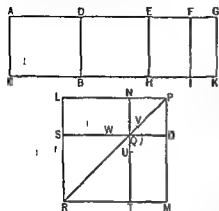
For let DG be the annex to AD

therefore AG , GD are rational straight lines commensurable in square only [v 73]

and the annex DG is commensurable with the rational straight line AC set out, while the square on the whole AG is greater than the square on the annex GD by the square on a straight line commensurable in length with AG [v 8, Def m 1]

[v 8, Def m 1]

Since, then the square on AG is greater than the square on GD by the square on a straight line commensurable with AG ,



therefore, if there be applied to AG a parallelogram equal to the fourth part of the square on GD and deficient by a square figure it divides it into commensurable parts [x 17]

Let then DG be bisected at E , let there be applied to AG a parallelogram equal to the square on EG and deficient by a square figure, and let it be the rectangle AF, FG , therefore AF is commensurable in length with FG

Therefore AG is also commensurable in length with each of the straight lines AF, FG [x 15]

But AG is rational and incommensurable in length with AC , therefore each of the straight lines AF, FG is also rational and incommensurable in length with AC , [x 13]

therefore each of the rectangles AI, FK is medial [x 21]

Again since DE is commensurable with EG therefore DG is also commensurable with each of the straight lines DE, EG [x 15]

But DG is commensurable in length with AC

Therefore each of the rectangles DH, EK is rational [x 10]

Let then the square LM be constructed equal to AI , and let there be subtracted NO equal to FK and being about the same angle with LM namely the angle LPM ,

therefore the squares LM, NO are about the same diameter [vi 26]

Let PR be their diameter and let the figure be drawn

Since then AI, FK are medial and are equal to the squares on LP, PN ,

the squares on LP, PN are also medial

therefore LP, PN are also medial straight lines commensurable in square only

And since the rectangle AF, FG is equal to the square on EG ,

therefore, as AF is to EG so is EG to FG , [vi 17]

while, as AF is to EG so is AI to EK

and as EG is to FG so is EK to FK [vi 11]

therefore EK is a mean proportional between AI, FK [v 11]

But MN is also a mean proportional between the squares LM, NO

and AI is equal to LM and FK to NO ,

therefore MN is also equal to EK

But DH is equal to EK and LO equal to MN

therefore the whole DA is equal to the gnomon UVW and ΛO

Since then the whole AK is equal to $LM, \Lambda O$

and, in these DA is equal to the gnomon UVW and NO

therefore the remainder AB is equal to TS

But TS is the square on LN

therefore the square on LN is equal to the area AB ,

therefore LN is the "side" of the area AB

I say that LN is a first apotome of a medial straight line

For, since EK is rational and is equal to LO

therefore LO that is the rectangle LP, PN , is rational

But NO was proved medial,

therefore LO is incommensurable with NO

But, as LO is to NO , so is LP to PN , [VI 1]

therefore LP, PN are incommensurable in length [X 11]

Therefore LP, PN are medial straight lines commensurable in square only which contain a rational rectangle,

therefore LN is a first apotome of a medial straight line [X 74]

And it is the "side" of the area AB

Therefore the "side" of the area AB is a first apotome of a medial straight line
Q E D

PROPOSITION 93

If an area be contained by a rational straight line and a third apotome the "side" of the area is a second apotome of a medial straight line

For let the area AB be contained by the rational straight line AC and the third apotome AD

I say that the "side" of the area AB is a second apotome of a medial straight line

For let DG be the annex to AD

therefore AG, GD are rational straight lines commensurable in square only, and neither of the straight lines AG, GD is commensurable in length with the rational straight line AC set out

while the square on the whole AG is greater than the square on the annex DG by the square on a straight line commensurable with AG [X Def III 3]

Since, then, the square on AG is greater than the square on DG by the square on a straight line commensurable with AG ,

therefore if there be applied to AG a parallelogram equal to the fourth part of the square on DG and deficient by a square figure it will divide it into commensurable parts [X 17]

Let then DG be bisected at E

let there be applied to AG a parallelogram equal to the square on EG and deficient by a square figure,

and let it be the rectangle AF, FG

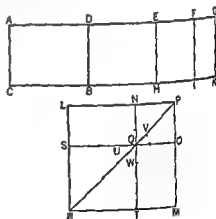
Let EH, FI, GK be drawn through the points E, F, G parallel to AC

Therefore AF, FG are commensurable

rational
therefore AI is also commensurable with FA

[VI 1 X 11]

And, since AF, FG are commensurable in length, therefore AG is also commensurable in length with each of the straight lines AF, FG [X 13]



But AG is rational and incommensurable in length with AC ,
so that AF FG are so also

[x 13]

Therefore each of the rectangles AI FA is medial

[x 21]

Again since DE is commensurable in length with EG

therefore DG is also commensurable in length with each of the straight lines DE EG

[x 15]

But GD is rational and incommensurable in length with AC

therefore each of the straight lines DE , EG is also rational and incommensurable in length with AC

[x 13]

therefore each of the rectangles DH EA is medial

[x 21]

And since AG GD are commensurable in square only

therefore AG is incommensurable in length with GD

But AG is commensurable in length with AF and DG with EG

therefore AF is incommensurable in length with EG

[x 13]

But as AF is to EG so is AI to EA

[vi 1]

therefore AI is incommensurable with EA

[x 11]

Now let the square LV be constructed equal to AI

and let there be subtracted AO equal to FK and being about the same angle with LV

therefore LM NO are about the same diameter

[vi 26]

Let PR be their diameter and let the figure be drawn

Now since the rectangle AF FG is equal to the square on EG

therefore as AF is to EG so is EG to FG

[vi 17]

But as AF is to EG , so is AI to EA

and as EG is to FG so is EK to FA ,

[vi 1]

therefore also as AI is to EA so is EA to FA

[v 11]

therefore EA is a mean proportional between AI FA

But MA is also a mean proportional between the squares LM NO

and AI is equal to LM and FA to AO

therefore EK is also equal to MN

But MA is equal to LO and EA equal to DH

therefore the whole DA is also equal to the gnomon UVW and AO

But AK is also equal to LM AO

therefore the remainder AB is equal to ST that is to the square on LN

therefore LN is the side of the area AB

I say that LN is a second apotome of a medial straight line

For since AI FA were proved medial and are equal to the squares on LP PN

therefore each of the squares on LP PN is also medial

therefore each of the straight lines LP PN is medial

And since AI is commensurable with FA

[vi 1 x 11]

therefore the square on LP is also commensurable with the square on PN

Again since AI was proved incommensurable with EA

therefore LM is also incommensurable with MA

that is the square on LP with the rectangle LP PN

so that LP is also incommensurable in length with LV ,

[vi 1, x 11]

therefore LP PN are medial straight lines commensurable in square only

I say next that they also contain a medial rectangle

For since EA was proved medial and is equal to the rectangle LP , PN

therefore the rectangle LP , PN is also medial
so that LP PN are medial straight lines commensurable in square only which contain a medial rectangle

Therefore LN is a second apotome of a medial straight line, [x 75]
and it is the side of the area AB

Therefore the 'side' of the area AB is a second apotome of a medial straight line Q E D

PROPOSITION 94

If an area be contained by a rational straight line and a fourth apotome, the 'side of the area is minor

For let the area AB be contained by the rational straight line AC and the fourth apotome AD ,

I say that the side of the area AB = minor

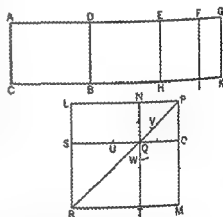
For let DG be the annex to AD

therefore AG GD are rational straight lines commensurable in square only, AG = commensurable in length with the rational straight line AC set out and the square on the whole AG is greater than the square on the annex DG by the square on a straight line incommensurable in length with AG

[x Def iii 4]

Since then the square on AG is greater than the square on GD by the square on a straight line incommensurable in length with AG

therefore if there be applied to AG a parallelogram equal to the fourth part of the square on DG and deficient by a square figure it will divide it into incommensurable parts [x 18]



Let then DG be bisected at E
let there be applied to AG a parallelogram equal to the square on EG and deficient by a square figure

and let it be the rectangle AF FG
therefore AF is incommensurable in length with FG

Let EH FI GA be drawn through E F G parallel to AC BD

Since then AG is rational and commensurable in length with AC ,
therefore the whole AG is rational [x 10]

Again since DG is incommensurable in length with AC and both are rational

therefore DA is medial [x 11]

Again since AF is incommensurable in length with FG

therefore AI is also incommensurable with FA [vi 1 x 11]

Now let the square LM be constructed equal to AI
and let there be subtracted VO equal to FA and about the same angle the angle LPM

Therefore the squares LM AO are about the same diameter [vi 10]

Let PR be their diameter and let the figure be drawn

Since then, the rectangle AF FG is equal to the square on EG ,
therefore, proportionally, as AF is to EG so is EG to FG [vi 17]

But as AF is to EG , so is AI to EK ,
and as EG is to FG so is EK to FA , [vi 1]

therefore EK is a mean proportional between AI FA [v 11]

But MN is also a mean proportional between the squares LM , NO ,
and AI is equal to LM and FA to NO ,

therefore EK is also equal to MN

But DH is equal to EK , and LO is equal to MN ,
therefore the whole DA is equal to the gnomon UVW and NO

Since then the whole AA is equal to the squares LM , NO ,
and in these DA is equal to the gnomon UVW and the square NO
therefore the remainder AB is equal to ST , that is to the square on LN ,
therefore LN is the "side" of the area AB

I say that LN is the irrational straight line called minor

For, since AK is rational and is equal to the squares on LP , PN ,
therefore the sum of the squares on LP , PN is rational

Again since DA is medial,

and DA is equal to twice the rectangle LP , PN ,
therefore twice the rectangle LP PN is medial

And since AI was proved incommensurable with FA
therefore the square on LP is also incommensurable with the square on PN
Therefore LP , PN are straight lines incommensurable in square which make
the sum of the squares on them rational but twice the rectangle contained by
them medial

Therefore LN is the irrational straight line called minor, [x 76]
and it is the "side" of the area AB

Therefore the "side" of the area AB is minor Q E D

PROPOSITION 95

If an area be contained by a rational straight line and a fifth apotome the "side" of the area is a straight line which produces with a rational area a medial whole

For let the area AB be contained by the rational straight line AC and the fifth apotome AD

I say that the "side" of the area AB is a straight line which produces with a rational area a medial whole

For let DG be the annex to AD ,
therefore AG GD are rational straight lines commensurable in square only
the annex GD is commensurable in length with the rational straight line AC
set out

and the square on the whole AG is greater than the square on the annex DG by
the square on a straight line incommensurable with AG [x Def III 5]

Therefore, if there be applied to AG a parallelogram equal to the fourth part
of the square on DG and deficient by a square figure it will divide it into in
commensurable parts [x 18]

Let then DG be bisected at the point E ,
let there be applied to AG a parallelogram equal to the square on EG and deficient by a square figure and let it be the rectangle AF , FG ,
therefore AF is incommensurable in length with FG

Now since AG is incommensurable in length with CA , and both are rational
therefore AK is medial [x 21]

Again since DG is rational and commensurable in length with AC

DA is rational [x 19]

Now let the square LM be constructed equal to AI and let the square NO equal to FK and about the same angle the angle LPM be subtracted

therefore the squares LM , NO are about the same diameter [vi 26]

Let PR be their diameter and let the figure be drawn

Similarly then we can prove that LN is the 'side' of the area AB

I say that LN is the straight line which produces with a rational area a medial whole

For since AK was proved medial and is equal to the squares on LP , PN ,
therefore the sum of the squares on LP , PN is medial

Again, since DA is rational and is equal to twice the rectangle LP , PN ,
the latter is itself also rational

And since AI is incommensurable with FK ,
therefore the square on LP is also incommensurable with the square on PN ,
therefore LP , PN are straight lines incommensurable in square which make
the sum of the squares on them medial but twice the rectangle contained by
them rational

Therefore the remainder LN is the irrational straight line called that which
produces with a rational area a medial whole [x 77]

and it is the 'side' of the area AB

Therefore the 'side' of the area AB is a straight line which produces with a
rational area a medial whole Q E D

PROPOSITION 96

If an area be contained by a rational straight line and a sixth apotome the "side
of the area is a straight line which produces with a medial area a medial whole

For let the area AB be contained by the rational straight line AC and the
sixth apotome AD

I say that the 'side' of the area AB is a straight line which produces with
a medial area a medial whole

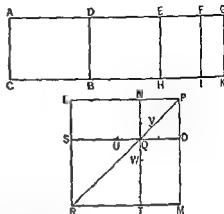
For let DG be the annex to AD

therefore AG , GD are rational straight lines commensurable in square only
neither of them is commensurable in length with the rational straight line AC
set out

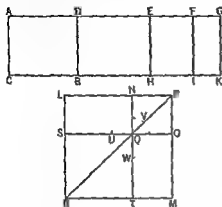
and the square on the whole AG is greater than the square on the annex DG by
the square on a straight line incommensurable in length with AG

[x Def iii 6]

Since then the square on AG is greater than the square on GD by the square
on a straight line incommensurable in length with AG ,



therefore if there be applied to $4G$ a parallelogram equal to the fourth part of the square on DG and deficient by a square figure it will divide it into incommensurable parts [x 18]



Let then DG be bisected at E
let there be applied to AG a parallelogram equal to the square on EG and deficient by a square figure, and let it be the rectangle AF, FG , therefore AF is incommensurable in length with FG

But as AF is to FG so is AI to FK [vi 1]
therefore AI is incommensurable with FK [x 11]

And since AG, AC are rational straight lines commensurable in square only

AK is medial [x 21]

Again since AC, DG are rational straight lines and incommensurable in length

DK is also medial [x 21]

Now, since AG, GD are commensurable in square only
therefore AG is incommensurable in length with GD

But $4G$ is to GD so is AK to KD , [vi 1]
therefore AK is incommensurable with KD [x 11]

Now let the square LM be constructed equal to AI ,
and let NO equal to FK and about the same angle be subtracted,
therefore the squares LM, NO are about the same diameter [vi 26]

Let PR be their diameter and let the figure be drawn

Then in manner similar to the above we can prove that LN is the 'side' of the area AB

I say that LV is a straight line which produces with a medial area a medial whole

For since AK was proved medial and is equal to the squares on LP, PN
therefore the sum of the squares on LP, PN is medial

Again since DK was proved medial and is equal to twice the rectangle LP, PN

twice the rectangle LP, PN is also medial

And since AK was proved incommensurable with DK
the squares on LP, PN are also incommensurable with twice the rectangle LP, PN

And since AI is incommensurable with FK
therefore the square on LP is also incommensurable with the square on PN
therefore LP, PN are straight lines incommensurable in square which make the sum of the squares on them medial twice the rectangle contained by them medial and further the squares on them incommensurable with twice the rectangle contained by them

Therefore LV is the irrational straight line called that which produces with a medial area a medial whole [x 7^c]

and it is the "side" of the area AB

Therefore the "side" of the area is a straight line which produces with a medial area a medial whole Q E D

PROPOSITION 97

The square on an apotome applied to a rational straight line produces as breadth a first apotome

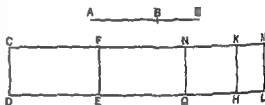
Let AB be an apotome and CD rational,
and to CD let there be applied CE equal to the square on AB and producing CF as breadth

I say that CF is a first apotome

For let BG be the annex to AB ,

therefore AG , GB are rational straight lines commensurable in square only [x 73]

To CD let there be applied CH equal to the square on AG and KL equal to the square on BG



Therefore the whole CL is equal to the squares on AG , GB , and, in these, CE is equal to the square on AB ,

therefore the remainder FL is equal to twice the rectangle AG , GB [x 7]

Let FM be bisected at the point N ,

and let NO be drawn through N parallel to CD

therefore each of the rectangles FO , LN is equal to the rectangle AG , GB

Now since the squares on AG , GB are rational

and DM is equal to the squares on AG , GB ,

therefore DM is rational

And it has been applied to the rational straight line CD , producing CM as breadth

therefore CM is rational and commensurable in length with CD [x 90]

Again since twice the rectangle AG , GB is medial and FL is equal to twice the rectangle AG , GB ,

therefore FL is medial

And it is applied to the rational straight line CD producing FM as breadth

therefore FM is rational and incommensurable in length with CD [x 22]

And since the squares on AG , GB are rational

while twice the rectangle AG , GB is medial,

therefore the squares on AG , GB are incommensurable with twice the rectangle AG , GB

And CL is equal to the squares on AG , GB ,

and FL to twice the rectangle AG , GB ,

therefore DM is incommensurable with FL

But, as DM is to FL so is CM to FM , [vi 1]

therefore CM is incommensurable in length with FM [x 11]

And both are rational

therefore CM , MF are rational straight lines commensurable in square only

therefore CF is an apotome [x 73]

I say next that it is also a first apotome

For since the rectangle AG, GB is a mean proportional between the squares on AG, GB

and CH is equal to the square on AG ,

KL equal to the square on GB ,

and NL equal to the rectangle AG, GB ,

therefore NL is also a mean proportional between CH, KL ,

therefore as CH is to NL , so is NL to KL

But as CH is to NL , so is CA to NM ,

and, as NL is to KL so is NM to KM , [vi 1]

therefore the rectangle CK, KM is equal to the square on NM [vi 17] that is, to the fourth part of the square on FM

And since the square on AG is commensurable with the square on GB ,

CH is also commensurable with KL

But, as CH is to KL so is CA to KM [vi 11]

therefore CA is commensurable with KM [x 11]

Since, then, CM, MF are two unequal straight lines

and to CM there has been applied the rectangle CK, KM equal to the fourth part of the square on FM and deficient by a square figure

while CA is commensurable with KM

therefore the square on CM is greater than the square on MF by the square on a straight line commensurable in length with CM [x 17]

And CM is commensurable in length with the rational straight line CD set out,

therefore CF is a first apotome [x Def III 1]

Therefore etc

Q E D

PROPOSITION 98

The square on a first apotome of a medial straight line applied to a rational straight line produces as breadth a second apotome

Let AB be a first apotome of a medial straight line and CD a rational straight line

and to CD let there be applied CE equal to the square on AB , producing CF as breadth

I say that CF is a second apotome

For let BG be the annex to AB ,

therefore AG, GB are medial straight lines commensurable in square only which contain a rational rectangle [x 74]

A ——— G



To CD let there be applied CH equal to the square on AG , producing CA as breadth, and KL equal to the square on GB , producing KM as breadth therefore the whole CL is equal to the squares on AG, GB ,

therefore CL is also medial [x 15 and 23 Por]

And it is applied to the rational straight line CD producing CM as breadth therefore CM is rational and incommensurable in length with CD [x 22]

Now since CL is equal to the squares on AG GB

and in these the square on AB is equal to CE

therefore the remainder twice the rectangle AG GB , is equal to FL [x 7]

But twice the rectangle AG GB is rational

therefore FL is rational

And it is applied to the rational straight line FE producing FM as breadth
therefore FM is also rational and commensurable in length with CD [x 20]

Now since the sum of the squares on AG GB that is CL is medial while
twice the rectangle AG GB that is FL is rational

therefore CL is incommensurable with FL

But as CL is to FL so is CM to FM , [vi 1]

therefore CM is incommensurable in length with FM [x 11]

And both are rational,

therefore CM MF are rational straight lines commensurable in square only,

therefore CF is an apotome [x 73]

I say next that it is also a second apotome

For let FM be bisected at N

and let NO be drawn through N parallel to CD ,

therefore each of the rectangles FO NL is equal to the rectangle AG GB

Now since the rectangle AG GB is a mean proportional between the squares
on AG GB

and the square on AG is equal to CH

the rectangle AG , GB to NL

and the square on GB to KL

therefore NL is also a mean proportional between CH KL

therefore as CH is to NL so is NL to KL

But as CH is to VL so is CK to NM

and as VL is to KL so is NM to MK [vi 1]

therefore as CK is to NM so is NM to KM [v 11]

therefore the rectangle CK KM is equal to the square on NM [vi 17] that is
to the fourth part of the square on FM

Since then CM MF are two unequal straight lines and the rectangle CK ,
 KM equal to the fourth part of the square on MF and deficient by a square
figure has been applied to the greater CM and divides it into commensurable
parts

therefore the square on CM is greater than the square on MF by the square on
a straight line commensurable in length with CM [x 11]

And the annex FM is commensurable in length with the rational straight
line CD set out

therefore CF is a second apotome [x Def m 2]

Therefore etc

Q E D

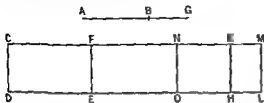
PROPOSITION 99

The square on a second apotome of a medial straight line applied to a rational
straight line produces as breadth a third apotome

Let AB be a second apotome of a medial straight line and CD rational
and to CD let there be applied CE equal to the square on AB producing CF as
breadth,

I say that CF is a third apotome

For let BG be the annex to AB ,
therefore AG GB are medial straight lines commensurable in square only
which contain a medial rectangle [x 75]



Let CH equal to the square
on AG be applied to CD , pro-
ducing CA as breadth
and let KL equal to the square
on BG be applied to KH pro-
ducing KM as breadth,
therefore the whole CL is equal
to the squares on AG , GB ,

therefore CL is also medial [x 15 and 23, Por.]

And it is applied to the rational straight line CD producing CM as breadth
therefore CM is rational and incommensurable in length with CD [x 22]

Now, since the whole CL is equal to the squares on AG GB and in these
 CE is equal to the square on AB

therefore the remainder LF is equal to twice the rectangle AG GB [ii 7]

Let then FM be bisected at the point N

and let NO be drawn parallel to CD ,

therefore each of the rectangles FO NL is equal to the rectangle AG , GB

But the rectangle AG , GB is medial,

therefore FL is also medial

And it is applied to the rational straight line EF producing FM as breadth
therefore FM is also rational and incommensurable in length with CD [x 22]

And since AG , GB are commensurable in square only,

therefore AG is incommensurable in length with GB

therefore the square on AG is also incommensurable with the rectangle AG
 GB [vi 1 x 11]

But the squares on AG GB are commensurable with the square on AG ,

and twice the rectangle AG GB with the rectangle AG , GB ,

therefore the squares on AG GB are incommensurable with twice the rectangle
 AG GB [x 13]

But CL is equal to the squares on AG GB

and FL is equal to twice the rectangle AG GB ,

therefore CL is also incommensurable with FL

But as CL is to FL so is CM to FM , [vi 1]

therefore CM is incommensurable in length with FM [x 11]

And both are rational,

therefore CM , MF are rational straight lines commensurable in square only

therefore CF is an apotome [x 73]

I say next that it is also a third apotome

For since the square on AG is commensurable with the square on GB

therefore CH is also commensurable with KL

so that CA is also commensurable with KM [vi 1 x 11]

And since the rectangle AG GB is a mean proportional between the squares
on AG GB

and CH is equal to the square on AG

KL equal to the square on GB

and NL equal to the rectangle AG GB

therefore NL is also a mean proportional between CH ; KL ,
 therefore as CH is to NL so is NL to KL

But, as CH is to NL so is CK to NM

and, as NL is to KL , so is NM to KM , [vi 1]

therefore as CK is to NM , so is NM to KM , [v 11]

therefore the rectangle CK KM is equal to [the square on NM , that is to] the fourth part of the square on FM

Since then CM MF are two unequal straight lines and a parallelogram equal to the fourth part of the square on FM and deficient by a square figure has been applied to CM and divides it into commensurable parts - -
 therefore the square on CM is greater than the square on MF by the square on a straight line commensurable with CM [x 17]

And neither of the straight lines CM MF is commensurable in length with the rational straight line CD set out

therefore CF is a third apotome [x Def iii 3]

Therefore etc

Q E D

PROPOSITION 100

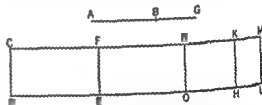
The square on a minor straight line applied to a rational straight line produces as breadth a fourth apotome

Let AB be a minor and CD a rational straight line and to the rational straight line CD let CE be applied equal to the square on AB and producing CF as breadth

I say that CF is a fourth apotome

For let BG be the annex to AB

therefore AG GB are straight lines incommensurable in square which make the sum of the squares on AG GB rational but twice the rectangle AG GB medial [x 76]



To CD let there be applied CH equal to the square on AG and producing CA as breadth

and KL equal to the square on BG producing KM as breadth,

therefore the whole CL is equal to the squares on AG , GB

And the sum of the squares on AG GB is rational

therefore CL is also rational

And it is applied to the rational straight line CD producing CM as breadth, therefore CM is also rational and commensurable in length with CD [x 20]

And since the whole CL is equal to the squares on AG GB and in these, CE is equal to the square on AB

therefore the remainder FL is equal to twice the rectangle AG GB [ii 7]

Let then FM be bisected at the point N

and let AO be drawn through A parallel to either of the straight lines CD ML , therefore each of the rectangles FO NL is equal to the rectangle AG GB

And since twice the rectangle AG GB is medial and is equal to FL

therefore FL is also medial

And it is applied to the rational straight line FE producing FM as breadth

therefore FM is rational and incommensurable in length with CD [x 22]

And since the sum of the squares on AG GB is rational

while twice the rectangle AG GB is medial,

the squares on AG GB are incommensurable with twice the rectangle AG GB

But GL is equal to the squares on AG , GB ,

and FL equal to twice the rectangle AG GB ,

therefore CL is incommensurable with FL

But as CL is to FL so is CM to MF ,

[vi 1]

therefore CM is incommensurable in length with MF

[x 11]

And both are rational,

therefore CM , MF are rational straight lines commensurable in square only

therefore CF is an apotome

[x 73]

I say that it is also a fourth apotome

For, since AG , GB are incommensurable in square

therefore the square on AG is also incommensurable with the square on GB

And CH is equal to the square on AG ,

and KL equal to the square on GB ,

therefore CH is incommensurable with KL

But as CH is to KL , so is CK to KM ,

[vi 1]

therefore CK is incommensurable in length with KM

[x 11]

And since the rectangle AG , GB is a mean proportional between the squares on AG , GB ,

and the square on AG is equal to CH ,

the square on GB to KL

and the rectangle AG GB to NL

therefore NL is a mean proportional between CH , KL

therefore as CH is to NL , so is NL to KL

But as CH is to NL so is CK to NM

and as NL is to KL so is NM to KM ,

[vi 1]

therefore as CK is to NM so is NM to KM

[v 11]

therefore the rectangle CK , KM is equal to the square on NM [vi 17] that is to the fourth part of the square on FM

Since then CM MF are two unequal straight lines and the rectangle CK , KM equal to the fourth part of the square on MF and deficient by a square figure has been applied to CM and divides it into incommensurable parts, therefore the square on CM is greater than the square on MF by the square on a straight line incommensurable with CM

[x 18]

And the whole CM is commensurable in length with the rational straight line CD set out,

therefore CF is a fourth apotome

[x Def. III 4]

Therefore etc

Q E D

PROPOSITION 101

The square on the straight line which produces with a rational area a medial whole, if applied to a rational straight line produces as breadth a fifth apotome

Let AB be the straight line which produces with a rational area a medial whole and CD a rational straight line and to CD let CE be applied equal to the square on AB and producing CF as breadth

I say that CF is a fifth apotome

For let BG be the annex to AB

therefore AG, GB are straight lines incommensurable in square which make the sum of the squares on them medial but twice the rectangle contained by them rational [x 7]

To CD let there be applied CH equal to the square on AG and KL equal to the square on GB ,

therefore the whole CL is equal to the squares on AG, GB

But the sum of the squares on AG, GB together is medial, therefore CL is medial

And it is applied to the rational straight line CD , producing CM as breadth therefore CM is rational and incommensurable with CD [x 20]

And, since the whole CL is equal to the squares on AG, GB , and in these CE is equal to the square on AB ,

therefore the remainder FL is equal to twice the rectangle AG, GB [ii 4]

Let then FM be bisected at N ,

and through N let NO be drawn parallel to either of the straight lines CD, ML , therefore each of the rectangles FO, NL is equal to the rectangle AG, GB

And, since twice the rectangle AG, GB is rational and equal to FL therefore FL is rational

And it is applied to the rational straight line EF producing FM as breadth therefore FM is rational and commensurable in length with CD [x 20]

Now since CL is medial and FL rational,

therefore CL is incommensurable with FL

But as CL is to FL so is CM to MF [vi 1]

therefore CM is incommensurable in length with MF [x 11]

And both are rational,

therefore CM, MF are rational straight lines commensurable in square only therefore CF is an apotome [x 73]

I say next that it is also a fifth apotome

For we can prove similarly that the rectangle CA, KM is equal to the square on NM that is to the fourth part of the square on FM

And since the square on AG is incommensurable with the square on GB , while the square on AG is equal to CH ,

and the square on GB to KL

therefore CH is incommensurable with KL

But as CH is to KL so is CA to KM , [vi 1]

therefore CK is incommensurable in length with KM [x 11]

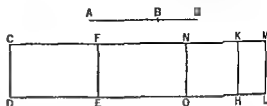
Since then CM, MF are two unequal straight lines

and a parallelogram equal to the fourth part of the square on FM and deficient by a square figure has been applied to CM , and divides it into incommensurable parts

therefore the square on CM is greater than the square on MF by the square on a straight line incommensurable with CM [x 18]

And the annex FM is commensurable with the rational straight line CD set out therefore CF is a fifth apotome [x Def III 5]

Q E D



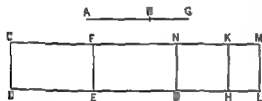
PROPOSITION 102

The square on the straight line which produces with a medial area a medial whole if applied to a rational straight line produces as breadth a sixth apotome

Let AB be the straight line which produces with a medial area a medial whole and CD a rational straight line, and to CD let CE be applied equal to the square on AB and producing CF as breadth

I say that CF is a sixth apotome

For let BG be the annex to AB ,



therefore AG GB are straight lines incommensurable in square which make the sum of the squares on them medial, twice the rectangle AG GB medial and the squares on AG GB incommensurable with twice the rectangle AG GB

[x 78]

Now to CD let there be applied CH equal to the square on AG and producing CA as breadth

and KL equal to the square on BG ,

therefore the whole CL is equal to the squares on AG GB

therefore CL is also medial

And it is applied to the rational straight line CD producing CM as breadth, therefore CM is rational and incommensurable in length with CD [x 22]

Since now CL is equal to the squares on AG GB ,

and in these CE is equal to the square on AB

therefore the remainder FL is equal to twice the rectangle AG GB [ii 7]

And twice the rectangle AG GB is medial,

therefore FL is also medial

And it is applied to the rational straight line FE producing FM as breadth, therefore FM is rational and incommensurable in length with CD [x 22]

And since the squares on AG GB are incommensurable with twice the rectangle AG GB

and CL is equal to the squares on AG GB

and FL equal to twice the rectangle AG GB

therefore CL is incommensurable with FL

But as CL is to FL so is CM to MF [vi 1]

therefore CM is incommensurable in length with MF [x 11]

And both are rational

Therefore CM MF are rational straight lines commensurable in square only

therefore CF is an apotome [x 73]

I say next that it is also a sixth apotome

For since FL is equal to twice the rectangle AG GB

let FU be bisected at N

and let NO be drawn through N parallel to CD

therefore each of the rectangles FO NL is equal to the rectangle AG GB

And since AG GB are incommensurable in square

therefore the square on AG is incommensurable with the square on GB
 But CH is equal to the square on AG ,

and KL is equal to the square on GB ,

therefore CH is incommensurable with KL

But as CH is to KL so is CA to KM [vi 1]

therefore CA is incommensurable with KM [x 11]

And since the rectangle AG GB is a mean proportional between the squares on AG , GB

and CH is equal to the square on AG ,

KL equal to the square on GB ,

and NL equal to the rectangle AG GB

therefore NL is also a mean proportional between CH , KL ,

therefore as CH is to NL so is NL to KL

And for the same reason as before the square on CM is greater than the square on MF by the square on a straight line incommensurable with CM [x 18]

And neither of them is commensurable with the rational straight line CD set out

therefore CF is a sixth apotome [x Def iii 6]
 $q \equiv p$

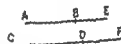
PROPOSITION 103

A straight line commensurable in length with an apotome is an apotome and the same in order

Let AB be an apotome

and let CD be commensurable in length with AB

I say that CD is also an apotome and the same in order with AB



For since AB is an apotome let BE be the annex to it

therefore AE EB are rational straight lines commensurable in square only [x 77]

Let it be contrived that the ratio of BE to DF is the same as the ratio of AE to CD [x 17]

therefore also as one is to one so are all to all [x 12]

therefore also as the whole AF is to the whole CF so is AB to CD

But AB is commensurable in length with CD

Therefore AE is also commensurable with CF and BE with DF [x 11]

And AE EB are rational straight lines commensurable in square only
 therefore CF FD are also rational straight lines commensurable in square only [x 13]

Now since as AE is to CF so is BE to DF

alternately therefore as AE is to EB so is CF to FD [x 16]

And the square on AE is greater than the square on EB either by the square on a straight line commensurable with AE or by the square on a straight line incommensurable with it

If then the square on AE is greater than the square on EB by the square on a straight line commensurable with AE the square on CF will also be greater than the square on FD by the square on a straight line commensurable with CF [x 14]

And, if AE is commensurable in length with the rational straight line set out,

CF is so also [x 12]

if BE then DF also [id]

and if neither of the straight lines AF , EB , then neither of the straight lines CF , FD [x 13]

But if the square on AE is greater than the square on FB by the square on a straight line incommensurable with AE ,
the square on CF will also be greater than the square on FD by the square on a straight line incommensurable with CF [x 14]

And if AE is commensurable in length with the rational straight line set out,

CF is so also

if BF then DF also [x 12]

and if neither of the straight lines AE , EB , then neither of the straight lines CF , FD [x 13]

Therefore CD is an apotome and the same in order with AB Q E D

PROPOSITION 104

A straight line commensurable with an apotome of a medial straight line is an apotome of a medial straight line and the same in order

Let AB be an apotome of a medial straight line

and let CD be commensurable in length with AB ,

I say that CD is also an apotome of a medial straight line and the same in order with AB

For, since AB is an apotome of a medial straight line let EB be the annex to it

Therefore AE , EB are medial straight lines commensurable in square only [x 74 75]

Let it be contrived that as AB is to CD , so is BE to DF [v 12]

therefore AE is also commensurable with CF and BE with DF [v 12 x 11]

But AE , EB are medial straight lines commensurable in square only

therefore CF , FD are also medial straight lines [x 23] commensurable in square only [x 13]

therefore CD is an apotome of a medial straight line [x 74 75]

I say next that it is also the same in order with AB

Since as AE is to EB so is CF to FD

therefore also as the square on AE is to the rectangle AE , EB so is the square on CF to the rectangle CF , FD

But the square on AE is commensurable with the square on CF

therefore the rectangle AE , EB is also commensurable with the rectangle CF , FD [v 16 x 11]

Therefore, if the rectangle AE , EB is rational the rectangle CF , FD will also be rational [x Def 4]

and if the rectangle AE , EB is medial the rectangle CF , FD is also medial [x Por]

Therefore CD is an apotome of a medial straight line and the same in order with AB [x 74 75]

Q E D

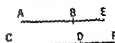
PROPOSITION 105

A straight line commensurable with a minor straight line is minor

Let AB be a minor straight line and CD commensurable with AB ,

I say that CD is also minor

Let the same construction be made as before,
then since AE EB are incommensurable in square



[x 76]

therefore CF FD are also incommensurable in square [x 13]

Now since as AE is to EB so is CF to FD [v 12 v 16]

therefore also as the square on AE is to the square on EB so is the square on CF to the square on FD [vi 27]

Therefore *componendo* as the squares on AE EB are to the square on EB so are the squares on CF FD to the square on FD [v 18]

But the square on BE is commensurable with the square on DF
therefore the sum of the squares on AE , EB is also commensurable with the sum of the squares on CF FD [v 16 x 11]

But the sum of the squares on AE EB is rational, [x 6]

therefore the sum of the squares on CF FD is also rational [x Def 4]

Again since as the square on AE is to the rectangle AE , EB so is the square on CF to the rectangle CF FD

while the square on AE is commensurable with the square on CF
therefore the rectangle AE EB is also commensurable with the rectangle CF , FD

But the rectangle AE EB is medial [x 16]

therefore the rectangle CF FD is also medial, [x 23 Por 1]
therefore CF FD are straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial

Therefore CD is minor [x 16]

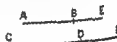
Q E D

PROPOSITION 106

A straight line commensurable with that which produces with a rational area a medial whole is a straight line which produces with a rational area a medial whole

Let AB be a straight line which produces with a rational area a medial whole
and CD commensurable with AB

I say that CD is also a straight line which produces with a rational area a medial whole



For let BE be the annex to AB

therefore AL LB are straight lines incommensurable in square which make the sum of the squares on AL PB medial but the rectangle contained by them rational [x 17]

Let the same construction be made

Then we can prove in manner similar to the foregoing that CF FD are in the same ratio as AE EB

the sum of the squares on AE EB is commensurable with the sum of the squares on CF FD

and the rectangle AE EB with the rectangle CF FD

so that CF, FD are also straight lines incommensurable in square which make the sum of the squares on CF, FD medial but the rectangle contained by them rational

Therefore CD is a straight line which produces with a rational area a medial whole

[x 77]

Q E D

PROPOSITION 107

A straight line commensurable with that which produces with a medial area a medial whole is itself also a straight line which produces with a medial area a medial whole

Let AB be a straight line which produces with a medial area a medial whole and let CD be commensurable with AB

I say that CD is also a straight line which produces with a medial area a medial whole

For let BE be the annex to AB ,

and let the same construction be made

therefore AE, EB are straight lines incommensurable in square which make the sum of the squares on them medial the rectangle contained by them medial and further the sum of the squares on them incommensurable with the rectangle contained by them

[x 78]

Now, as was proved, AE, EB are commensurable with CF, FD , the sum of the squares on AE, EB with the sum of the squares on CF, FD ,

and the rectangle AE, EB with the rectangle CF, FD ,

therefore CF, FD are also straight lines incommensurable in square which make the sum of the squares on them medial the rectangle contained by them medial and further, the sum of the squares on them incommensurable with the rectangle contained by them

Therefore CD is a straight line which produces with a medial area a medial whole

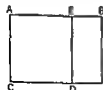
[x 78]

Q E D

PROPOSITION 108

If from a rational area a medial area be subtracted the 'side' of the remaining area becomes one of two irrational straight lines either an apotome or a minor straight line

For from the rational area BC let the medial area BD be subtracted



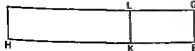
I say that the 'side' of the remainder EC becomes one of two irrational straight lines either an apotome or a minor straight line

For let a rational straight line FG be set out

to FG let there be applied the rectangular parallelogram GH equal to BC

and let GK equal to DB be subtracted therefore the remainder EC is equal to LH

Since then BC is rational and BD medial



while BC is equal to GH , and BD to GK ,
therefore GH is rational and GK medial

And they are applied to the rational straight line FG ,
therefore FH is rational and commensurable in length with FG , [x 73]
while FA is rational and incommensurable in length with FG , [x 27]
therefore FH is incommensurable in length with FK [x 13]

Therefore FH , FA are rational straight lines commensurable in square only,
therefore KH is an apotome [x 73] and AF the annex to it

Now the square on HF is greater than the square on FA by the square on a straight line either commensurable with HF or not commensurable

First let the square on it be greater by the square on a straight line commensurable with it

Now the whole HF is commensurable in length with the rational straight line FG set out,

therefore AH is a first apotome [x Def III 1]

But the side of the rectangle contained by a rational straight line and a first apotome is an apotome [x 91]

Therefore the side of LH that is of EC is an apotome

But if the square on HF is greater than the square on FK by the square on a straight line incommensurable with HF
while the whole FH is commensurable in length with the rational straight line FG set out

AH is a fourth apotome [x Def III 4]

But the side of the rectangle contained by a rational straight line and a fourth apotome is minor [x 94]

Q E D

PROPOSITION 109

If from a medial area a rational area be subtracted there arise two other irrational straight lines either a first apotome of a medial straight line or a straight line which produces with a rational area a medial whole

For from the medial area BC let the rational area BD be subtracted

I say that the side of the remainder EC becomes one of two irrational straight lines either a first apotome of a medial straight line or a straight line which produces with a rational area a medial whole

For let a rational straight line FG be set out

and let the areas be similarly applied

It follows then that FH is rational and incommensurable in length with FG
while FA is rational and commensurable in length with FG
therefore FH , FA are rational straight lines commensurable in square only [x 13]



therefore KH is an apotome and FA the annex to it [x 73]

Now the square on HF is greater than the square on FA either by the square on a straight line commensurable with HF or by the square on a straight line incommensurable with it

If then the square on HF is greater than the square on FA by the square on a straight line commensurable with HF , while the annex FA is commensurable in length with the rational straight line FG set out

KH is a second apotome

[x Def III 2]

But FG is rational, so that the 'side' of LH , that is of LC , is a first apotome of a medial straight line

[x 92]

But, if the square on HF is greater than the square on FA by the square on a straight line incommensurable with HF , while the annex FA is commensurable in length with the rational straight line FG set out,

KH is a fifth apotome,

[x Def III 5]

so that the 'side' of EC is a straight line which produces with a rational area a medial whole

[x 93]

Q E D

PROPOSITION 110

If from a medial area there be subtracted a medial area incommensurable with the whole the two remaining irrational straight lines arise, either a second apotome of a medial straight line or a straight line which produces with a medial area a medial whole

For as in the foregoing figures let there be subtracted from the medial area BC the medial area BD incommensurable with the whole



I say that the 'side' of EC is one of two irrational straight lines either a second apotome of a medial straight line or a straight line which produces with a medial area a medial whole

For since each of the rectangles BC , BD is medial

and BC is incommensurable with BD

it follows that each of the straight lines FH , FK will be rational and incommen-

[x 22]

surable in length with FG

And since BC is incommensurable with BD

that is GH with GK

HF is also incommensurable with FK

[vi 1 x 11]

therefore FH , FK are rational straight lines commensurable in square only

therefore KH is an apotome

[x 73]

If then the square on FH is greater than the square on FK by the square on a straight line commensurable with FH

while neither of the straight lines FH , FK is commensurable in length with the rational straight line FG set out

KH is a third apotome

[x Def III 3]

But KL is rational

and the rectangle contained by a rational straight line and a third apotome is irrational,

and the "side" of it is irrational and is called a second apotome of a medial straight line, [x 93]
so that the 'side' of LH , that is, of EC , is a second apotome of a medial straight line

But if the square on FH is greater than the square on FK by the square on a straight line incommensurable with FH while neither of the straight lines HF FK is commensurable in length with FG

But the "side" of the rectangle contained by a rational straight line and a sixth apotome is a straight line which produces with a medial area a medial whole. [x 96]

Therefore the "side" of LH that is of EC is a straight line which produces with a medial area a medial whole QED

PROPOSITION 111

The apolome is not the same with the binomial straight line

Let AB be an apotome

I say that AB is not the same with the binomial straight line

For if possible let it be so,
let a rational straight line DC be set out and to CD
let there be applied the rectangle CE equal to the
square on AB and producing DE as breadth

Then since AB is an apotome

DE is a first apotome {v 97}

Let EF be the annex to it

therefore DF FE are rational straight lines commen-
surable in square only

the square on DF is greater than the square on FE by the square on a straight line commensurable with DF

and DF is commensurable in length with the rational straight line DC set out
[x Delf iii 1]

Again since AB is binormal

therefore DE is a first binomial straight line

Let it be divided into its terms at G

and let DG be the greater term

therefore DG GE are rational straight lines commensurable in square only the square on DG is greater than the square on GE by the square on a straight line commensurable with DG and the greater term DG is commensurable in length with the rational straight line DC set out [v. Def. 11.]

Therefore DF is also commensurable in length with DG .

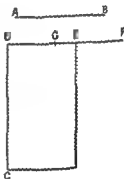
But DF is incommensurable in length with EF

Therefore GF and IP are rational straight lines commensurable in square only. [x 13]

But it is also rational

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which is impossible



Therefore the apotome is not the same with the binomial straight line

Q E D

The apotome and the irrational straight lines following it are neither the same with the medial straight line nor with one another

For the square on a medial straight line if applied to a rational straight line produces as breadth a straight line rational and incommensurable in length with that to which it is applied [x 22]

while the square on an apotome, if applied to a rational straight line produces as breadth a first apotome, [x 97]

the square on a first apotome of a medial straight line if applied to a rational straight line produces as breadth a second apotome [x 98]

the square on a second apotome of a medial straight line if applied to a rational straight line produces as breadth a third apotome, [x 99]

the square on a minor straight line if applied to a rational straight line, produces as breadth a fourth apotome [x 100]

the square on the straight line which produces with a rational area a medial whole if applied to a rational straight line produces as breadth a fifth apotome [x 101]

and the square on the straight line which produces with a medial area a medial whole, if applied to a rational straight line produces as breadth a sixth apotome [x 102]

Since then the said breadths differ from the first and from one another from the first because it is rational, and from one another since they are not the same in order

it is clear that the irrational straight lines themselves also differ from one another

And, since the apotome has been proved not to be the same as the binomial straight line [x 111]

but if applied to a rational straight line the straight lines following the apotome produce as breadths each according to its own order, apotomes and those following the binomial straight line themselves also according to their order produce the binomials as breadths

therefore those following the apotome are different and those following the binomial straight line are different so that there are, in order, thirteen irrational straight lines in all

Medial

Binomial

First bimedial

Second bimedial

Major

Side of a rational plus a medial area

Side of the sum of two medial areas

Apotome

First apotome of a medial straight line

Second apotome of a medial straight line

Minor

Producing with a rational area a medial whole

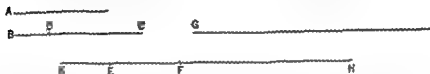
Producing with a medial area a medial whole

PROPOSITION 112

The square on a rational straight line applied to the binomial straight line produces as breadth an apotome the terms of which are commensurable with the terms of the binomial and moreover in the same ratio and further the apotome so arising will have the same order as the binomial straight line

Let A be a rational straight line

let BC be a binomial and let DC be its greater term,
let the rectangle BC, EF be equal to the square on A ,



I say that EF is an apotome the terms of which are commensurable with CD , DB and in the same ratio and further EF will have the same order as BC

For again let the rectangle BD, G be equal to the square on A

Since then the rectangle BC, EF is equal to the rectangle BD, G

therefore as CB is to BD so is G to EF [vi 16]

But CB is greater than BD

therefore G is also greater than EF [v 16 v 14]

Let EH be equal to G

therefore as CB is to BD so is HE to EF

therefore separando as CD is to BD so is HF to FE [v 17]

Let it be contrived that as HF is to FE so is FA to AE

therefore also the whole HA is to the whole KE as FA is to AE

for as one of the antecedents is to one of the consequents so are all the antecedents to all the consequents [v 12]

But as FA is to AE so is CD to DB [v 11]

therefore also as HA is to AE so is CD to DB [v 11]

But the square on CD is commensurable with the square on DB [x 36]

therefore the square on HA is also commensurable with the square on AE [x 36]

[vi 22 v 11]

And as the square on HA is to the square on AE so is HA to AE , since the three straight lines HA, AE, AE are proportional [v Del 9]

Therefore HA is commensurable in length with AE

so that HE is also commensurable in length with EA [x 13]

Now since the square on 1 is equal to the rectangle EH, BD

while the square on A is rational

therefore the rectangle EH, BD is also rational

And it is applied to the rational straight line BD

therefore EH is rational and commensurable in length with BD [x 20]

so that EA being commensurable with it is also rational and commensurable in length with BD

Since then as CD is to DB so is FA to AE

while CD, DB are straight lines commensurable in square only,

therefore FA, AE are also commensurable in square only [x 11]

But AE is rational,

therefore FA is also rational

Therefore FA KE are rational straight lines commensurable in square only
therefore EF is an apotome [x 73]

Now the square on CD is greater than the square on DB either by the square on a straight line commensurable with CD or by the square on a straight line incommensurable with it

If then the square on CD is greater than the square on DB by the square on a straight line commensurable with CD the square on FA is also greater than the square on KE by the square on a straight line commensurable with FA . [x 14]

And if CD is commensurable in length with the rational straight line set out,
so also is FK [x 11 12]

if BD is so commensurable
so also is KE [x 12]

but, if neither of the straight lines CD DB is so commensurable
neither of the straight lines FK KE is so

But if the square on CD is greater than the square on DB by the square on a straight line incommensurable with CD
the square on FA is also greater than the square on KE by the square on a straight line incommensurable with FA [x 14]

And, if CD is commensurable with the rational straight line set out,
so also is FA ,
if BD is so commensurable,
so also is KE ,

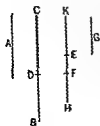
but, if neither of the straight lines CD DB is so commensurable,
neither of the straight lines FK KE is so

so that FE is an apotome the terms of which FK KE are commensurable with the terms CD DB of the binomial straight line and in the same ratio and it has the same order as BC Q E D

PROPOSITION 113

The square on a rational straight line if applied to an apotome produces as breadth the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio and further the binomial so arising has the same order as the apotome

Let A be a rational straight line and BD an apotome and let the rectangle BD KH be equal to the square on A so that the square on the rational straight line A when applied to the apotome BD produces KH as breadth



I say that AH is a binomial straight line the terms of which are commensurable with the terms of BD and in the same ratio and further AH has the same order as BD

For let DC be the annex to BD
therefore BC CD are rational straight lines commensurable in square only [x 73]

Let the rectangle BC G be also equal to the square on A

But the square on A is rational

therefore the rectangle BC, G is also rational

And it has been applied to the rational straight line BC

therefore G is rational and commensurable in length with BC [x 20]

Since now the rectangle BC, G is equal to the rectangle BD, KH ,

therefore, proportionally, as CB is to BD , so is AH to G [vi 16]

But BC is greater than BD ,

therefore AH is also greater than G [v 15 v 14]

Let AE be made equal to G

therefore AE is commensurable in length with BC

And since as CB is to BD so is HA to AE

therefore *convertendo* as BC is to CD so is AH to HE [v 19 Por]

Let it be contrived that, as AH is to HE so is HF to FE

therefore also the remainder AF is to FH as AH is to HE that is as BC is to CD [v 19]

But BC, CD are commensurable in square only,

therefore AF, FH are also commensurable in square only [x 11]

And since as AH is to HE so is AF to FH

while as AH is to HE so is HF to FE

therefore also as AF is to FH so is HF to FE [v 11]

so that also as the first is to the third so is the square on the first to the square on the second [v Def 9]

therefore also as AF is to FE so is the square on AF to the square on FH

But the square on KF is commensurable with the square on FH

for AF, FH are commensurable in square,

therefore AF is also commensurable in length with FE , [x 11]

so that AF is also commensurable in length with KE [x 13]

But AE is rational and commensurable in length with BC ,

therefore AF is also rational and commensurable in length with BC [x 12]

And since as BC is to CD so is AF to FH

alternately as BC is to AF so is DC to FH [v 16]

But BC is commensurable with AF

therefore FH is also commensurable in length with CD [x 11]

But BC, CD are rational straight lines commensurable in square only

therefore AF, FH are also rational straight lines [x Def 3] commensurable in square only

therefore AH is binomial [x 36]

If now the square on BC is greater than the square on CD by the square on a straight line commensurable with BC

the square on AF will also be greater than the square on FH by the square on a straight line commensurable with AF [x 14]

And, if BC is commensurable in length with the rational straight line set out

so also is AF

if CD is commensurable in length with the rational straight line set out,

so also is FH

but if neither of the straight lines BC, CD

then neither of the straight lines AF, FH

But, if the square on BC is greater than the square on CD by the square on a straight line incommensurable with BC

the square on AF is also greater than the square on FH by the square on a

straight line incommensurable with KF

[x 14]

And if BC is commensurable with the rational straight line set out

so also is ΔF

if CD is so commensurable

so also is ΓH

but if neither of the straight lines BC , CD ,

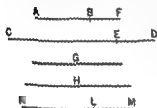
then neither of the straight lines ΔF , ΓH

Therefore KH is a binomial straight line the terms of which ΔF , ΓH are commensurable with the terms BC , CD of the apotome and in the same ratio and further ΔH has the same order as BD Q E D

PROPOSITION 114

If an area be contained by an apotome and the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio the side of the area is rational

For let an area the rectangle AB , CD be contained by the apotome AB and the binomial straight line CD



and let CE be the greater term of the latter let the terms CE , ED of the binomial straight line be commensurable with the terms AF , FB of the apotome and in the same ratio

and let the side of the rectangle AB , CD be G

I say that G is rational

For let a rational straight line H be set out and to CD let there be applied a rectangle equal to the square on H and producing KL as breadth

Therefore KL is an apotome

Let its terms be KM , ML commensurable with the terms CE , ED of the binomial straight line and in the same ratio [x 112]

But CE , ED are also commensurable with AF , FB and in the same ratio therefore as AF is to FB so is KM to ML

Therefore alternately as AF is to KM so is BF to LM

therefore also the remainder AB is to the remainder KL as AF is to KM [v 19]

But AF is commensurable with KM [x 12]

therefore AB is also commensurable with KL [x 11]

And as AB is to KL so is the rectangle CD , AB to the rectangle CD , KL [vi 1]

therefore the rectangle CD , AB is also commensurable with the rectangle CD , KL [x 11]

But the rectangle CD , KL is equal to the square on H

therefore the rectangle CD , AB is commensurable with the square on H

But the square on G is equal to the rectangle CD , AB

therefore the square on G is commensurable with the square on H

But the square on H is rational

therefore the square on G is also rational

therefore G is rational

And it is the side of the rectangle CD , AB

Therefore etc

POINCE. And it is made manifest to us by this also that it is possible for a rational area to be contained by irrational straight lines : Q E D

PROPOSITION II5

From a medial straight line there arise irrational straight lines infinite in number, and none of them is the same as any of the preceding

Let A be a medial straight line

I say that from A there arise irrational straight lines infinite in number, and none of them is the same as any of the preceding

Let a rational straight line B be set out and let the square on C be equal to the rectangle $B A$,



therefore C is irrational [x Def 4]

for that which is contained by an irrational and a rational straight line is irrational [deduction from x 20]

And it is not the same with any of the preceding for the square on none of the preceding if applied to a rational straight line produces as breadth a medial straight line

Again let the square on D be equal to the rectangle B, C , therefore the square on D is irrational [deduction from x 20]

Therefore D is irrational [x Def 4] and it is not the same with any of the preceding for the square on none of the preceding if applied to a rational straight line produces C as breadth

Similarly if this arrangement proceeds *ad infinitum*, it is manifest that from the medial straight line there arise irrational straight lines infinite in number and none is the same with any of the preceding Q E D

BOOK ELEVEN

DEFINITIONS

- 1 A *solid* is that which has length breadth and depth
- 2 An extremity of a solid is a surface
- 3 A *straight line* is at right angles to a plane when it makes right angles with all the straight lines which meet it and are in the plane
- 4 A plane is at right angles to a plane when the straight lines drawn in one of the planes at right angles to the common section of the planes are at right angles to the remaining plane
- 5 The inclination of a straight line to a plane is assuming a perpendicular drawn from the extremity of the straight line which is elevated above the plane to the plane and a straight line joined from the point thus arising to the extremity of the straight line which is in the plane the angle contained by the straight line so drawn and the straight line standing up
- 6 The inclination of a plane to a plane is the acute angle contained by the straight lines drawn at right angles to the common section at the same point one in each of the planes
- 7 A plane is said to be *similarly inclined* to a plane as another is to another when the said angles of the inclinations are equal to one another
- 8 *Parallel planes* are those which do not meet
- 9 *Similar solid figures* are those contained by similar planes equal in multitude
- 10 *Equal and similar solid figures* are those contained by similar planes equal in multitude and in magnitude
- 11 A *solid angle* is the inclination constituted by more than two lines which meet one another and are not in the same surface towards all the lines
Otherwise A *solid angle* is that which is contained by more than two plane angles which are not in the same plane and are constructed to one point
- 12 A *pyramid* is a solid figure contained by planes which is constructed from one plane to one point
- 13 A *prism* is a solid figure contained by planes two of which namely those which are opposite are equal similar and parallel while the rest are parallelograms
- 14 When the diameter of a semicircle remaining fixed the semicircle is carried round and restored again to the same position from which it began to be moved the figure so comprehended is a *sphere*
- 15 The *axis of the sphere* is the straight line which remains fixed and about which the semicircle is turned
- 16 The *centre of the sphere* is the same as that of the semicircle
- 17 A *diameter of the sphere* is any straight line drawn through the centre and

terminated in both directions by the surface of the sphere

18 When one side of those about the right angle in a right angled triangle remaining fixed the triangle is carried round and restored again to the same position from which it began to be moved the figure so comprehended is a *cone*

And if the straight line which remains fixed be equal to the remaining side about the right angle which is carried round, the cone will be *right-angled*, if less *obtuse angled*, and if greater *acute-angled*

19 The *axis of the cone* is the straight line which remains fixed and about which the triangle is turned

20 And the *base* is the circle described by the straight line which is carried round

21 When one side of those about the right angle in a rectangular parallelogram remaining fixed, the parallelogram is carried round and restored again to the same position from which it began to be moved the figure so comprehended is a *cylinder*

22 The *axis of the cylinder* is the straight line which remains fixed and about which the parallelogram is turned

23 And the *bases* are the circles described by the two sides opposite to one another which are carried round

24 *Similar cones and cylinders* are those in which the axes and the diameters of the bases are proportional

25 A *cube* is a solid figure contained by six equal squares

26 An *octahedron* is a solid figure contained by eight equal and equilateral triangles

27 An *icosahedron* is a solid figure contained by twenty equal and equilateral triangles

28 A *dodecahedron* is a solid figure contained by twelve equal equilateral and equiangular pentagons

BOOK VI PROPOSITIONS

PROPOSITION I

A part of a straight line cannot be in the plane of reference and a part in a plane more elevated

I or if possible let a part AB of the straight line ABC be in the plane of reference and a part BC in a plane more elevated

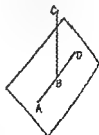
There will then be in the plane of reference some straight line continuous with AB in a straight line

Let it be BD

therefore AB is a common segment of the two straight lines ABC ABD

which is impossible inasmuch as if we describe a circle with centre B and distance AB the diameters will cut off unequal circumferences of the circle

Therefore a part of a straight line cannot be in the plane of reference and a part in a plane more elevated



Q E D

PROPOSITION 2

If two straight lines cut one another, they are in one plane and every triangle is in one plane

For let the two straight lines AB , CD cut one another at the point E ,
I say that AB , CD are in one plane and every triangle is in one plane

For let points F , G be taken at random on EC , EB ,

let CB , FG be joined

and let FH , GK be drawn across

I say first that the triangle ECB is in one plane

For if part of the triangle ECB either FHC or GBK is in the plane of reference and the rest in another

a part also of one of the straight lines EC , EB will be in the plane of reference and a part in another

But if the part $FCBG$ of the triangle ECB be in the plane of reference and the rest in another

a part also of both the straight lines EC , EB will be in the plane of reference and a part in another

which was proved absurd (xi 1)

Therefore the triangle ECB is in one plane

But in whatever plane the triangle ECB is in that plane also is each of the straight lines EC , EB

and in whatever plane each of the straight lines EC , EB is in that plane are AB , CD also (xi 1)

Therefore the straight lines AB , CD are in one plane

and every triangle is in one plane

Q E D

PROPOSITION 3

If two planes cut one another their common section is a straight line

For let the two planes AB , BC cut one another

and let the line DB be their common section

I say that the line DB is a straight line

For if not from D to B let the straight line DEB be joined in the plane AB ,

and in the plane BC the straight line DFB

Then the two straight lines DEB , DFB will have the same extremities and will clearly enclose an area

which is absurd

Therefore DEB , DFB are not straight lines

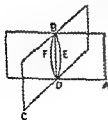
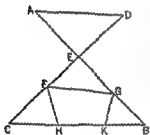
Similarly we can prove that neither will there be any other straight line joined from D to B except DB the common section of the planes AB , BC

Therefore etc

Q E D

PROPOSITION 4

If a straight line be set up at right angles to two straight lines which cut one another, at their common point of section it will also be at right angles to the plane through them



For let a straight line EF be set up at right angles to the two straight lines AB , CD , which cut one another at the point E , from E ,

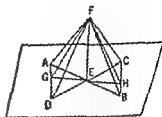
I say that EF is also at right angles to the plane through AB , CD

For let AE , EB , CE , ED be cut off equal to one another,

and let any straight line GEH be drawn across through E at random,

let AD , CB be joined,

and further, let FA , FG , FD , FC , FH , FB be joined from the point F taken at random <on EF >



Now since the two straight lines AE , ED are equal to the two straight lines CE , EB , and contain equal angles [15]

therefore the base AD is equal to the base CB ,

and the triangle AED will be equal to the triangle CEB , [4]

so that the angle DAE is also equal to the angle EBC

But the angle ACG is also equal to the angle BEH [15]

therefore AGE , BEH are two triangles which have two angles equal to two angles respectively and one side equal to one side namely that adjacent to the equal angles that is to say AE to EB ,

therefore they will also have the remaining sides equal to the remaining sides [4]

Therefore GE is equal to EH and AG to BH

And since AE is equal to EB

while FE is common and at right angles

therefore the base FA is equal to the base FB [4]

For the same reason

FC is also equal to FD

And since AD is equal to CB

and FA is also equal to FB

the two sides FA , AD are equal to the two sides FB , BC respectively,

and the base FD was proved equal to the base FC

therefore the angle FAD is also equal to the angle FBC [8]

And since again AG was proved equal to BH

and further FA also equal to FB

the two sides FA , AG are equal to the two sides FB , BH

And the angle FAG was proved equal to the angle FBH ,

therefore the base FG is equal to the base FH [4]

Now since again GE was proved equal to EH

and EF is common

the two sides GE , EF are equal to the two sides HE , EF ,

and the base FG is equal to the base FH

therefore the angle GFE is equal to the angle HEF [8]

Therefore each of the angles CEP , HFP is right

Therefore FE is at right angles to GH drawn at random through E

Similarly we can prove that FE will also make right angles with all the straight lines which meet it and are in the plane of reference

But a straight line is at right angles to a plane when it makes right angles

with all the straight lines which meet it and are in that same plane, [XI Def 3]
 therefore FE is at right angles to the plane of reference
 But the plane of reference is the plane through the straight lines AB CD
 Therefore FE is at right angles to the plane through AB , CD
 Therefore etc

Q E D

PROPOSITION 5

If a straight line be set up at right angles to three straight lines which meet one another, at their common point of section the three straight lines are in one plane

For let a straight line AB be set up at right angles to the three straight lines BC , BD , BE , at their point of meeting at B ,

I say that BC BD , BE are in one plane

For suppose they are not but, if possible, let BD , BE be in the plane of reference and BC in one more elevated

let the plane through AB , BC be produced, it will thus make as common section in the plane of reference, a straight line [XI 3]

Let it make BF

Therefore the three straight lines AB , BC , BF are in one plane, namely that drawn through AB BC

Now, since AB is at right angles to each of the straight lines BD , BE therefore AB is also at right angles to the plane through BD , BE [XI 4]

But the plane through BD , BE is the plane of reference, therefore AB is at right angles to the plane of reference

Thus AB will also make right angles with all the straight lines which meet it and are in the plane of reference [XI Def 3]

But BF which is in the plane of reference meets it, therefore the angle ABF is right

But by hypothesis the angle ABC is also right, therefore the angle ABF is equal to the angle ABC

And they are in one plane

which is impossible

Therefore the straight line BC is not in a more elevated plane, therefore the three straight lines BC , BD BE are in one plane

Therefore if a straight line be set up at right angles to three straight lines, at their point of meeting, the three straight lines are in one plane Q E D

PROPOSITION 6

If two straight lines be at right angles to the same plane the straight lines will be parallel

For let the two straight lines AB CD be at right angles to the plane of reference

I say that AB is parallel to CD

For let them meet the plane of reference at the points B , D

let the straight line BD be joined

let DE be drawn in the plane of reference at right angles to BD ,

let DE be made equal to AB

and let BE , AE , AD be joined

Now, since AB is at right angles to the plane of reference it will also make right angles with all the straight lines which meet it and are in the plane of reference [xi Def 3]

But each of the straight lines BD , BE is in the plane of reference and meets AB ,

therefore each of the angles ABD ABE is right

For the same reason

each of the angles CDB , CDE is also right

And since $AB \equiv$ equal to DE

and BD is common,

the two sides AB , BD are equal to the two sides

ED , DB ,

and they include right angles,

therefore the base AD is equal to the base BE [I 4]

And since AB is equal to DE ,

while AD is also equal to BE ,

the two sides AB , BE are equal to the two sides ED , DA ,

and $AE \equiv$ their common base,

therefore the angle ABE is equal to the angle EDA [I 8]

But the angle ABE is right

therefore the angle EDA is also right,

therefore ED is at right angles to DA

But it is also at right angles to each of the straight lines BD , DC , therefore ED is set up at right angles to the three straight lines BD DA DC at their point of meeting

therefore the three straight lines BD DA , DC are in one plane [xi 5]

But, in whatever plane DB DA are in that plane is AB also

for every triangle is in one plane, [xi 2]

therefore the straight lines AB BD , DC are in one plane

And each of the angles ABD BDC is right

therefore $AB \equiv$ parallel to CD [I 28]

Therefore etc

Q E D

PROPOSITION 7

If two straight lines be parallel and points be taken at random on each of them the straight line joining the points is in the same plane with the parallel straight lines

Let AB CD be two parallel straight lines

and let points E , F be taken at random on them respectively

I say that the straight line joining the points E , F is in the same plane with the parallel straight lines

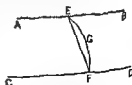
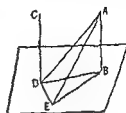
For suppose it is not, but if possible let it be in a more elevated plane as EGF

and let a plane be drawn through EGF

it will then make as section in the plane of reference a straight line [xi 3]

Let it make it as EF ,

therefore the two straight lines EGF EF will enclose an area which is impossible



Therefore the straight line joined from E to F is not in a plane more elevated, therefore the straight line joined from E to F is in the plane through the parallel straight lines AB , CD

Therefore etc

Q E D

PROPOSITION 8

If two straight lines be parallel and one of them be at right angles to any plane, the remaining one will also be at right angles to the same plane

Let AB , CD be two parallel straight lines,

and let one of them, AB , be at right angles to the plane of reference,

I say that the remaining one, CD , will also be at right angles to the same plane

For let AB , CD meet the plane of reference at the points B , D ,

and let BD be joined,

therefore AB , CD , BD are in one plane [XI 7]

Let DE be drawn in the plane of reference, at right angles to BD ,

let DE be made equal to AB

and let BE , AE , AD be joined

Now since AB is at right angles to the plane of reference therefore AB is also at right angles to all the straight lines which meet it and are in the plane of reference [XI Def 3]

therefore each of the angles ABD , ABE is right

And since the straight line BD has fallen on the parallels AB , CD ,

therefore the angles ABD , CDB are equal to two right angles [I 29]

But the angle ABD is right

therefore the angle CDB is also right

therefore CD is at right angles to BD

And since AB is equal to DE

and BD is common

the two sides AB , BD are equal to the two sides ED , DB ,

and the angle ABD is equal to the angle EDB

for each is right

therefore the base AD is equal to the base BE

And, since AB is equal to DE ,

and BE to AD ,

the two sides AB , BE are equal to the two sides ED , DA respectively,

and AE is their common base,

therefore the angle ABE is equal to the angle EDA

But the angle ABE is right

therefore the angle EDA is also right

therefore ED is at right angles to AD

But it is also at right angles to DB

therefore ED is also at right angles to the plane through BD , DA [XI 4]

Therefore CD will also make right angles with all the straight lines which meet it and are in the plane through BD , DA

But DC is in the plane through BD , DA , inasmuch as AB , BD are in the

plane through BD , DA ,

[xi 7]

and DC is also in the plane in which AB BD are

Therefore ED is at right angles to DC ,

so that CD is also at right angles to DE

But CD is also at right angles to BD

Therefore CD is set up at right angles to the two straight lines DE , DB which cut one another from the point of section at D ,

so that CD is also at right angles to the plane through DE , DB [xi 4]

But the plane through DE , DB is the plane of reference,
therefore CD is at right angles to the plane of reference

Therefore etc

Q E D

PROPOSITION 9

Straight lines which are parallel to the same straight line and are not in the same plane with it are also parallel to one another

For let each of the straight lines AB , CD be parallel to EF , not being in the same plane with it

I say that AB is parallel to CD

For let a point G be taken at random on EF

and from it let there be drawn GH in the plane through EF AB , at right angles to EF and GK in the plane through FE CD again at right angles to EF

Now since EF is at right angles to each of the straight lines GH GK

therefore EF is also at right angles to the plane through GH GK [xi 4]

And EF is parallel to AB

therefore AB is also at right angles to the plane through HG GK [xi 8]

For the same reason

CD is also at right angles to the plane through HG GK ,
therefore each of the straight lines AB , CD is at right angles to the plane through HG GK

But if two straight lines be at right angles to the same plane the straight lines are parallel [xi 6]

therefore AB is parallel to CD

Q E D

PROPOSITION 10

If two straight lines meeting one another be parallel to two straight lines meeting one another not in the same plane they will contain equal angles

For let the two straight lines AB , BC meeting one another be parallel to the two straight lines DE EF meeting one another not in the same plane,

I say that the angle ABC is equal to the angle DEF

For let BA BC ED EF be cut off equal to one another and let AD CF , BE AC DF be joined

Now since BA is equal and parallel to ED

therefore AD is also equal and parallel to BE

[i 33]

For the same reason

CF is also equal and parallel to BE

Therefore each of the straight lines AD , CF is equal and parallel to BE

But straight lines which are parallel to the same straight line and are not in the same plane with it are parallel to one another, [xi 9]

therefore AD is parallel and equal to CF

And AC , DF join them,
therefore AC is also equal and parallel to DF [i 33]

Now, since the two sides AB , BC are equal to the two sides DE , EF ,

and the base AC is equal to the base DF
therefore the angle $\angle ABC$ is equal to the angle $\angle DEF$ [i 8]

Therefore etc

Q E D

PROPOSITION 11

From a given elevated point to draw a straight line perpendicular to a given plane

Let A be the given elevated point and the plane of reference the given plane, thus it is required to draw from the point A a straight line perpendicular to the plane of reference

Let any straight line BC be drawn at random in the plane of reference

and let AD be drawn from the point A perpendicular to BC [i 12]

If then AD is also perpendicular to the plane of reference that which was enjoined will have been done

But if not let DE be drawn from the point D at right angles to BC and in the plane of reference, [i 11]

let AF be drawn from A perpendicular to DE [i 12]

and let GH be drawn through the point F parallel to BC [i 31]

Now since BC is at right angles to each of the straight lines DA , DE ,
therefore BC is also at right angles to the plane through ED , DA [xi 4]

And GH is parallel to it,
but if two straight lines be parallel and one of them be at right angles to any plane the remaining one will also be at right angles to the same plane [xi 8]

therefore GH is also at right angles to the plane through ED , DA
Therefore GH is also at right angles to all the straight lines which meet it and are in the plane through ED , DA [xi Def 3]

But AF meets it and is in the plane through ED , DA
therefore GH is at right angles to FA ,
so that FA is also at right angles to GH

But AF is also at right angles to DE
therefore AF is at right angles to each of the straight lines GH , DE

But if a straight line be set up at right angles to two straight lines which cut one another at the point of section it will also be at right angles to the plane through them [xi 4]

therefore FA is at right angles to the plane through ED , GH

But the plane through ED , GH is the plane of reference

therefore AF is at right angles to the plane of reference

Therefore from the given elevated point A the straight line AF has been drawn perpendicular to the plane of reference Q E F

PROPOSITION 12

To set up a straight line at right angles to a given plane from a given point in it

Let the plane of reference be the given plane,

and A the point in it,

thus it is required to set up from the point A a straight line at right angles to the plane of reference

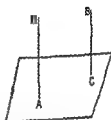
Let any elevated point B be conceived

from B let BC be drawn perpendicular to

the plane of reference [xi 11]

and through the point A let AD be drawn

parallel to BC [i 31]



Then since AD CB are two parallel straight lines while one of them BC is at right angles to the plane of reference therefore the remaining one AD is also at right angles to the plane of reference [xi 8]

Therefore AD has been set up at right angles to the given plane from the point A in it Q E F

PROPOSITION 13

From the same point two straight lines cannot be set up at right angles to the same plane on the same side

For if possible from the same point A let the two straight lines AB , AC be set up at right angles to the plane of reference and on the same side

and let a plane be drawn through BA AC it will then make as section through A in the plane of reference a straight line [xi 3]

Let it make DAE

therefore the straight lines AB AC , DAE are in one plane

And since CA is at right angles to the plane of reference it will also make right angles with all the straight lines which meet it and are in the plane of reference [xi Def 3]

But DAE meets it and is in the plane of reference therefore the angle CAE is right

For the same reason

the angle BAE is also right

therefore the angle CAE is equal to the angle BAE

And they are in one plane

which is impossible

Therefore etc

Q E D

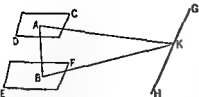
PROPOSITION 14

Planes to which the same straight line is at right angles will be parallel

For let any straight line AB be at right angles to each of the planes CD , EF ,

I say that the planes are parallel

For, if not they will meet when produced



Let them meet,
they will then make as common section,
a straight line [XI 3]

Let them make GH ,
let a point K be taken at random on GH ,
and let AK , BK be joined

Now, since AB is at right angles to
the plane EF

therefore AB is also at right angles to BK which is a straight line in the plane
 EF produced, [XI Def 3]

therefore the angle ABK is right

For the same reason

the angle BAK is also right

Thus in the triangle ABK the two angles ABK BAK are equal to two
right angles

which is impossible [I 17]

Therefore the planes CD EF will not meet when produced

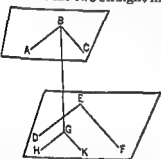
therefore the planes CD EF are parallel [XI Def 8]

Therefore planes to which the same straight line is at right angles are parallel
Q E D

PROPOSITION 15

If two straight lines meeting one another be parallel to two straight lines meeting
one another not being in the same plane, the planes through them are parallel

For let the two straight lines AB , BC meeting one another be parallel to the
two straight lines DE EF meeting one another
not being in the same plane, I say that the
planes produced through AB BC and DE , EF
will not meet one another



For let BG be drawn from the point B per
pendicular to the plane through DE EF [XI 11],

and let it meet the plane at the point G ,
through G let GH be drawn parallel to ED and
 GK parallel to EF [I 31]

Now since BG is at right angles to the plane
through DE EF

therefore it will also make right angles with all the straight lines which meet it
and are in the plane through DE EF [XI Def 3]

But each of the straight lines GH GK meets it and is in the plane through
 DE EF

therefore each of the angles BGH , BGK is right

And since BA is parallel to GH , [XI 9]

therefore the angles GBA , BGH are equal to two right angles [I 29]

But the angle BGH is right

therefore the angle GBA is also right

therefore GB is at right angles to BA

For the same reason

GB is also at right angles to *BC*

Since then the straight line *GB* is set up at right angles to the two straight lines *BA*, *BC* which cut one another, therefore *GB* is also at right angles to the plane through *BA*, *BC* [xi 4]

But planes to which the same straight line is at right angles are parallel, [xi 14]

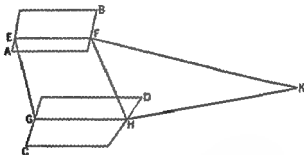
therefore the plane through *AB*, *BC* is parallel to the plane through *DE*, *EF*

Therefore if two straight lines meeting one another be parallel to two straight lines meeting one another not in the same plane, the planes through them are parallel Q E D

PROPOSITION 16

If two parallel planes be cut by any plane their common sections are parallel

For let the two parallel planes *AB*, *CD* be cut by the plane *EFGH*,
and let *EF*, *GH* be their common sections,
I say that *EF* is parallel to *GH*



For if not *EF* *GH* will when produced meet either in the direction of *FH* or of *EG*

Let them be produced as in the direction of *FH* and let them first, meet at *K*

Now since *EFA* is in the plane *AB*

therefore all the points on *EFA* are also in the plane *AB* [xi 1]

But *A* is one of the points on the straight line *EFA*,

therefore *K* is in the plane *AB*

For the same reason

K is also in the plane *CD*

therefore the planes *AB* *CD* will meet when produced

But they do not meet because they are by hypothesis parallel therefore the straight lines *EF* *GH* will not meet when produced in the direction of *FH*

Similarly we can prove that neither will the straight lines *EF*, *GH* meet when produced in the direction of *EG*

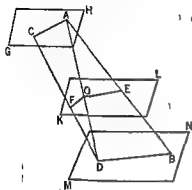
But straight lines which do not meet in either direction are parallel [i Def 23]

Therefore *EF* is parallel to *GH*

Therefore etc Q E D

PROPOSITION 17

If two straight lines be cut by parallel planes they will be cut in the same ratios
 For let the two straight lines AB CD be cut by the parallel planes GH , KL , MN at the points A , E , B and C , F , D ,



I say that, as the straight line AE is to EB so is CF to FD

For let AC BD , AD be joined,
 let AD meet the plane KL at the point O ,
 and let EO OF be joined

Now, since the two parallel planes KL MN are cut by the plane $EBDO$,
 their common sections EO BD are parallel [XI 16]

For the same reason since the two parallel planes GH KL are cut by the plane $AOFC$,
 their common sections AC , OF are parallel [id]

And, since the straight line EO has been drawn parallel to BD , one of the sides of the triangle ABD

therefore proportionally, as AE is to EB so is AO to OD [VI 2]

Again since the straight line OF has been drawn parallel to AC one of the sides of the triangle ADC

proportionally as AO is to OD , so is CF to FD [id]

But it was also proved that as AO is to OD so is AE to EB ,
 therefore also as AE is to EB , so is CF to FD [V 11]

Therefore etc

Q E D

PROPOSITION 18

If a straight line be at right angles to any plane, all the planes through it will also be at right angles to the same plane

For let any straight line AB be at right angles to the plane of reference

I say that all the planes through AB are also at right angles to the plane of reference

For let the plane DE be drawn through AB
 let CE be the common section of the plane DE
 and the plane of reference

let a point F be taken at random on CE
 and from F let FG be drawn in the plane DE
 at right angles to CE [I 11]

Now since AB is at right angles to the plane of reference AB is also at right angles to all the straight lines which meet it and are in the plane of reference [XI Def 3]

so that it is also at right angles to CE

therefore the angle ABF is right

But the angle GFB is also right

therefore AB is parallel to FG [I 28]

But AB is at right angles to the plane of reference,

therefore FG is also at right angles to the plane of reference [xi 8]

Now a plane is at right angles to a plane when the straight lines drawn in one of the planes at right angles to the common section of the planes are at right angles to the remaining plane [xi Def 4]

And FG drawn in one of the planes DE at right angles to CE the common section of the planes was proved to be at right angles to the plane of reference therefore the plane DE is at right angles to the plane of reference

Similarly also it can be proved that all the planes through AB are at right angles to the plane of reference

Therefore etc

Q E D

PROPOSITION 19

If two planes which cut one another be at right angles to any plane their common section will also be at right angles to the same plane

For let the two planes AB BC be at right angles to the plane of reference and let BD be their common section

I say that BD is at right angles to the plane of reference

For suppose it is not and from the point D let DE be drawn in the plane AB at right angles to the straight line AD and DF in the plane BC at right angles to CD

Now since the plane AB is at right angles to the plane of reference and DE has been drawn in the plane AB at right angles to AD their common section

therefore DF is at right angles to the plane of reference

Similarly we can prove that

DF is also at right angles to the plane of reference

Therefore from the same point D two straight lines have been set up at right angles to the plane of reference on the same side which is impossible

Therefore no straight line except the common section DB of the planes AB , BC can be set up from the point D at right angles to the plane of reference therefore etc

[xi Def 4]

[xi 13]

Q E D

PROPOSITION 20

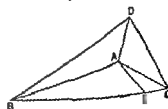
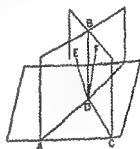
If a solid angle be contained by three plane angles any two taken together in any manner are greater than the remaining one

For let the solid angle at A be contained by the three plane angles BAC CAD DAB

I say that any two of the angles BAC CAD DAB taken together in any manner are greater than the remaining one

If now the angles BAC CAD DAB are equal to one another it is manifest that any two are greater than the remaining one

But if not let BAC be greater and on the straight line AB and at the point A on it let the angle BAE be



constructed in the plane through BA AC , equal to the angle DAB

let AE be made equal to AD ,

and let BEC , drawn across through the point E cut the straight lines AB , AC at the points B C ,

let DB , DC be joined

Now since DA is equal to AE ,

and AB is common

two sides are equal to two sides,

and the angle DAB is equal to the angle BAE ,

therefore the base DB is equal to the base BE

[I 4]

And, since the two sides BD , DC are greater than BC

[I 20]

and of these DB was proved equal to BE

therefore the remainder DC is greater than the remainder EC

Now, since DA is equal to AE ,

and AC is common

and the base DC is greater than the base EC ,

therefore the angle DAC is greater than the angle EAC

[I 25]

But the angle DAB was made equal to the angle BAE ,

therefore the angles DAB , DAC are greater than the angle BAC

Similarly we can prove that the remaining angles also taken together two and two, are greater than the remaining one

Therefore etc

Q. E. D.

PROPOSITION 21

Any solid angle is contained by plane angles less than four right angles

Let the angle at A be a solid angle contained by the plane angles BAC , CAD DAB ,

I say that the angles BAC CAD DAB are less than four right angles

For let points B , C , D be taken at random on the straight lines AB , AC AD respectively,

and let BC , CD , DB be joined

Now since the solid angle at B is contained by the three plane angles CBA ABD CBD any two are greater than the remaining one

[XI 20]

therefore the angles CBA ABD are greater than the angle CBD

For the same reason

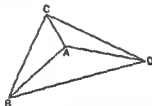
the angles BCA ACD are also greater than the angle BCD , and the angles CDA ADB are greater than the angle CDB

therefore the six angles CBA ABD BCA ACD CDA ADB are greater than the three angles CBD BCD CDB

But the three angles CBD BDC BCD are equal to two right angles [I 32] therefore the six angles CBA ABD BCA , ACD CDA ADB are greater than two right angles

And since the three angles of each of the triangles ABC ACD ADB are equal to two right angles

therefore the nine angles of the three triangles the angles CBA ACB , BAC , ACD CDA CAD ADB , DBA BAD are equal to six right angles



and of them the six angles $ABC, BCA, ACD, CDA, ADB, DBA$ are greater than two right angles,
therefore the remaining three angles BAC, CAD, DAB containing the solid angle are less than four right angles

Therefore etc

Q E D

PROPOSITION 22

If there be three plane angles of which two taken together in any manner are greater than the remaining one, and they are contained by equal straight lines, it is possible to construct a triangle out of the straight lines joining the extremities of the equal straight lines

Let there be three plane angles ABC, DEF, GHA , of which two taken together in any manner are greater than the remaining one, namely

the angles ABC, DEF greater than the angle GHA ,

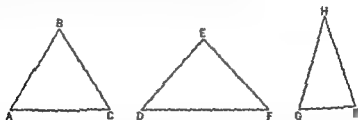
the angles DEF, GHA greater than the angle ABC

and further, the angles GHA, ABC greater than the angle DEF ,

let the straight lines AB, BC, DE, EF, GH, HA be equal,

and let AC, DF, GA be joined,

I say that it is possible to construct a triangle out of straight lines equal to AC, DF, GA that is that any two of the straight lines AC, DF, GA are greater than the remaining one



Now if the angles ABC, DEF, GHA are equal to one another it is manifest that AC, DF, GA being equal also it is possible to construct a triangle out of straight lines equal to AC, DF, GA

But if not let them be unequal

and on the straight line HA and at the point H on it let the angle KHL be constructed equal to the angle ABC ,

let HL be made equal to one of the straight lines AB, BC, DE, EF, GH, HA

and let KL, GL be joined

Now since the two sides AB, BC are equal to the two sides KH, HL

and the angle at B is equal to the angle KHL

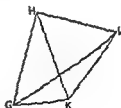
therefore the base AC is equal to the base KL

And since the angles ABC, GHA are greater than the angle DEF ,

while the angle ABC is equal to the angle KHL

therefore the angle GHL is greater than the angle DEF

And since the two sides GH, HL are equal to the two sides DE, EF



{ 4 }

and the angle GHL is greater than the angle DEF ,
therefore the base GL is greater than the base DF

[1 24]

But GK, KL are greater than GL

Therefore GK, AL are much greater than DF

But KL is equal to AC ,

therefore AC, GK are greater than the remaining straight line DF

Similarly we can prove that

AC, DF are greater than GK ,

and further DF, GK are greater than AC

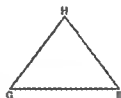
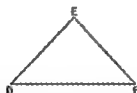
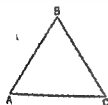
Therefore it is possible to construct a triangle out of straight lines equal to
 AC, DF, GK

Q E D

PROPOSITION 23

To construct a solid angle out of three plane angles two of which taken together in any manner, are greater than the remaining one thus the three angles must be less than four right angles

Let the angles ABC, DEF, GHA be the three given plane angles and let two of these taken together in any manner be greater than the remaining one, while further the three are less than four right angles
thus it is required to construct a solid angle out of angles equal to the angles ABC, DEF, GHA



Let AB, BC, DE, EF, GH, HA be cut off equal to one another

and let AC, DF, GK be joined

it is therefore possible to construct a triangle out of straight lines equal to AC, DF, GK

[xi 22]

Let LMN be so constructed that AC is equal to LM, DF to MN and further, GK to NL ,
let the circle LMN be described about the triangle LMN

let its centre be taken and let it be O ,

let LO, MO, NO be joined

I say that $\angle B$ is greater than $\angle O$

For if not $\angle B$ is either equal to $\angle O$, or less

First let it be equal

Then since AB is equal to LO

while AB is equal to BC and OL to OM

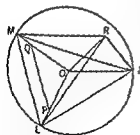
the two sides AB, BC are equal to the two sides LO, OM respectively,

and by hypothesis the base AC is equal to the base LM ,

therefore the angle ABC is equal to the angle LOM

[1 8]

For the same reason



the angle DEF is also equal to the angle MON ,
 and further the angle GKH to the angle NOL ,
 therefore the three angles ABC, DEF, GHK are equal to the three angles LOM, MOV, NOL

But the three angles LOM, MON, NOL are equal to four right angles,
 therefore the angles ABC, DEF, GHK are equal to four right angles

But they are also, by hypothesis, less than four right angles
 which is absurd

Therefore AB is not equal to LO

I say next that neither is AB less than LO

For, if possible let it be so

and let OP be made equal to AB and OQ equal to BC ,
 and let PQ be joined

Then, since AB is equal to BC ,

OP is also equal to OQ ,

so that the remainder LP is equal to QV

Therefore LM is parallel to PQ

and LMO is equiangular with PQO ,

therefore, as OL is to LM , so is OP to PQ ,

and alternately as LO is to OP , so is LM to PQ

But LO is greater than OP ,

therefore LM is also greater than PQ

But LM was made equal to AC ,

therefore AC is also greater than PQ

Since then the two sides AB, BC are equal to the two sides PO, OQ ,

and the base AC is greater than the base PQ

therefore the angle ABC is greater than the angle POQ [I 25]

Similarly we can prove that

the angle DEF is also greater than the angle MON ,

and the angle GKH greater than the angle NOL

Therefore the three angles ABC, DEF, GHK are greater than the three angles LOM, MON, NOL

But by hypothesis the angles ABC, DEF, GHK are less than four right angles

therefore the angles LOM, MON, NOL are much less than four right angles

But they are also equal to four right angles

which is absurd

Therefore AB is not less than LO

And it was proved that neither is it equal,

therefore AB is greater than LO

Let then OR be set up from the point O at right angles to the plane of the circle LMN [XI 17]

and let the square on OR be equal to that area by which the square on AB is greater than the square on LO [Lemma]

let RL, RM, RN be joined

Then since RO is at right angles to the plane of the circle LMN ,
 therefore RO is also at right angles to each of the straight lines LO, MO, NO

And since LO is equal to OV

while OR is common and at right angles

therefore the base RL is equal to the base RM

[1 4]

For the same reason

RN is also equal to each of the straight lines RL RM ,

therefore the three straight lines RL RM , RN are equal to one another

Next, since by hypothesis the square on OR is equal to that area by which the square on AB is greater than the square on LO

therefore the square on AB is equal to the squares on LO OR

But the square on LR is equal to the squares on LO OR , for the angle LOR is right,

[1 47]

therefore the square on AB is equal to the square on RL ,

therefore AB is equal to RL

But each of the straight lines BC DE EF , GH HA is equal to AB ,

while each of the straight lines RM , RN is equal to RL ,

therefore each of the straight lines AB , BC , DE EF GH HA

is equal to each of the straight lines RL , RM RN

And since the two sides LR RM are equal to the two sides AB , BC ,

and the base LM is by hypothesis equal to the base AC ,

therefore the angle LRM is equal to the angle ABC

[1 8]

For the same reason

the angle MRN is also equal to the angle DEF ,

and the angle LRN to the angle GHA

Therefore, out of the three plane angles LRM , MRN , LRN , which are equal to the three given angles ABC , DEF GHA the solid angle at R has been constructed which is contained by the angles LRM MRN LRN $Q E F$

LEMMA

But how it is possible to take the square on OR equal to that area by which the square on AB is greater than the square on LO we can show as follows

Let the straight lines AB LO be set out,

and let AB be the greater,

let the semicircle ABC be described on AB

and into the semicircle ABC let AC be fitted equal to the straight line LO not being greater than the diameter AB

[IV 1]

let CB be joined

Since then the angle ACB is an angle in the semicircle ACB

therefore the angle ACB is right

[III 31]

Therefore the square on AB is equal to the squares on AC CB

[1 47]

Hence the square on AB is greater than the square on LO in the square on CB

But AC is equal to LO

Therefore the square on AB is greater than the square on LO by the square on CB

If then we cut off OR equal to BC the square on AB will be greater than the square on LO by the square on OR

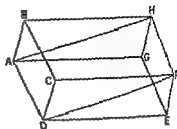
PROPOSITION 21

If a solid be contained by parallel planes the opposite planes are parallelogrammic

For let the solid $CDHG$ be contained by the parallel planes AC , GF AH DF BF , AE ,

I say that the opposite planes in it are equal and parallelogrammic

For since the two parallel planes BG CE are cut by the plane AC their common sections are parallel [xi 16]



Therefore AB is parallel to DC
Again since the two parallel planes BF AE are cut by the plane AC their common sections are parallel [xi 16]

Therefore BC is parallel to AD
But AB was also proved parallel to DC

therefore AC is a parallelogram
Similarly we can prove that each of the planes DF , FG , GB , BF , AE is a parallelogram

Let AH DF be joined

Then since AB is parallel to DC and BH to CF the two straight lines AB BH which meet one another are parallel to the two straight lines DC CF which meet one another not in the same plane, therefore they will contain equal angles, [xi 10]

therefore the angle ABH is equal to the angle DCF
And since the two sides AB BH are equal to the two sides DC CF , [i 30]

and the angle ABH is equal to the angle DCF
therefore the base AH is equal to the base DF
and the triangle ABH is equal to the triangle DCF [i 4]
And the parallelogram BG is double of the triangle ABH and the parallelogram CE double of the triangle DCF [i 34]

therefore the parallelogram BG is equal to the parallelogram CE
Similarly we can prove that

AC is also equal to GF
and AD to BF

Therefore etc

Q E D

PROPOSITION 25

If a parallelepipedal solid be cut by a plane which is parallel to the opposite planes then as the base is to the base so will the solid be to the solid

For let the parallelepipedal solid $ABCD$ be cut by the plane FG which is parallel to the opposite planes RA DH

I say that as the base $APFV$ is to the base $EHCF$ so is the solid $ABFU$ to the solid $EICD$

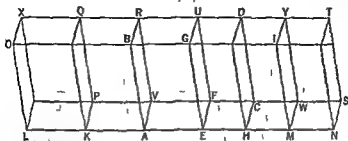
For let AH be produced in each direction

let any number of straight lines whatever AK KL be made equal to AE ,
and any number whatever HM MN equal to EH ,

and let the parallelograms LP KV HW MS and the solids LQ , KR , DM , VT be completed

Then since the straight lines LA KA AE are equal to one another the parallelograms LP KV IF are also equal to one another,

KO , KB , AG are equal to one another,
and further, LV , KQ , AR are equal to one another, for they are opposite
[XI 24]



For the same reason

the parallelograms EC , HW , MS are also equal to one another,

— — — HG , HI , IN are equal to one another

and further, DH , MI , NT are equal to one another

Therefore in the solids LQ , KR , AU three planes are equal to three planes

But the three planes are equal to the three opposite,

therefore the three solids LQ , KR , AU are equal to one another

For the same reason

the three solids ED , DM , MT are also equal to one another

Therefore whatever multiple the base LF is of the base AF , the same multiple also is the solid LU of the solid AU

For the same reason

whatever multiple the base NF is of the base FH the same multiple also is the solid NU of the solid HU

And if the base LF is equal to the base NF the solid LU is also equal to the solid NU

if the base LF exceeds the base NF the solid LU also exceeds the solid NU ,
and, if one falls short, the other falls short

Therefore there being four magnitudes the two bases AF , FH and the two solids AU , UH

equimultiples have been taken of the base AF and the solid AU namely the base LF and the solid LU ,

and equimultiples of the base FH and the solid HU , namely the base NF and the solid NU

and it has been proved that if the base LF exceeds the base NF the solid LU also exceeds the solid NU

if the bases are equal the solids are equal,

and if the base falls short the solid falls short

Therefore as the base AF is to the base FH , so is the solid AU to the solid UH

[v Def 5]

Q E D

PROPOSITION 26

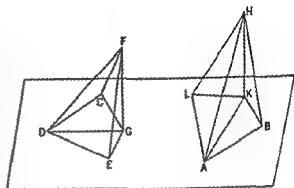
On a given straight line and at a given point on it, to construct a solid angle equal to a given solid angle

Let AB be the given straight line, A the given point on it and the angle at D contained by the angles EDC , EDF , FDC , the given solid angle, thus it is required to construct on the straight line AB , and at the point A on it \equiv solid angle equal to the solid angle at D

For let a point F be taken at random on DF

let FG be drawn from F perpendicular to the plane through ED DC , and let it meet the plane at G [xi 11]

let DG be joined, let there be constructed on the straight line AB and at the point A on it the angle BAL equal to the angle EDC , and the angle BAK equal to the angle EDG



[i 23]

let AK be made equal to DG

let AH be set up from the point A at right angles to the plane through BA , AL , [xi 12]

let AH be made equal to GF

and let HA be joined

I say that the solid angle at A , contained by the angles BAL , BAH , HAL is equal to the solid angle at D contained by the angles EDC , EDF , FDC

For let AB DE be cut off equal to one another

and let HB KB FE GE be joined

Then since FG is at right angles to the plane of reference it will also make right angles with all the straight lines which meet it and are in the plane of reference [xi Def 3]

therefore each of the angles FGD FGE is right

For the same reason

each of the angles HKA HKB is also right

And since the two sides KA AB are equal to the two sides GD DE respectively

and they contain equal angles

therefore the base KB \equiv equal to the base GE [i 4]

But AH is also equal to GF

and they contain right angles

therefore HB is also equal to FE [i 4]

Again since the two sides AK AH are equal to the two sides DG GF ,

and they contain right angles

therefore the base AB is equal to the base FD [i 4]

But AB \equiv also equal to DE

therefore the two sides HA AB are equal to the two sides DF DE

And the base HB is equal to the base FE

therefore the angle BAH is equal to the angle EDF [i 8]

For the same reason

the angle HAL is also equal to the angle FDC

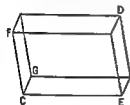
And the angle BAL is also equal to the angle EDC

Therefore on the straight line AB , and at the point A on it a solid angle has been constructed equal to the given solid angle at D Q E F

PROPOSITION 27

On a given straight line to describe a parallelepipedal solid similar and similarly situated to a given parallelepipedal solid

Let AB be the given straight line and CD the given parallelepipedal solid thus it is required to describe on the given straight line AB a parallelepipedal solid similar and similarly situated to the given parallelepipedal solid CD



For on the straight line AB and at the point A on it let the solid angle contained by the angles BAH HAK KAB , be constructed equal to the solid angle at C , so that the angle BAH is equal to the angle ECF the angle BAK equal to the

angle ECG , and the angle KAH to the angle GCF

and let it be contrived that

as EC is to CG , so is BA to AK ,

and as GC is to CF , so is KA to AH

[VI 12]

Therefore also *ex aequali*

as EC is to CF , so is BA to AH

[V 22]

Let the parallelogram HB and the solid AL be completed

Now since, as EC is to CG so is BA to AK ,

and the sides about the equal angles ECG BAK are thus proportional

therefore the parallelogram GE is similar to the parallelogram AB

For the same reason

the parallelogram AH is also similar to the parallelogram GF and further FE to HB ,

therefore three parallelograms of the solid CD are similar to three parallelograms of the solid AL

But the former three are both equal and similar to the three opposite parallelograms

and the latter three are both equal and similar to the three opposite parallelograms,

therefore the whole solid CD is similar to the whole solid AL [XI Def 9]

Therefore on the given straight line AB there has been described AL similar and similarly situated to the given parallelepipedal solid CD Q E F

PROPOSITION 28

If a parallelepipedal solid be cut by a plane through the diagonals of the opposite planes the solid will be bisected by the plane

For let the parallelepipedal solid AB be cut by the plane $CDEF$ through the diagonals CF , DE of opposite planes

I say that the solid AB will be bisected by the plane $CDEF$

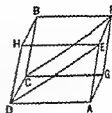
For since the triangle CGF is equal to the triangle CFB ,

[I 34]

and ADE to DEH ,
 while the parallelogram CA is also equal to the parallelogram EB , for they are opposite and GE to CH ,
 therefore the prism contained by the two triangles CGF , ADE and the three parallelograms GE , AC , CE
 is also equal to the prism contained by the two triangles CFB , DEH and the three parallelograms CH , BE , CE ,
 for they are contained by planes equal both in multitude and in magnitude [11 Def 10]

Hence the whole solid AB is bisected by the plane $CDEF$

Q E D



PROPOSITION 29

Parallelepipedal solids which are on the same base and of the same height and in which the extremities of the sides which stand up are on the same straight lines are equal to one another

Let CM CN be parallelepipedal solids on the same base AB and of the same height

and let the extremities of their sides which stand up namely AG AF LM LN , CD CE BH BA , be on the same straight lines FN DA

I say that the solid CM is equal to the solid CN

For since each of the figures CH CA is a parallelogram CB is equal to each of the straight lines DH EA [1 34]

hence DH is also equal to EA

Let EH be subtracted from each

therefore the remainder DE is equal to the remainder HK

Hence the triangle DCE is also equal to the triangle HBK [1 8 4]

and the parallelogram DG to the parallelogram HN [1 36]

For the same reason

the triangle AFG is also equal to the triangle MLN

But the parallelogram CF is equal to the parallelogram BM and CG to BN , for they are opposite

therefore the prism contained by the two triangles AFG DCE and the three parallelograms AD DG CG is equal to the prism contained by the two triangles MLN HBK and the three parallelograms BM HN BN

Let there be added to each the solid of which the parallelogram AB is the base and $GEHM$ its opposite

therefore the whole parallelepipedal solid CM is equal to the whole parallelepipedal solid CN

Therefore etc

Q E D

PROPOSITION 30

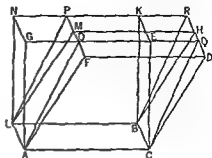
Parallelepipedal solids which are on the same base and of the same height and in which the extremities of the sides which stand up are not on the same straight lines are equal to one another

Let CM CN be parallelepipedal solids on the same base AB and of the same height, and let the extremities of their sides which stand up namely $AF, AG, LM, LN, CD, CE, BH, BK$, not be on the same straight lines,

I say that the solid CM is equal to the solid CN

For let NK DH be produced and meet one another at R

and further, let FM, GE be produced to P, Q



let AO, LP, CQ, BR be joined

Then the solid CM , of which the parallelogram $ACBL$ is the base, and $FDHM$ its opposite is equal to the solid CP , of which the parallelogram $ACBL$ is the base and $OQRP$ its opposite

for they are on the same base $ACBL$ and of the same height, and the extremities of their sides which stand up namely $AF, AO, LM, LP, CD, CQ, BH, BR$, are on the same straight lines FP, DR [XI 29]

But the solid CP , of which the parallelogram $ACBL$ is the base and $OQRP$ its opposite, is equal to the solid CN , of which the parallelogram $ACBL$ is the base and $GEKN$ its opposite,

for they are again on the same base $ACBL$ and of the same height and the extremities of their sides which stand up namely $AO, CE, CQ, LN, LP, BK, BR$ are on the same straight lines GQ, NR

Hence the solid CM is also equal to the solid CN

Therefore etc

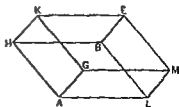
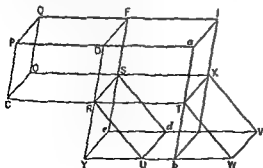
Q E D

PROPOSITION 31

Parallelepipedal solids which are on equal bases and of the same height are equal to one another

Let the parallelepipedal solids AE CF , of the same height be on equal bases AB, CD

I say that the solid AE is equal to the solid CF



First let the sides which stand up $HK, BE, AG, LM, PQ, DF, CO, RS$, be at right angles to the bases AB, CD

let the straight line RT be produced in a straight line with CR on the straight line RT and at the point R on it let the angle TRU be constructed equal to the angle ALB , [r 23]

let RT be made equal to AL , and RU equal to LB ,

and let the base RW and the solid λU be completed

Now since the two sides TR RU are equal to the two sides AL LB , and they contain equal angles, therefore the parallelogram RW is equal and similar to the parallelogram HL

Since again AL is equal to RT , and LM to RS

and they contain right angles

therefore the parallelogram $R\lambda$ is equal and similar to the parallelogram λM

For the same reason

LE is also equal and similar to SU ,

therefore three parallelograms of the solid AE are equal and similar to three parallelograms of the solid λU

But the former three are equal and similar to the three opposite and the latter three to the three opposite [xi 21]

therefore the whole parallelepipedal solid AE is equal to the whole parallelepipedal solid XU [xi Def 10]

Let DR , WU be drawn through and meet one another at Y ,

let aTb be drawn through T parallel to DY ,

let PD be produced to a

and let the solids $\lambda\lambda$, RI be completed

Then the solid $\lambda\lambda$ of which the parallelogram $R\lambda$ is the base and Ye its opposite is equal to the solid XU of which the parallelogram $R\lambda$ is the base and $U\lambda$ its opposite

for they are on the same base $R\lambda$ and of the same height and the extremities of their sides which stand up namely $R\lambda$ RU , Tb TH , Se Sd λc $\lambda\lambda$, are on the same straight lines YW eV [xi 29]

But the solid λU is equal to AE

therefore the solid $\lambda\lambda$ is also equal to the solid AE

And since the parallelogram $RUWT$ is equal to the parallelogram λT for they are on the same base RT and in the same parallels RT , λW , [i 35]

while $RUWT$ is equal to CD since it is also equal to AB ,

therefore the parallelogram YT is also equal to CD

But DT is another parallelogram,

therefore as the base CD is to DT so is λT to DT [v 7]

And since the parallelepipedal solid CI has been cut by the plane RF which is parallel to opposite planes

as the base CD is to the base DT so is the solid CF to the solid RI [xi 25]

For the same reason

since the parallelepipedal solid λI has been cut by the plane RY which is parallel to opposite planes

as the base λT is to the base TD so is the solid λY to the solid RI [xi 25]

But as the base CD is to DT so is λT to DT

therefore also as the solid CF is to the solid RI so is the solid $\lambda\lambda$ to RI [v 11]

Therefore each of the solids CF $\lambda\lambda$ has to RI the same ratio

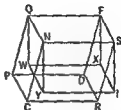
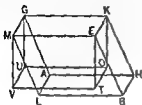
therefore the solid CF is equal to the solid λY [v 9]

But XY was proved equal to AE

therefore AE is also equal to CF

Next, let the sides standing up $AG, HK, BE, LU, CN, PQ, DF, RS$ not be at right angles to the bases AB, CD ,

I say again that the solid AE is equal to the solid CF



For from the points A, E, G, M, Q, F, N, S let $KO, ET, GU, MV, QW, FX, NY, SI$ be drawn perpendicular to the plane of reference and let them meet the plane at the points O, T, U, V, W, X, Y, I

and let $OT, OU, UV, TV, WX, WY, YI, XI$ be joined

Then the solid AV is equal to the solid QI ,
for they are on the equal bases AM, QS and of the same height and their sides which stand up are at right angles to their bases [First part of this Prop]

But the solid AV is equal to the solid AE

and QI to CF

for they are on the same base and of the same height while the extremities of their sides which stand up are not on the same straight lines [xi 30]

Therefore the solid AE is also equal to the solid CF

Therefore etc

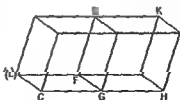
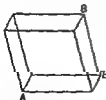
Q E D

PROPOSITION 32

Parallelepipedal solids which are of the same height are to one another as their bases

Let AB, CD be parallelepipedal solids of the same height

I say that the parallelepipedal solids AB, CD are to one another as their bases that is that as the base AE is to the base CF so is the solid AB to the solid CD



For let FH equal to AE be applied to FG [i 45]
and on FH as base and with the same height as that of CD let the parallelepipedal solid GH be completed

Then the solid AB is equal to the solid GH
for they are on equal bases AE, FH and of the same height [xi 31]

And since the parallelepipedal solid CH is cut by the plane DG which is parallel to opposite planes

But the former three parallelograms are equal and similar to their opposites and the latter three to their opposites, [xi 24]
 therefore the whole solid AP is equal and similar to the whole solid CD [xi Def 10]

Let the parallelogram GK be completed,
 and on the parallelograms GA, KL as bases and with the same height as that of AB , let the solids EO, LQ be completed

Then since owing to the similarity of the solids AB, CD
 as AE is to CF so is EG to FN , and EH to FR
 while CF is equal to EK , FN to EL , and FR to EM
 therefore as AE is to EK , so is GE to FI , and HE to EM

But, as AE is to EK so is AG to the parallelogram GK ,
 as GE is to EL , so is GK to KL
 and as HE is to EM , so is QE to KM , [vi 1]
 therefore also as the parallelogram AG is to GK so is GK to KL and QE to KM

But, as AG is to GK , so is the solid AB to the solid EO ,
 as GK is to KL so is the solid OE to the solid QL
 and, as QE is to KM , so is the solid QL to the solid AP , [xi 32]
 therefore also as the solid AB is to EO so is EO to QL and QL to AP

But if four magnitudes be continuously proportional, the first has to the fourth the ratio triplicate of that which it has to the second [v Def 10]
 therefore the solid AB has to AP the ratio triplicate of that which AB has to EO

But as AB is to EO , so is the parallelogram AG to GK and the straight line AE to EK [vi 1]
 hence the solid AB has also to KP the ratio triplicate of that which AE has to EK

But the solid KP is equal to the solid CD ,
 and the straight line EK to CF
 therefore the solid AB has also to the solid CD the ratio triplicate of that which the corresponding side of it AE has to the corresponding side CF

Therefore etc

Q E D

PROPOSITION 34
 From this it is manifest that if four straight lines be continuously proportional as the first is to the fourth so will a parallelepipedal solid on the first be to the similar and similarly described parallelepipedal solid on the second inasmuch as the first has to the fourth the ratio triplicate of that which it has to the second

PROPOSITION 34

In equal parallelepipedal solids the bases are reciprocally proportional to the heights and those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal

Let AB, CD be equal parallelepipedal solids,

I say that in the parallelepipedal solids AB, CD the bases are reciprocally proportional to the heights
 that is as the base EH is to the base NQ so is the height of the solid CD to the height of the solid AB

First, let the sides which stand up namely $AG, EF, LB, HA, CM, NO, PD$

QR be at right angles to their bases,

I say that, as the base EH is to the base NQ , so is CM to AG

If now the base EH is equal to the base NQ

while the solid AB is also equal to the solid CD ,

CM will also be equal to AG

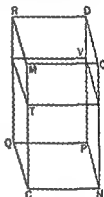
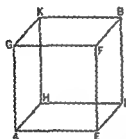
For parallelepipedal solids of the same height are to one another as the bases, [xi 32]

and as the base EH is to NQ so will CM be to AG

and it is manifest that in the parallelepipedal solids AB , CD the bases are reciprocally proportional to the heights

Next, let the base EH not be equal to the base NQ ,

but let EH be greater



Now the solid AB is equal to the solid CD

therefore CM is also greater than AG

Let then CT be made equal to AG ,

and let the parallelepipedal solid VC be completed on NQ as base and with CT as height

Now since the solid AB is equal to the solid CD

and CV is outside them

while equals have to the same the same ratio

therefore as the solid AB is to the solid CV so is the solid CD to the solid CV [v 1]

But as the solid AB is to the solid CV so is the base EH to the base NQ

for the solids AB , CV are of equal height [xi 32]

and as the solid CD is to the solid CV so is the base MQ to the base TQ [xi 23] and CM to CT [vi 1]

therefore also as the base EH is to the base NQ so is MC to CT

But CT is equal to AG

therefore also as the base EH is to the base NQ so is MC to AG

Therefore in the parallelepipedal solids AB , CD the bases are reciprocally proportional to the heights

Again in the parallelepipedal solids AB , CD let the bases be reciprocally proportional to the heights that is as the base EH is to the base NQ so let the height of the solid CD be to the height of the solid AB ,

I say that the solid AB is equal to the solid CD

Let the sides which stand up be again at right angles to the bases

Now if the base EH is equal to the base NQ

and as the base EH is to the base NQ so is the height of the solid CD to the height of the solid AB ,

therefore the height of the solid CD is also equal to the height of the solid AB

But parallelepipedal solids on equal bases and of the same height are equal to one another, [xi 31]

therefore the solid AB is equal to the solid CD

Next let the base EH not be equal to the base NQ ,

but let EH be greater,

therefore the height of the solid CD is also greater than the height of the solid AB ,

that is CM is greater than AG

Let CT be again made equal to AG

and let the solid CV be similarly completed

Since, as the base EH is to the base NQ so is MC to AG

while AG is equal to CT

therefore as the base EH is to the base NQ so is CM to CT

But as the base EH is to the base NQ , so is the solid AB to the solid CV ,

for the solids AB , CV are of equal height, [xi 32]

and, as CM is to CT , so is the base MQ to the base QT [xi 1]

and the solid CD to the solid CV [xi 25]

Therefore also as the solid AB is to the solid CV , so is the solid CD to the solid CV ,

therefore each of the solids AB CD has to CV the same ratio

Therefore the solid AB is equal to the solid CD [v 9]

Now let the sides which stand up FE , BL , GA , HA , ON , DP , MC , RQ not be at right angles to their bases

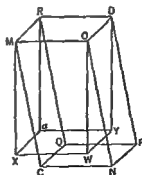
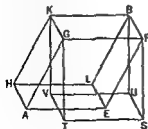
let perpendiculars be drawn from the points F , G , B , K , O , M , D , R to the planes through EH , NQ

and let them meet the planes at S , T , U , V , W , X , Y , a

and let the solids FV Oa be completed,

I say that in this case too if the solids AB CD are equal the bases are reciprocally proportional to the heights that is as the base EH is to the base NQ , so is the height of the solid CD to the height of the solid AB

Since the solid AB is equal to the solid CD



while AB is equal to BT ,

for they are on the same base FA and of the same height

[xi 29 30]

and the solid CD is equal to DX

for they are again on the same base RO and of the same height, [id]

therefore the solid BT is also equal to the solid DX

Therefore, as the base FK is to the base OR , so \equiv the height of the solid DX to the height of the solid BT [Part 1]

But the base FA is equal to the base EH ,

and the base OR to the base NQ ,

therefore as the base EH is to the base NQ , so \equiv the height of the solid DX to the height of the solid BT

But the solids DX , BT and the solids DC , BA have the same heights respectively,

therefore as the base $EH \equiv$ to the base AQ so is the height of the solid DC to the height of the solid AB

Therefore in the parallelepipedal solids AB , CD the bases are reciprocally proportional to the heights

Again in the parallelepipedal solids AB , CD let the bases be reciprocally proportional to the heights

that is as the base EH is to the base NQ , so let the height of the solid CD be to the height of the solid AB ,

I say that the solid AB is equal to the solid CD

For with the same construction

since as the base EH is to the base AQ so is the height of the solid CD to the height of the solid AB

while the base EH is equal to the base FK ,

and NQ to OR

therefore as the base FK is to the base OR so is the height of the solid CD to the height of the solid AB

But the solids AB , CD and BT , DX have the same heights respectively therefore as the base $FK \equiv$ to the base OR , so is the height of the solid DX to the height of the solid BT

Therefore in the parallelepipedal solids BT , DX the bases are reciprocally proportional to the heights

therefore the solid BT is equal to the solid DX [Part 1]

But BT is equal to BA

for they are on the same base FA and of the same height, [xi 29 30]

and the solid DX is equal to the solid DC [id]

Therefore the solid AB is also equal to the solid CD Q E D

PROPOSITION 25

If there be two equal plane angles and on their vertices there be set up elevated straight lines containing equal angles with the original straight lines respectively if on the elevated straight lines points be taken at random and perpendiculars be drawn from them to the planes in which the original angles are, and if from the points so arising in the planes straight lines be joined to the vertices of the original angles they will contain with the elevated straight lines equal angles

Let the angles BAC , EDF be two equal rectilineal angles and from the points A , D let the elevated straight lines AG , DM be set up containing with the original straight lines equal angles respectively, namely the angle MAE to the angle GAB and the angle MDF to the angle GAC ,

let points G, M be taken at random on AG, DM ,
 let GL, MN be drawn from the points G, M perpendicular to the planes through
 BA, AC and ED, DF , and let them meet the planes at L, N ,
 and let LA, ND be joined

I say that the angle GAL is equal to the angle MDN

Let AH be made equal
 to DM ,
 and let HA be drawn
 through the point H paral-
 lel to GL

But GL is perpendicular
 to the plane through BA ,
 AC ,
 therefore HK is also per-
 pendicular to the plane
 through BA, AC [x1 8]



From the points K, N let KC, NF KB, NE be drawn perpendicular to the
 straight lines AC, DF, AB, DE :

and let HC, CB MF, FE be joined

Since the square on HA is equal to the squares on HK, KA ,

and the squares on KC, CA are equal to the square on KA , [1 47]

therefore the square on HA is also equal to the squares on HK, KC, CA

But the square on HC is equal to the squares on HK, KC , [1 47]

therefore the square on HA is equal to the squares on HC, CA

Therefore the angle HCA is right [1 48]

For the same reason

the angle DFM is also right

Therefore the angle ACH is equal to the angle DFM

But the angle HAC is also equal to the angle MDF

Therefore MDF, HAC are two triangles which have two angles equal to two
 angles respectively and one side equal to one side namely that subtending
 one of the equal angles that is, HA equal to MD ,

therefore they will also have the remaining sides equal to the remaining sides
 respectively [1 26]

Therefore AC is equal to DF

Similarly we can prove that AB is also equal to DE

Since then AC is equal to DF and AB to DE

the two sides CA, AB are equal to the two sides FD, DE

But the angle CAB is also equal to the angle FDE

therefore the base BC is equal to the base EF , the triangle to the triangle, and
 the remaining angles to the remaining angles [1 4]

therefore the angle ACB is equal to the angle DFE

But the right angle ACK is also equal to the right angle DFN

therefore the remaining angle BCK is also equal to the remaining angle EFN

For the same reason

the angle CBK is also equal to the angle FEN

Therefore BCK, FEN are two triangles which have two angles equal to two
 angles respectively and one side equal to one side namely that adjacent to
 the equal angles that is BC equal to EF

therefore they will also have the remaining sides equal to the remaining sides [1 26]

Therefore CH is equal to FA

But AC is also equal to DF ,

therefore the two sides AC CH are equal to the two sides DF FA ,
and they contain right angles

Therefore the base AK is equal to the base DN

[1 4]

And since AH is equal to DV

the square on AH is also equal to the square on DV

But the squares on AK KH are equal to the square on AH ,

for the angle AKH is right,

[1 4]

and the squares on DN NV are equal to the square on DV

for the angle DNV is right,

[1 47]

therefore the squares on AK KH are equal to the squares on DN NV ,

and of these the square on AK is equal to the square on DV

therefore the remaining square on KH is equal to the square on NV

therefore HK is equal to VN

And since the two sides HA AK are equal to the two sides MD DV respectively,

and the base HA was proved equal to the base MN ,

therefore the angle HAK is equal to the angle MDN

[1 8]

Therefore etc

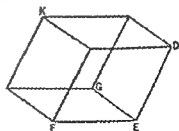
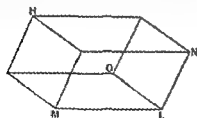
PORISM From this it is manifest that if there be two equal plane angles and if there be set up on them elevated straight lines which are equal and contain equal angles with the original straight lines respectively, the perpendiculars drawn from their extremities to the planes in which are the original angles are equal to one another

Q E D

PROPOSITION 36

If three straight lines be proportional the parallelepipedal solid formed out of the three is equal to the parallelepipedal solid on the mean which is equilateral but equiangular with the aforesaid solid

Let A B C be three straight lines in proportion so that as A is to B so is B to C



I say that the solid formed out of A B C is equal to the solid on B which is equilateral but equiangular with the aforesaid solid

Let there be set out the solid angle at E contained by the angles DEG , GEF , FED ,

let each of the straight lines DE , GB , EF be made equal to B , and let the parallelepipedal solid EK be completed

let LM be made equal to A ,
and on the straight line LM , and at the point L on it, let there be constructed a solid angle equal to the solid angle at E namely that contained by NLO , OLM , MLN

let LO be made equal to B , and LN equal to C

Now since as A is to B so is B to C
while A is equal to LM B to each of the straight lines LO ED and C to LN ,
therefore as LM is to EF so is DE to LN

Thus the sides about the equal angles NLM , DEF are reciprocally proportional,
therefore the parallelogram MN is equal to the parallelogram DF [xi 14]

And since the angles DEF , NLM are two plane rectilineal angles and on them the elevated straight lines LO , EG are set up which are equal to one another and contain equal angles with the original straight lines respectively therefore the perpendiculars drawn from the points G O to the planes through NL , LM and DE EF are equal to one another, [xi 35 Por.]

hence the solids LH EK are of the same height

But parallelepipedal solids on equal bases and of the same height are equal to one another [xi 31]

therefore the solid HL is equal to the solid EK

And LH is the solid formed out of A B , C and EK the solid on B
therefore the parallelepipedal solid formed out of A B , C is equal to the solid on B which is equilateral but equiangular with the aforesaid solid

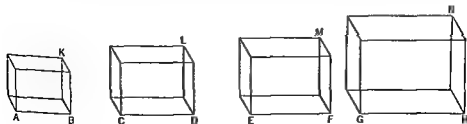
Q E D

PROPOSITION 37

If four straight lines be proportional the parallelepipedal solids on them which are similar and similarly described will also be proportional and if the parallelepipedal solids on them which are similar and similarly described be proportional the straight lines will themselves also be proportional

Let AB , CD EF , GH be four straight lines in proportion so that as AB is to CD so is EF to GH ,

and let there be described on AB , CD , EF GH the similar and similarly situ



ated parallelepipedal solids KA LC , ME NG ,

I say that as KA is to LC so is ME to NG

For since the parallelepipedal solid KA is similar to LC ,
therefore KA has to LC the ratio triplicate of that which AB has to CD [xi 33]

For the same reason
 ME also has to NG the ratio triplicate of that which EF has to GH [id]
And as AB is to CD so is EF to GH

Therefore also as KA is to LC , so is ME to NG

Next as the solid KA is to the solid LC so let the solid ME be to the solid NG ,

I say that as the straight line AB is to CD so is EF to GH

For since again KA has to LC the ratio triplicate of that which AB has to CD [xi 33]

and ME also has to NG the ratio triplicate of that which EF has to GH [id]
and as KA is to LC , so is ME to NG

therefore also as AB is to CD so is EF to GH

Therefore etc

Q E D

PROPOSITION 38

If the sides of the opposite planes of a cube be bisected and planes be carried through the points of section the common section of the planes and the diameter of the cube bisect one another

For let the sides of the opposite planes CF AH of the cube AF be bisected at the points K L M N O Q P R and through the points of section let the planes KN OR be carried
let US be the common section of the planes and DG the diameter of the cube AF

I say that UT is equal to TS and DT to TG

For let DL LE BS SG be joined

Then since DO is parallel to PE

the alternate angles DOU UPE are equal to one another [i 29]

And since DO is equal to PE and OU to UP

and they contain equal angles
therefore the base DU is equal to the base UE

the triangle DOU is equal to the triangle PUE

and the remaining angles are equal to the remaining angles [i 4]

therefore the angle ODU is equal to the angle PLE

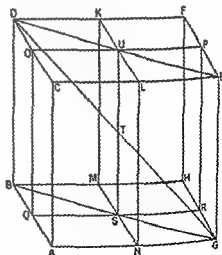
For this reason DLE is a straight line [i 14]

For the same reason BSG is also a straight line

and BS is equal to SG

Now since CA is equal and parallel to DB ,

while CA is also equal and parallel to EG



therefore DB is also equal and parallel to EG

[VI 9]

And the straight lines DE BG join their extremities,

therefore DE is parallel to BG

[I 33]

Therefore the angle EDT is equal to the angle BGT ,

for they are alternate,

[I 29]

and the angle DTU is equal to the angle GTS

[I 13]

Therefore DTU , GTS are two triangles which have two angles equal to two angles and one side equal to one side namely that subtending one of the equal angles that is DU equal to GS

for they are the halves of DE BG

therefore they will also have the remaining sides equal to the remaining sides

[I 26]

Therefore DT is equal to TG and UT to TS

Therefore etc

Q E D

PROPOSITION 39

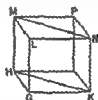
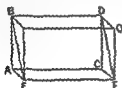
If there be two prisms of equal height and one have a parallelogram as base and the other a triangle and if the parallelogram be double of the triangle the prisms will be equal

Let $ABCDEF$ $GHLUN$ be two prisms of equal height

let one have the parallelogram AF as base and the other the triangle GHA ,

and let the parallelogram AF be double of the triangle GHA

I say that the prism $ABCDEF$ is equal to the prism $GHALUN$



For let the solids AO GP be completed

Since the parallelogram AF is double of the triangle GHA

while the parallelogram HK is also double of the triangle GHA

[I 34]

therefore the parallelogram AF is equal to the parallelogram HK

But parallelepipedal solids which are on equal bases and of the same height are equal to one another

[XI 31]

therefore the solid AO is equal to the solid GP

And the prism $ABCDEF$ is half of the solid AO

and the prism $GHALMN$ is half of the solid GP

[XI 28]

therefore the prism $ABCDEF$ is equal to the prism $GHALMN$

Therefore etc

Q E D

BOOK TWELVE

PROPOSITIONS

PROPOSITION 1

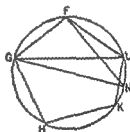
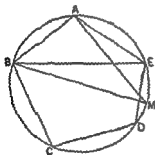
Similar polygons inscribed in circles are to one another as the squares on the diameters

Let ABC FGH be circles
let $ABCDE$ $FGHKL$ be similar polygons inscribed in them and let BM GN be diameters of the circles,

I say that as the square on BM is to the square on GN , so is the polygon $ABCDE$ to the polygon $FGHKL$

For let BE AM GL FN be joined

Now since the polygon $ABCDE$ is similar to the polygon $FGHKL$,



the angle BAE is equal to the angle GFL

and as BA is to AE so is GF to FL

[vi Def 1]

Thus BAE GFL are two triangles which have one angle equal to one angle namely the angle BAE to the angle GFL and the sides about the equal angles proportional

therefore the triangle BAE is equiangular with the triangle GFL

[vi 6]

Therefore the angle AEB is equal to the angle FLG

But the angle AEB is equal to the angle AMB

for they stand on the same circumference,

[iii 27]

and the angle FLG to the angle FNH

therefore the angle AMB is also equal to the angle FNH

But the right angle BAM is also equal to the right angle GFN

[iii 31]

therefore the remaining angle is equal to the remaining angle

[i 32]

Therefore the triangle ABM is equiangular with the triangle FGN

Therefore proportionally as BM is to GN so is BA to GF

[vi 4]

But the ratio of the square on BM to the square on GN is duplicate of the ratio of BM to GN
 and the ratio of the polygon $ABCDE$ to the polygon $FGHKL$ is duplicate of the ratio of BA to GF , [VI 20]
 therefore also as the square on BM is to the square on GN so is the polygon $ABCDE$ to the polygon $FGHKL$.

Therefore etc

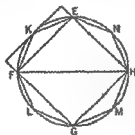
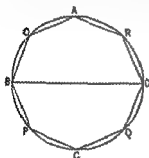
Q E D

PROPOSITION 2

Circles are to one another as the squares on the diameters

Let $ABCD$ $EFGH$ be circles and BD FH their diameters

I say that as the circle $ABCD$ is to the circle $EFGH$, so is the square on BD to the square on FH



For if the square on BD is not to the square on FH as the circle $ABCD$ is to the circle $EFGH$
 then as the square on BD is to the square on FH so will the circle $ABCD$ be either to some less area than the circle $EFGH$ or to a greater

First let it be in that ratio to a less area S

Let the square $EFGH$ be inscribed in the circle $EFGH$ then the inscribed square is greater than the half of the circle $EFGH$ inasmuch as if through the points $E F G H$ we draw tangents to the circle the square $EFGH$ is half the square circumscribed about the circle and the circle is less than the circumscribed square

hence the inscribed square $EFGH$ is greater than the half of the circle $EFGH$

Let the circumferences EF FG GH HE be bisected at the points K L M N

and let EK KF FL LG GM MH HN NE be joined

therefore each of the triangles EKF FLG GMH HNE is also greater than the half of the segment of the circle about it inasmuch as if through the points K L M N we draw tangents to the circle and complete the parallelograms on the straight lines EF FG GH HE each of the triangles EKF FLG GMH HNE will be half of the parallelogram about it

while the segment about it is less than the parallelogram
 hence each of the triangles EKF FLG GMH HNE is greater than the half of the segment of the circle about it

Thus by bisecting the remaining circumferences and joining straight lines

and by doing this continually we shall leave some segments of the circle which will be less than the excess by which the circle $EFGH$ exceeds the area S

For it was proved in the first theorem of the tenth book that if two unequal magnitudes be set out and if from the greater there be subtracted a magnitude greater than the half and from that which is left a greater than the half and if this be done continually there will be left some magnitude which will be less than the lesser magnitude set out

Let segments be left such as described and let the segments of the circle $EFGH$ on $EK KF FL LG GM MH HN NE$ be less than the excess by which the circle $EFGH$ exceeds the area S

Therefore the remainder the polygon $EKFLGMHN$ is greater than the area S

Let there be inscribed also in the circle $ABCD$ the polygon $AOBPCQDR$ similar to the polygon $EKFLGMHN$
therefore as the square on BD is to the square on FH so is the polygon $AOBPCQDR$ to the polygon $EKFLGMHN$ [xi 1]

But as the square on BD is to the square on FH so also is the circle $ABCD$ to the area S

therefore also as the circle $ABCD$ is to the area S so is the polygon $AOBPCQDR$ to the polygon $EKFLGMHN$ [v 11]

therefore alternately as the circle $ABCD$ is to the polygon inscribed in it so is the area S to the polygon $EKFLGMHN$ [v 16]

But the circle $ABCD$ is greater than the polygon inscribed in it

therefore the area S is also greater than the polygon $EKFLGMHN$

But it is also less

which is impossible

Therefore as the square on BD is to the square on FH so is not the circle $ABCD$ to any area less than the circle $EFGH$

Similarly we can prove that neither is the circle $EFGH$ to any area less than the circle $ABCD$ as the square on FH is to the square on BD

I say next that neither is the circle $ABCD$ to any area greater than the circle $EFGH$ as the square on BD is to the square on FH

For if possible let it be in that ratio to a greater area S

Therefore inversely as the square on FH is to the square on BD so is the area S to the circle $ABCD$

But as the area S is to the circle $ABCD$ so is the circle $EFGH$ to some area less than the circle $ABCD$

therefore also as the square on FH is to the square on BD so is the circle $EFGH$ to some area less than the circle $ABCD$ [v 11]

which was proved impossible

Therefore as the square on BD is to the square on FH so is not the circle $ABCD$ to any area greater than the circle $EFGH$

And it was proved that neither is it in that ratio to any area less than the circle $EFGH$

therefore as the square on BD is to the square on FH so is the circle $ABCD$ to the circle $EFGH$

Therefore etc

Q E D

LEMMA

I say that the area S being greater than the circle $EFGH$, as the area S is to the circle $ABCD$, so is the circle $EFGH$ to some area less than the circle $ABCD$

For let it be contrived that as the area S is to the circle $ABCD$ so is the circle $EFGH$ to the area T

I say that the area T is less than the circle $ABCD$

For since as the area S is to the circle $ABCD$, so is the circle $EFGH$ to the area T ,

therefore alternately, as the area S is to the circle $EFGH$, so is the circle $ABCD$ to the area T [v 16]

But the area S is greater than the circle $EFGH$,

therefore the circle $ABCD$ is also greater than the area T

Hence as the area S is to the circle $ABCD$ so is the circle $EFGH$ to some area less than the circle $ABCD$ Q E D

PROPOSITION 3

Any pyramid which has a triangular base is divided into two pyramids equal and similar to one another similar to the whole and having triangular bases and into two equal prisms and the two prisms are greater than the half of the whole pyramid

Let there be a pyramid of which the triangle ABC is the base and the point D the vertex

I say that the pyramid $ABCD$ is divided into two pyramids equal to one another having triangular bases and similar to the whole pyramid and into two equal prisms and the two prisms are greater than the half of the whole pyramid

For let AB BC CA AD DB DC be bisected at the points E F G H , K L and let HE , EG GH HA AL LH AF FG be joined

Since AE is equal to EB and AH to DH

therefore EH is parallel to DB [vi 2]

For the same reason

HK is also parallel to AB

Therefore $HEBK$ is a parallelogram

therefore HK is equal to EB

But EB is equal to EA

therefore AE is also equal to HA

But AH is also equal to HD

therefore the two sides EA AH are equal to the two sides AH HD respectively

and the angle EAH is equal to the angle KHD ,

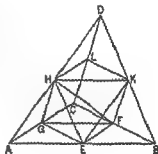
therefore the base EH is equal to the base KD [i 4]

Therefore the triangle AEH is equal and similar to the triangle HKD

For the same reason

the triangle AHG is also equal and similar to the triangle HLD

Now since two straight lines EH HG meeting one another are parallel to two straight lines AD DL meeting one another and are not in the same plane they will contain equal angles [xi 10]



Therefore the angle EHG is equal to the angle ADL

And since the two straight lines EH, HG are equal to the two KD, DL respectively

and the angle EHG is equal to the angle ADL

therefore the base EG is equal to the base KL , [1 4]

therefore the triangle EHG is equal and similar to the triangle ADL

For the same reason

the triangle AEG is also equal and similar to the triangle HKL

Therefore the pyramid of which the triangle AEG is the base and the point H the vertex is equal and similar to the pyramid of which the triangle HKL is the base and the point D the vertex [XI Def 10]

And since HA has been drawn parallel to AB , one of the sides of the triangle ADB

the triangle ADB is equiangular to the triangle DHA [1 29]

and they have their sides proportional

therefore the triangle ADB is similar to the triangle DHK [XI Def 1]

For the same reason

the triangle DBC is also similar to the triangle DAL and the triangle ADC to the triangle DLH

Now since the two straight lines BA, AC meeting one another are parallel to the two straight lines HA, HL meeting one another not in the same plane, they will contain equal angles [XI 10]

Therefore the angle BAC is equal to the angle AHL

And as BA is to AC so is HA to HL

therefore the triangle ABC is similar to the triangle HKL

Therefore also the pyramid of which the triangle ABC is the base and the point D the vertex is similar to the pyramid of which the triangle HKL is the base and the point D the vertex.

But the pyramid of which the triangle HKL is the base and the point D the vertex was proved similar to the pyramid of which the triangle AEG is the base and the point H the vertex.

Therefore each of the pyramids $AEGH, HKLD$ is similar to the whole pyramid $ABCD$

Next since BF is equal to FC

the parallelogram $EBFC$ is double of the triangle GFC

And since if there be two prisms of equal height and one have a parallelogram as base and the other a triangle and if the parallelogram be double of the triangle the prisms are equal [XI 39]

therefore the prism contained by the two triangles BKF, FHG and the three parallelograms $FBFG, FBKH, HKFG$ is equal to the prism contained by the two triangles GFC, HKL and the three parallelograms $KFCL, LCGH, HKFG$

And it is manifest that each of the prisms namely that in which the parallelogram $EBFC$ is the base and the straight line HA is its opposite and that in which the triangle GFC is the base and the triangle HKL its opposite is greater than each of the pyramids of which the triangles LEG, HKL are the bases and the points H, D the vertices

inasmuch as if we join the straight lines IF, FA the prism in which the parallelogram $EBFC$ is the base and the straight line HA its opposite is greater than the pyramid of which the triangle FHF is the base and the point A the vertex

But the pyramid of which the triangle EBF is the base and the point K the vertex is equal to the pyramid of which the triangle AEG is the base and the point H the vertex

for they are contained by equal and similar planes

Hence also the prism in which the parallelogram $PBFG$ is the base and the straight line HA its opposite is greater than the pyramid of which the triangle AEG is the base and the point H the vertex

But the prism in which the parallelogram $EBFG$ is the base and the straight line HA its opposite is equal to the prism in which the triangle GFC is the base and the triangle HKL its opposite

and the pyramid of which the triangle AEG is the base and the point H the vertex is equal to the pyramid of which the triangle HKL is the base and the point D the vertex

Therefore the said two prisms are greater than the said two pyramids of which the triangles AEG HKL are the bases and the points H D the vertices

Therefore the whole pyramid of which the triangle ABC is the base and the point D the vertex has been divided into two pyramids equal to one another and into two equal prisms and the two prisms are greater than the half of the whole pyramid

Q E D

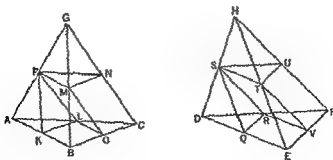
PROPOSITION 4

If there be two pyramids of the same height which have triangular bases and each of them be divided into two pyramids equal to one another and similar to the whole and into two equal prisms then as the base of the one pyramid is to the base of the other pyramid so will all the prisms in the one pyramid be to all the prisms being equal in multitude in the other pyramid

Let there be two pyramids of the same height which have the triangular bases ABC DEF and vertices the points G H and let each of them be divided into two pyramids equal to one another and similar to the whole and into two equal prisms

[XII 3]

I say that as the base ABC is to the base DEF so are all the prisms in the pyramid $ABCG$ to all the prisms being equal in multitude in the pyramid $DEFH$



For since BO is equal to OC and AL to LC
therefore LO is parallel to AB
and the triangle ABC is similar to the triangle LOC

For the same reason

the triangle DEF is also similar to the triangle RVF

And since BC is double of CO and EF of FV ,

therefore as BC is to CO so is EF to FV

And on BC CO are described the similar and similarly situated rectilineal figures ABC , LOC ,

and on EF FV the similar and similarly situated figures DEF , RVF ,
therefore as the triangle ABC is to the triangle LOC so is the triangle DEF to the triangle RVF , [v 22]

therefore alternately as the triangle ABC is to the triangle DEF , so is the triangle LOC to the triangle RVF [v 16]

But as the triangle LOC is to the triangle RVF so is the prism in which the triangle LOC is the base and PMN its opposite to the prism in which the triangle RVF is the base and STU its opposite [Lemma following]

therefore also as the triangle ABC is to the triangle DEF , so is the prism in which the triangle LOC is the base and PMN its opposite to the prism in which the triangle RVF is the base and STU its opposite

But as the said prisms are to one another so is the prism in which the parallelogram $KBOL$ is the base and the straight line PM its opposite to the prism in which the parallelogram $QEV R$ is the base and the straight line ST its opposite [xi 39 of xii 3]

Therefore also the two prisms that in which the parallelogram $KBOL$ is the base and PM its opposite and that in which the triangle LOC is the base and PMN its opposite are to the prisms in which $QEV R$ is the base and the straight line ST its opposite and in which the triangle RVF is the base and STU its opposite in the same ratio [v 12]

Therefore also as the base ABC is to the base DEF so are the said two prisms to the said two prisms

And similarly if the pyramids $PMNG$ $STUH$ be divided into two prisms and two pyramids

as the base PMN is to the base STU so will the two prisms in the pyramid $PMNG$ be to the two prisms in the pyramid $STUH$

But as the base PMN is to the base STU so is the base ABC to the base DEF

for the triangles PMN STU are equal to the triangles LOC RVF respectively

Therefore also as the base ABC is to the base DEF so are the four prisms to the four prisms

And similarly also if we divide the remaining pyramids into two pyramids and into two prisms then as the base ABC is to base the DEF so will all the prisms in the pyramid $ABCG$ be to all the prisms being equal in multitude in the pyramid $DEFH$ Q E D

LEMMA

But that as the triangle LOC is to the triangle RVF so is the prism in which the triangle LOC is the base and PMN its opposite to the prism in which the triangle RVF is the base and STU its opposite we must prove as follows

For in the same figure let perpendiculars be conceived drawn from G , H to

the planes ABC , DEF , these are of course equal because by hypothesis, the pyramids are of equal height

Now, since the two straight lines GC and the perpendicular from G are cut by the parallel planes ABC , PMN

they will be cut in the same ratios [XI 17]

And GC is bisected by the plane PMN at N , therefore the perpendicular from G to the plane ABC will also be bisected by the plane PMN

For the same reason the perpendicular from H to the plane DEF will also be bisected by the plane STU

And the perpendiculars from G H to the planes ABC , DEF are equal, therefore the perpendiculars from the triangles PMN , STU to the planes ABC , DEF are also equal

Therefore the prisms in which the triangles LOC , RVF are bases, and PMN , STU their opposites, are of equal height

Hence also the parallelepipedal solids described from the said prisms are of equal height and are to one another as their bases, [XI 32]

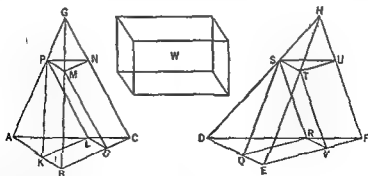
therefore their halves namely the said prisms, are to one another as the base LOC is to the base RVF Q E D

PROPOSITION 5

Pyramids which are of the same height and have triangular bases are to one another as the bases

Let there be pyramids of the same height of which the triangles ABC , DEF are the bases and the points G H the vertices,

I say that as the base ABC is to the base DEF , so is the pyramid $ABCG$ to the pyramid $DEFH$



For if the pyramid $ABCG$ is not to the pyramid $DEFH$ as the base ABC is to the base DEF , then as the base ABC is to the base DEF so will the pyramid $ABCG$ be either to some solid less than the pyramid $DEFH$ or to a greater

Let it first be in that ratio to a less solid W and let the pyramid $DEFH$ be divided into two pyramids equal to one another and similar to the whole and into two equal prisms

then the two prisms are greater than the half of the whole pyramid [xii 3]

Again let the pyramids arising from the division be similarly divided and let this be done continually until there are left over from the pyramid $DEFH$ some pyramids which are less than the excess by which the pyramid $DEFH$ exceeds the solid Π [x 1]

Let such be left and let them be for the sake of argument, $DQRS$ $STUH$ therefore the remainders, the prisms in the pyramid $DEFH$, are greater than the solid Π

Let the pyramid $ABCG$ also be divided similarly, and a similar number of times with the pyramid $DEFH$

therefore as the base ABC is to the base DEF so are the prisms in the pyramid $ABCG$ to the prisms in the pyramid $DEFH$ [xii 4]

But as the base ABC is to the base DEF so also is the pyramid $ABCG$ to the solid Π

therefore also as the pyramid $ABCG$ is to the solid Π , so are the prisms in the pyramid $ABCG$ to the prisms in the pyramid $DEFH$, [v 11]

therefore alternately as the pyramid $ABCG$ is to the prisms in it so is the solid Π to the prisms in the pyramid $DEFH$ [v 16]

But the pyramid $ABCG$ is greater than the prisms in it, therefore the solid Π is also greater than the prisms in the pyramid $DEFH$

But it is also less

which is impossible

Therefore the prism $ABCG$ is not to any solid less than the pyramid $DEFH$ as the base ABC is to the base DEF

Similarly it can be proved that neither is the pyramid $DEFH$ to any solid less than the pyramid $ABCG$ as the base DEF is to the base ABC

I say next that neither is the pyramid $ABCG$ to any solid greater than the pyramid $DEFH$ as the base ABC is to the base DEF

For if possible let it be in that ratio to a greater solid Π therefore inversely as the base DEF is to the base ABC , so is the solid Π to the pyramid $ABCG$

But as the solid Π is to the solid $ABCG$ so is the pyramid $DEFH$ to some solid less than the pyramid $ABCG$ as was before proved [xii 2 Lemma]

therefore also as the base DEF is to the base ABC , so is the pyramid $DEFH$ to some solid less than the pyramid $ABCG$ [v 11]

which was proved absurd

Therefore the pyramid $ABCG$ is not to any solid greater than the pyramid $DEFH$ as the base ABC is to the base DEF

But it was proved that neither is it in that ratio to a less solid

Therefore as the base ABC is to the base DEF so is the pyramid $ABCG$ to the pyramid $DEFH$ Q E D

PROPOSITION 6

Pyramids which are of the same height and have polygonal bases are to one another as the bases

Let there be pyramids of the same height of which the polygons $ABCDE$ $FGHKL$ are the bases and the points M N the vertices

I say that as the base $ABCD$ is to the base $FGHL$, so is the pyramid $ABCDM$ to the pyramid $FGHLN$

For let AC AD FH , FA be joined

Since then $ABCM$, $ACDM$ are two pyramids which have triangular bases and equal height

they are to one another as the bases, [xii 5]

therefore as the base ABC to the base ACD so is the pyramid $ABCM$ to the pyramid $ACDM$

And *componendo* as the base $ABCD$ is to the base ACD so is the pyramid $ABCDM$ to the pyramid $ACDM$ [v 18]

But also as the base ACD is to the base ADE so is the pyramid $ACDM$ to the pyramid $ADEM$ [xii 5]

Therefore *ex aequali* as the base $ABCD$ is to the base ADE , so is the pyramid $ABCDM$ to the pyramid $ADEM$ [v 22]

And again *componendo* as the base $ABCDE$ is to the base ADE , so is the pyramid $ABCDEM$ to the pyramid $ADEM$ [v 18]

Similarly also it can be proved that as the base $FGHKL$ is to the base FGH , so is the pyramid $FGHKLN$ to the pyramid $FGHN$

And since $ADEM$ $FGHN$ are two pyramids which have triangular bases and equal height

therefore as the base ADE is to the base FGH so is the pyramid $ADEM$ to the pyramid $FGHN$ [xii 5]

But as the base ADE is to the base $ABCDE$ so was the pyramid $ADEM$ to the pyramid $ABCDEM$

Therefore also *ex aequali* as the base $ABCDE$ is to the base FGH , so is the pyramid $ABCDEM$ to the pyramid $FGHN$ [v 22]

But further, as the base FGH is to the base $FGHKL$ so also was the pyramid $FGHN$ to the pyramid $FGHKLN$

Therefore also *ex aequali* as the base $ABCDE$ is to the base $FGHKL$ so is the pyramid $ABCDEM$ to the pyramid $FGHKLN$ [v 22]

Q E D

PROPOSITION 7

Any prism which has a triangular base is divided into three pyramids equal to one another which have triangular bases

Let there be a prism in which the triangle ABC is the base and DEF its opposite

I say that the prism $ABCDEF$ is divided into three pyramids equal to one another which have triangular bases

For let BD EC CD be joined

Since BED is a parallelogram and BD is its diameter

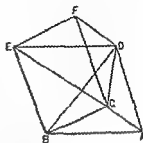
therefore the triangle ABD is equal to the triangle EBD [i 34]

therefore also the pyramid of which the triangle ABD is the base and the point C the vertex is equal to the pyramid of which the triangle DEB is the base and

the point C the vertex

[XII 5]

But the pyramid of which the triangle DEB is the base and the point C the vertex is the same with the pyramid of which the triangle EBC is the base and the point D the vertex



for they are contained by the same planes

Therefore the pyramid of which the triangle ABD is the base and the point C the vertex is also equal to the pyramid of which the triangle EBC is the base and the point D the vertex

Again since $FCBE$ is a parallelogram
and CE is its diameter

the triangle CEF is equal to the triangle CBE

[I 34]

Therefore also the pyramid of which the triangle BCE is the base and the point D the vertex is equal to the pyramid of which the triangle ECF is the base and the point D the vertex

[XII 5]

But the pyramid of which the triangle BCE is the base and the point D the vertex was proved equal to the pyramid of which the triangle ABD is the base and the point C the vertex

therefore also the pyramid of which the triangle CEF is the base and the point D the vertex is equal to the pyramid of which the triangle ABD is the base and the point C the vertex

therefore the prism $ABCDEF$ has been divided into three pyramids equal to one another which have triangular bases

And since the pyramid of which the triangle ABD is the base and the point C the vertex is the same with the pyramid of which the triangle CAB is the base and the point D the vertex

for they are contained by the same planes

while the pyramid of which the triangle ABD is the base and the point C the vertex was proved to be a third of the prism in which the triangle ABC is the base and DEF its opposite

therefore also the pyramid of which the triangle ABC is the base and the point D the vertex is a third of the prism which has the same base the triangle ABC , and DEF as its opposite

Proposition From this it is manifest that any pyramid is a third part of the prism which has the same base with it and equal height

Q.E.D.

PROPOSITION 8

Similar pyramids which have triangular bases are in the triplicate ratio of their corresponding sides

Let there be similar and similarly situated pyramids of which the triangles ABC DEF are the bases and the points G H the vertices

I say that the pyramid $ABCG$ has to the pyramid $DEFH$ the ratio triplicate of that which BC has to EF

For let the parallelepipedal solids $BGML$ $EHQP$ be completed

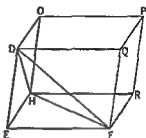
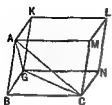
Now since the pyramid $ABCG$ is similar to the pyramid $DEFH$

therefore the angle ABC is equal to the angle DEF

the angle GBC to the angle HLF

and the angle ABG to the angle DEH

and as AB is to DE , so is BC to EF , and BG to EH
 And since as AB is to DE , so is BC to EF
 and the sides are proportional about equal angles,



therefore the parallelogram BM is similar to the parallelogram EQ
 For the same reason

BN is also similar to ER and BK to EO
 therefore the three parallelograms MB , BK , BN are similar to the three EQ , EO , ER

But the three parallelograms MB , BK , BN are equal and similar to their three opposites
 and the three EQ , EO , ER are equal and similar to their three opposites

[XI 24]

Therefore the solids $BGML$, $EHQP$ are contained by similar planes equal in multitude

Therefore the solid $BGML$ is similar to the solid $EHQP$

But similar parallelepipedal solids are in the triplicate ratio of their corresponding sides

[XI 33]

Therefore the solid $BGML$ has to the solid $EHQP$ the ratio triplicate of that which the corresponding side BC has to the corresponding side EF

But as the solid $BGML$ is to the solid $EHQP$, so is the pyramid $ABCG$ to the pyramid $DEFH$,

inasmuch as the pyramid is a sixth part of the solid because the prism which is half of the parallelepipedal solid [XI 26] is also triple of the pyramid [XII 7]

Therefore the pyramid $ABCG$ also has to the pyramid $DEFH$ the ratio triplicate of that which BC has to EF

Q E D

PROPOSITION From this it is manifest that similar pyramids which have polygonal bases are also to one another in the triplicate ratio of their corresponding sides

For, if they are divided into the pyramids contained in them which have triangular bases by virtue of the fact that the similar polygons forming their bases are also divided into similar triangles equal in multitude and corresponding to the wholes

[XI 20]

then as the one pyramid which has a triangular base in the one complete pyramid is to the one pyramid which has a triangular base in the other complete pyramid so also will all the pyramids which have triangular bases contained in the one pyramid be to all the pyramids which have triangular bases contained in the other pyramid [X 12] that is the pyramid itself which has a polygonal base to the pyramid which has a polygonal base

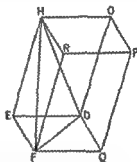
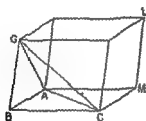
But the pyramid which has a triangular base is to the pyramid which has a triangular base in the triplicate ratio of the corresponding sides
therefore also the pyramid which has a polygonal base has to the pyramid which has a similar base the ratio triplicate of that which the side has to the side

PROPOSITION 9

In equal pyramids which have triangular bases the bases are reciprocally proportional to the heights and those pyramids in which the bases are reciprocally proportional to the heights are equal

For let there be equal pyramids which have the triangular bases ABC DEF and vertices the points G H

I say that in the pyramids $ABCG$ $DEFH$ the bases are reciprocally proportional to the heights that is as the base ABC is to the base DEF so is the height of the pyramid $DEFH$ to the height of the pyramid $ABCG$



For let the parallelepipedal solids $BGML$ $EHQP$ be completed
Now since the pyramid $ABCG$ is equal to the pyramid $DEFH$,
and the solid $BGML$ is six times the pyramid $ABCG$
and the solid $EHQP$ six times the pyramid $DEFH$
therefore the solid $BGML$ is equal to the solid $EHQP$

But in equal parallelepipedal solids the bases are reciprocally proportional to the heights
therefore as the base BM is to the base EQ so is the height of the solid $EHQP$ to the height of the solid $BGML$ [xi 31]

But as the base BM is to IQ so is the triangle ABC to the triangle DEF [i 31]

Therefore also as the triangle ABC is to the triangle DEF so is the height of the solid $IHPQ$ to the height of the solid $BGML$ [vi 11]

But the height of the solid $EHQI$ is the same with the height of the pyramid $DEFH$

and the height of the solid $BGVL$ is the same with the height of the pyramid $ABCG$

therefore as the base ABC is to the base DEF so is the height of the pyramid $DEFH$ to the height of the pyramid $ABCG$

Therefore in the pyramids $ABCG$ $DEFH$ the bases are reciprocally proportional to the heights

Next in the pyramids $ABCG$, $DEFH$ let the bases be reciprocally proportional to the heights,
that is as the base ABC is to the base DEF so let the height of the pyramid $DEFH$ be to the height of the pyramid $ABCG$,

I say that the pyramid $ABCG$ is equal to the pyramid $DEFH$

For with the same construction

since as the base ABC is to the base DEF , so is the height of the pyramid $DEFH$ to the height of the pyramid $ABCG$

while as the base ABC is to the base DEF , so is the parallelogram BM to the parallelogram EQ

therefore also, as the parallelogram BM is to the parallelogram EQ , so is the height of the pyramid $DEFH$ to the height of the pyramid $ABCG$ [I 11]

But the height of the pyramid $DEFH$ is the same with the height of the parallelepiped $EHQP$

and the height of the pyramid $ABCG$ is the same with the height of the parallelepiped $BGML$,

therefore as the base BM is to the base EQ so is the height of the parallelepiped $EHQP$ to the height of the parallelepiped $BGML$

But those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal, [XI 34]

therefore the parallelepipedal solid $BGML$ is equal to the parallelepipedal solid $EHQP$

And the pyramid $ABCG$ is a sixth part of $BGML$ and the pyramid $DEFH$ a sixth part of the parallelepiped $EHQP$,

therefore the pyramid $ABCG$ is equal to the pyramid $DEFH$

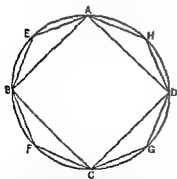
Therefore etc

Q E D

PROPOSITION 10

Any cone is a third part of the cylinder which has the same base with it and equal height

For let a cone have the same base, namely the circle $ABCD$, with a cylinder and equal height,



I say that the cone is a third part of the cylinder that is that the cylinder is triple of the cone

For if the cylinder is not triple of the cone the cylinder will be either greater than triple or less than triple of the cone

First let it be greater than triple and let the square $ABCD$ be inscribed in the circle $ABCD$, [IV 6]

then the square $ABCD$ is greater than the half of the circle $ABCD$

From the square $ABCD$ let there be set up a prism of equal height with the cylinder

Then the prism so set up is greater than the half of the cylinder inasmuch as if we also circumscribe a square about the circle $ABCD$ [IV 7] the square inscribed in the circle $ABCD$ is half of that circumscribed about it, and the solids set up from them are parallelepipedal prisms of equal height

while parallelepipedal solids which are of the same height are to one another as their bases [xi 3^o]

therefore also the prism set up on the square $ABCD$ is half of the prism set up from the square circumscribed about the circle $ABCD$

[cf xi 28 or xii 6 and 7 For]

and the cylinder = less than the prism set up from the square circumscribed about the circle $ABCD$

therefore the prism set up from the square $ABCD$ and of equal height with the cylinder is greater than the half of the cylinder

Let the circumferences AB BC CD DA be bisected at the points E , F G H

and let AE FB BF FC CG GD DH HA be joined

then each of the triangles AEB BFC CGD DHA is greater than the half of that segment of the circle $ABCD$ which is about it as we proved before

[xii 2]

On each of the triangles AEB BFC CGD DHA let prisms be set up of equal height with the cylinder

then each of the prisms so set up is greater than the half part of that segment of the cylinder which is about it

inasmuch as if we draw through the points E F G H parallels to AB BC , CD DA complete the parallelograms on AB BC CD DA and set up from them parallelepipedal solids of equal height with the cylinder the prisms on the triangles AEB BFC CGD DHA are halves of the several solids set up

and the segments of the cylinder are less than the parallelepipedal solids set up hence also the prisms on the triangles AEB BFC CGD DHA are greater than the half of the segments of the cylinder about them

Thus bisecting the circumferences that are left joining straight lines set ting up on each of the triangles prisms of equal height with the cylinder, and doing this continually

we shall leave some segments of the cylinder which will be less than the excess by which the cylinder exceeds the triple of the cone

[x 1]

Let such segments be left and let them be AE EB BF , FC CG , GD DH HA

therefore the remainder the prism of which the polygon $AEBFCGDH$ is the base and the height is the same as that of the cylinder is greater than triple of the cone

But the prism of which the polygon $AEBFCGDH$ is the base and the height the same as that of the cylinder is triple of the pyramid of which the polygon $AEBFCGDH$ is the base and the vertex is the same as that of the cone

[xii 7 For]

therefore also the pyramid of which the polygon $AEBFCGDH$ is the base and the vertex is the same as that of the cone is greater than the cone which has the circle $ABCD$ as base

But it is also less for it is enclosed by it

which is impossible

Therefore the cylinder is not greater than triple of the cone

I say next that neither is the cylinder less than triple of the cone

For if possible let the cylinder be less than triple of the cone

therefore inversely the cone is greater than a third part of the cylinder

Let the square $ABCD$ be inscribed in the circle $ABCD$,
therefore the square $ABCD$ is greater than the half of the circle $ABCD$

Now let there be set up from the square $ABCD$ a pyramid having the same vertex with the cone,
therefore the pyramid so set up is greater than the half part of the cone,
seeing that as we proved before, if we circumscribe a square about the circle,
the square $ABCD$ will be half of the square circumscribed about the circle,
and if we set up from the squares parallelepipedal solids of equal height with the cone which are also called prisms the solid set up from the square $ABCD$ will be half of that set up from the square circumscribed about the circle,
for they are to one another as their bases [xi 32]

Hence also the thirds of them are in that ratio
therefore also the pyramid of which the square $ABCD$ is the base is half of the pyramid set up from the square circumscribed about the circle

And the pyramid set up from the square about the circle is greater than the cone,

for it encloses it

Therefore the pyramid of which the square $ABCD$ is the base and the vertex is the same with that of the cone is greater than the half of the cone

Let the circumferences AB , BC , CD , DA be bisected at the points E , F , G , H ,

-and let AE , EB , BF , FC , CG , GD , DH , HA be joined
therefore also each of the triangles AEB , BFC , CGD , DHA is greater than the half part of that segment of the circle $ABCD$ which is about it

Now on each of the triangles AEB , BFC , CGD , DHA let pyramids be set up which have the same vertex as the cone,
therefore also each of the pyramids so set up is in the same manner, greater than the half part of that segment of the cone which is about it

Thus by bisecting the circumferences that are left, joining straight lines setting up on each of the triangles a pyramid which has the same vertex as the cone

and doing this continually,
we shall leave some segments of the cone which will be less than the excess by which the cone exceeds the third part of the cylinder [x 1]

Let such be left and let them be the segments on AE , EB , BF , FC , CG , GD , DH , HA ,
therefore the remainder the pyramid of which the polygon $AEBFCGDH$ is the base and the vertex the same with that of the cone is greater than a third part of the cylinder

But the pyramid of which the polygon $AEBFCGDH$ is the base and the vertex the same with that of the cone is a third part of the prism of which the polygon $AEBFCGDH$ is the base and the height is the same with that of the cylinder,

therefore the prism of which the polygon $AEBFCGDH$ is the base and the height is the same with that of the cylinder is greater than the cylinder of which the circle $ABCD$ is the base

But it is also less for it is enclosed by it

which is impossible

Therefore the cylinder is not less than triple of the cone

But it was proved that neither is it greater than triple,
therefore the cylinder is triple of the cone,
hence the cone is a third part of the cylinder

Therefore etc

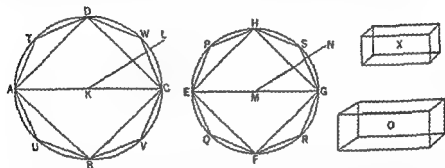
Q E D

PROPOSITION 11

Cones and cylinders which are of the same height are to one another as their bases

Let there be cones and cylinders of the same height,
let the circles $ABCD$ $EFGH$ be their bases AL MA their axes and AC EG
the diameters of their bases

I say that as the circle $ABCD$ is to the circle $EFGH$ so is the cone AL to
the cone EN



For if not then as the circle $ABCD$ is to the circle $EFGH$ so will the cone
 AL be either to some solid less than the cone EN or to a greater

First let it be in that ratio to a less solid O and let the solid X be equal to
that by which the solid O is less than the cone EN

therefore the cone EN is equal to the solids O X

Let the square $EFGH$ be inscribed in the circle $EFGH$

therefore the square is greater than the half of the circle

Let there be set up from the square $EFGH$ a pyramid of equal height with
the cone

therefore the pyramid so set up is greater than the half of the cone
inasmuch as if we circumscribe a square about the circle and set up from it a
pyramid of equal height with the cone the inscribed pyramid is half of the cir-
cumscripted pyramid

for they are to one another as their bases [XII 6]

while the cone is less than the circumscribed pyramid

Let the circumferences EF IG GH HE be bisected at the points P Q R S
and let HP PE IQ QF FR RG GS SH be joined

Therefore each of the triangles HPF FQF FRG GSH is greater than the
half of that segment of the circle which is about it

On each of the triangles HPF EQF FRG GSH let there be set up a pyra-
mid of equal height with the cone

therefore also each of the pyramids so set up is greater than the half of that
segment of the cone which is about it

Thus bisecting the circumferences which are left joining straight lines set

ting up on each of the triangles pyramids of equal height with the cone
and doing this continually
we shall leave some segments of the cone which will be less than the solid O [x 1]

Let such be left, and let them be the segments on $HP, PE, EQ, QF, FR, RG, GS, SH$,

therefore the remainder the pyramid of which the polygon $HPEQFRGS$ is the base and the height the same with that of the cone, is greater than the solid O

Let there also be inscribed in the circle $ABCD$ the polygon $DTAUBVCW$ similar and similarly situated to the polygon $HPEQFRGS$,

and on it let a pyramid be set up of equal height with the cone AL

Since then, as the square on AC is to the square on EG , so is the polygon $DTAUBVCW$ to the polygon $HPEQFRGS$ [xii 1]

while as the square on AC is to the square on EG , so is the circle $ABCD$ to the circle $EFGH$, [xii 2]

therefore also as the circle $ABCD$ is to the circle $EFGH$ so is the polygon $DTAUBVCW$ to the polygon $HPEQFRGS$

But as the circle $ABCD$ is to the circle $EFGH$, so is the cone AL to the solid O ,

and as the polygon $DTAUBVCW$ is to the polygon $HPEQFRGS$, so is the pyramid of which the polygon $DTAUBVCW$ is the base and the point L the vertex to the pyramid of which the polygon $HPEQFRGS$ is the base and the point N the vertex [xii 6]

Therefore also as the cone AL is to the solid O , so is the pyramid of which the polygon $DTAUBVCW$ is the base and the point L the vertex to the pyramid of which the polygon $HPEQFRGS$ is the base and the point N the vertex [v 11]

therefore alternately as the cone AL is to the pyramid in it so is the solid O to the pyramid in the cone EN [v 16]

But the cone AL is greater than the pyramid in it
therefore the solid O is also greater than the pyramid in the cone EN

But it is also less

which is absurd

Therefore the cone AL is not to any solid less than the cone EN as the circle $ABCD$ is to the circle $EFGH$

Similarly we can prove that neither is the cone EA to any solid less than the cone AL as the circle $EFGH$ is to the circle $ABCD$

I say next that neither is the cone AL to any solid greater than the cone EN as the circle $ABCD$ is to the circle $EFGH$

For if possible let it be in that ratio to a greater solid O
therefore inversely, as the circle $EFGH$ is to the circle $ABCD$ so is the solid O to the cone AL

But as the solid O is to the cone AL so is the cone EN to some solid less than the cone AL

therefore also as the circle $EFGH$ is to the circle $ABCD$ so is the cone EN to some solid less than the cone AL

which was proved impossible

Therefore the cone AL is not to any solid greater than the cone EN as the circle $ABCD$ is to the circle $EFGH$

But it was proved that neither is it in this ratio to a less solid,
therefore as the circle $ABCD$ is to the circle $EFGH$, so is the cone AL to the
cone EN

But as the cone is to the cone so is the cylinder to the cylinder,
for each is triple of each, [XII 10]

Therefore also as the circle $ABCD$ is to the circle $EFGH$ so are the cylinders
on them which are of equal height

Therefore etc

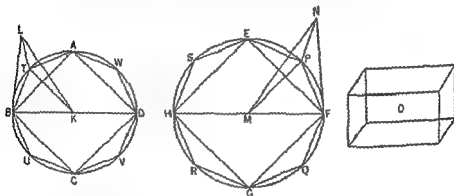
Q E D

PROPOSITION 12

*Similar cones and cylinders are to one another in the triplicate ratio of the diameters
in their bases*

Let there be similar cones and cylinders
let the circles $ABCD$, $EFGH$ be their bases BD FH the diameters of the bases,
and KL MN the axes of the cones and cylinders,

I say that the cone of which the circle $ABCD$ is the base and the point L the
vertex has to the cone of which the circle $EFGH$ is the base and the point N
the vertex the ratio triplicate of that which BD has to FH



For if the cone $ABCDL$ has not to the cone $EFGHN$ the ratio triplicate of
that which BD has to FH
the cone $ABCDL$ will have that triplicate ratio either to some solid less than
the cone $EFGHN$ or to a greater

First let it have that triplicate ratio to a less solid O

Let the square $FFCH$ be inscribed in the circle $EFGH$ [IV 6]
therefore the square $FFCH$ is greater than the half of the circle $EFGH$

Now let there be set up on the square $FFCH$ a pyramid having the same
vertex with the cone

therefore the pyramid so set up is greater than the half part of the cone

Let the circumferences FF IG CH HL be bisected at the points P Q R S
and let FI PP FQ QG GR RH HS SP be joined

Therefore each of the triangles FIP FQG GRH HSL is also greater than
the half part of that segment of the circle $FFGH$ which is about it

Now on each of the triangles FIP FQG GRH HSL let a pyramid be set up
having the same vertex with the cone,

therefore each of the pyramids so set up is also greater than the half part of that segment of the cone which is about it

Thus bisecting the circumferences so left joining straight lines, setting up on each of the triangles pyramids having the same vertex with the cone, and doing this continually,

we shall leave some segments of the cone which will be less than the excess by which the cone $EFGHN$ exceeds the solid O [x 1]

Let such be left and let them be the segments on $IP, PF, FQ, QG, GR, RH, HS, SE$

therefore the remainder, the pyramid of which the polygon $EPFQGRHS$ is the base and the point N the vertex is greater than the solid O

Let there be also inscribed in the circle $ABCD$ the polygon $ATBUCVDW$ similar and similarly situated to the polygon $EPFQGRHS$ and let there be set up on the polygon $ATBUCVDW$ a pyramid having the same vertex with the cone,

of the triangles containing the pyramid of which the polygon $ATBUCVDW$ is the base and the point L the vertex let LBT be one, and of the triangles containing the pyramid of which the polygon $EPFQGRHS$ is the base and the point N the vertex let NFP be one, and let KT, VP be joined

Now since the cone $ABCDL$ is similar to the cone $EFGHN$, therefore as BD is to FH , so is the axis KL to the axis MN [xi Def 24]

But, as BD is to FH , so is BK to FM ,

therefore also as BK is to FM so is KL to MN

And alternately, as BK is to KL , so is FM to MN [v 16]

And the sides are proportional about equal angles, namely the angles BAL, FMV ,

therefore the triangle BAL is similar to the triangle FMN [vi 6]

Again, since as BK is to KT , so is FM to MP

and they are about equal angles namely the angles BKT, FMP , inasmuch as whatever part the angle BAT is of the four right angles at the centre A , the same part also is the angle FMP of the four right angles at the centre M ,

since then the sides are proportional about equal angles

therefore the triangle BKT is similar to the triangle FMP [vi 6]

Again since it was proved that, as BK is to KL so is FM to MN

while BK is equal to KT and FM to PM

therefore as TK is to KL so is PM to MN

and the sides are proportional about equal angles namely the angles TAL, PMN for they are right,

therefore the triangle LKT is similar to the triangle NMP [vi 6]

And since owing to the similarity of the triangles LKB, NMF

as LB is to BK so is NT to FM

and owing to the similarity of the triangles BKT, FMP

as KB is to BT so is MF to FP

therefore *ex aequali* as LB is to BT , so is NT to FP [v 22]

Again since owing to the similarity of the triangles LTA, NPM

as LT is to TA so is NP to PM

and owing to the similarity of the triangles TAB, PMF ,

as KT is to TB so is MP to PF ,

therefore *ex aequali* as LT is to TB so is NP to PF [v 27]

But it was also proved that as TB is to BL , so is PF to FN

Therefore *ex aequali* as TL is to LB , so is PN to NF [v 27]

Therefore in the triangles LTB , NPF the sides are proportional,

therefore the triangles LTB , NPF are equiangular [vi 5]

hence they are also similar [vi Def 1]

Therefore the pyramid of which the triangle BKT is the base and the point L the vertex is also similar to the pyramid of which the triangle FMP is the base and the point N the vertex

for they are contained by similar planes equal in multitude [xi Def 9]

But similar pyramids which have triangular bases are to one another in the triplicate ratio of their corresponding sides [xii 8]

Therefore the pyramid $BA TL$ has to the pyramid $FMPN$ the ratio triplicate of that which BA has to FM

Similarly by joining straight lines from A , B , D , V , C , U to Λ , and from E , S , H , R , G , Q to M and setting up on each of the triangles pyramids which have the same vertex with the cones

we can prove that each of the similarly arranged pyramids will also have to each similarly arranged pyramid the ratio triplicate of that which the corresponding side BA has to the corresponding side FM that is which BD has to FH

And as one of the antecedents is to one of the consequents so are all the antecedents to all the consequents [v 17]

therefore also as the pyramid $BA TL$ is to the pyramid $FMPN$ so is the whole pyramid of which the polygon $ATBUCV D W$ is the base and the point L the vertex to the whole pyramid of which the polygon $EPFQGRHS$ is the base and the point Λ the vertex

hence also the pyramid of which $ATBUCVDW$ is the base and the point L the vertex has to the pyramid of which the polygon $EPFQGRHS$ is the base and the point Λ the vertex the ratio triplicate of that which BD has to FH

But by hypothesis the cone of which the circle $ABCD$ is the base and the point L the vertex has also to the solid O the ratio triplicate of that which BD has to FH

therefore as the cone of which the circle $ABCD$ is the base and the point L the vertex is to the solid O so is the pyramid of which the polygon $ATBUCV D W$ is the base and L the vertex to the pyramid of which the polygon $EPFQGRHS$ is the base and the point N the vertex

therefore alternately as the cone of which the circle $ABCD$ is the base and L the vertex is to the pyramid contained in it of which the polygon $ATBUCV D W$ is the base and L the vertex so is the solid O to the pyramid of which the polygon $EPFQGRHS$ is the base and Λ the vertex [v 16]

But the said cone is greater than the pyramid in it

for it encloses it

Therefore the solid O is also greater than the pyramid of which the polygon $EPFQGRHS$ is the base and Λ the vertex

But it is also less

which is impossible

Therefore the cone of which the circle $ABCD$ is the base and L the vertex

has not to any solid less than the cone of which the circle $EFGH$ is the base and the point N the vertex the ratio triplicate of that which BD has to FH

Similarly we can prove that neither has the cone $EFGHN$ to any solid less than the cone $ABCDL$ the ratio triplicate of that which FH has to BD

I say next that neither has the cone $ABCDL$ to any solid greater than the cone $EFGHN$ the ratio triplicate of that which BD has to FH

For if possible let it have that ratio to a greater solid O

Therefore, inversely, the solid O has to the cone $ABCDL$ the ratio triplicate of that which FH has to BD

But as the solid O is to the cone $ABCDL$ so is the cone $EFGHN$ to some solid less than the cone $ABCDL$

Therefore the cone $EFGHN$ also has to some solid less than the cone $ABCDL$ the ratio triplicate of that which FH has to BD

which was proved impossible

Therefore the cone $ABCDL$ has not to any solid greater than the cone $EFGHN$ the ratio triplicate of that which BD has to FH

But it was proved that neither has it this ratio to a less solid than the cone $EFGHN$

Therefore the cone $ABCDL$ has to the cone $EFGHN$ the ratio triplicate of that which BD has to FH

But, as the cone is to the cone so is the cylinder to the cylinder, for the cylinder which is on the same base as the cone and of equal height with it is triple of the cone, [XII 10]

therefore the cylinder also has to the cylinder the ratio triplicate of that which BD has to FH

Therefore etc

Q E D

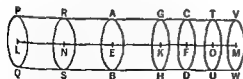
PROPOSITION 13

If a cylinder be cut by a plane which is parallel to its opposite planes then, as the cylinder is to the cylinder so will the axis be to the axis

For let the cylinder AD be cut by the plane GH which is parallel to the opposite planes AB CD

and let the plane GH meet the axis at the point K ,

I say that as the cylinder BG is to the cylinder GD , so is the axis EK to the axis KF



For let the axis EF be produced in both directions to the points L M

and let there be set out any number whatever of axes EN NL equal to the axis EK

and any number whatever FO OM equal to FK

and let the cylinder PW on the axis LM be conceived of which the circles PQ IJ are the bases

Let planes be carried through the points N , O parallel to AB CD and to the bases of the cylinder PW

and let them produce the circles RS TU about the centres N , O

Then since the axes LN NE EK are equal to one another,

therefore the cylinders QR RB BG are to one another as their bases [xii 11]

But the bases are equal,

therefore the cylinders QR RB BG are also equal to one another

Since then the axes LN NE EK are equal to one another

and the cylinders QR RB BG are also equal to one another,

and the multitude of the former is equal to the multitude of the latter, therefore whatever multiple the axis KL is of the axis EK , the same multiple also will the cylinder QG be of the cylinder BG

For the same reason whatever multiple the axis MK is of the axis KF , the same multiple also is the cylinder MG of the cylinder GD

And if the axis KL is equal to the axis KM , the cylinder QG will also be equal to the cylinder GM ,

if the axis is greater than the axis the cylinder will also be greater than the cylinder

and if less less

Thus there being four magnitudes the axes EK KF and the cylinders BG GD

there have been taken equimultiples of the axis EK and of the cylinder BG , namely the axis LA and the cylinder QG ,

and equimultiples of the axis KF and of the cylinder GD , namely the axis LM and the cylinder GM ,

and it has been proved that,

if the axis KL is in excess of the axis KM the cylinder QG is also in excess of the cylinder GM ,

if equal equal

and if less less

Therefore as the axis EK is to the axis KF so is the cylinder BG to the cylinder GD

[v Def 5]

Q E D

PROPOSITION 14

Cones and cylinders which are on equal bases are to one another as their heights

For let EB FD be cylinders on equal bases, the circles AB CD

I say that as the cylinder EB is to the cylinder FD so is the axis GH to the axis KL

For let the axis KL be produced to the point N

let LN be made equal to the axis GH , and let the cylinder CM be conceived about LN as axis

Since then the cylinders EB CM are of the same height they are to one another as their bases [xii 11]

But the bases are equal to one another

therefore the cylinders EB CM are also equal

And since the cylinder FM has been cut by the plane CD which is parallel to its opposite planes

therefore as the cylinder CM is to the cylinder FD so is the axis LN to the axis KL [vi 13]



But the cylinder CM is equal to the cylinder EB ,
 and the axis LN to the axis GH ,
 therefore as the cylinder EB is to the cylinder FD , so is the axis GH to the axis KL

But as the cylinder EB is to the cylinder FD , so is the cone ABG to the cone CDA [XII 10]

Therefore also as the axis GH is to the axis KL , so is the cone ABG to the cone CDA and the cylinder EB to the cylinder FD Q.E.D.

PROPOSITION 15

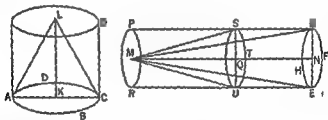
In equal cones and cylinders the bases are reciprocally proportional to the heights and those cones and cylinders in which the bases are reciprocally proportional to the heights are equal

Let there be equal cones and cylinders of which the circles $ABCD$, $EFGH$ are the bases,

let AC , EG be the diameters of the bases
 and AL , MN the axes which are also the heights of the cones or cylinders,

let the cylinders AO , EP be completed

I say that in the cylinders AO , EP the bases are reciprocally proportional to the heights,
 that is as the base $ABCD$ is to the base $EFGH$ so is the height MN to the height KL



For the height KL is either equal to the height MN or not equal

First let it be equal

Now the cylinder AO is also equal to the cylinder EP

But cones and cylinders which are of the same height are to one another as their bases [XII 11]

therefore the base $ABCD$ is also equal to the base $EFGH$

Hence also reciprocally as the base $ABCD$ is to the base $EFGH$, so is the height MN to the height KL

Next, let the height KL not be equal to MN ,

but let MN be greater,

from the height MN let QN be cut off equal to KL ,
 through the point Q let the cylinder EP be cut by the plane TUS parallel to the planes of the circles $EFGH$, RP

and let the cylinder ES be conceived erected from the circle $EFGH$ as base and with height NQ

Now since the cylinder AO is equal to the cylinder EP

therefore as the cylinder AO is to the cylinder ES so is the cylinder EP to the cylinder ES [v 7]

But as the cylinder AO is to the cylinder ES , so is the base $ABCD$ to the base $EFGH$,

for the cylinders AO , ES are of the same height [xi 11]

and, as the cylinder EP is to the cylinder ES so is the height MN to the height QV ,

for the cylinder EP has been cut by a plane which is parallel to its opposite planes [xi 13]

Therefore also as the base $ABCD$ is to the base $EFGH$, so is the height MN to the height QV [v 11]

But the height QV is equal to the height KL , therefore as the base $ABCD$ is to the base $EFGH$, so is the height MN to the height KL

Therefore in the cylinders AO , EP the bases are reciprocally proportional to the heights

Next in the cylinders AO , EP let the bases be reciprocally proportional to the heights

that is as the base $ABCD$ is to the base $EFGH$, so let the height MN be to the height KL

I say that the cylinder AO is equal to the cylinder EP

For with the same construction

since as the base $ABCD$ is to the base $EFGH$ so is the height MN to the height KL ,

while the height KL is equal to the height QV

therefore as the base $ABCD$ is to the base $EFGH$, so is the height MN to the height QV

But as the base $ABCD$ is to the base $EFGH$ so is the cylinder AO to the cylinder ES

for they are of the same height [xi 11]

and as the height MN is to QV so is the cylinder EP to the cylinder ES [xi 12]

therefore as the cylinder AO is to the cylinder ES so is the cylinder EP to the cylinder ES [v 11]

Therefore the cylinder AO is equal to the cylinder EP [v 9]

And the same is true for the cones also Q E D

PROPOSITION 16

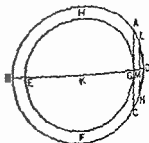
Given two circles about the same centre to inscribe in the greater circle an equilateral polygon with an even number of sides which does not touch the lesser circle

Let $ABCD$, $EFGH$ be the two given circles about the same centre K

thus it is required to inscribe in the greater circle $ABCD$ an equilateral polygon with an even number of sides which does not touch the circle $EFGH$

For let the straight line BKD be drawn through the centre K

and from the point G let GA be drawn at right angles to the straight line BD and carried through to C



therefore AC touches the circle $EFGH$ [III 16 Por.]

Then bisecting the circumference BAD bisecting the half of it and doing this continually, we shall leave a circumference less than AD

Let such be left and let it be LD ,
from L let LM be drawn perpendicular to BD and carried through to N
and let LD DN be joined
therefore LD is equal to DN (III 3, 1 4)

Now, since LN is parallel to AC ,

and AC touches the circle $EFGH$.

therefore LN does not touch the circle $EFGH$

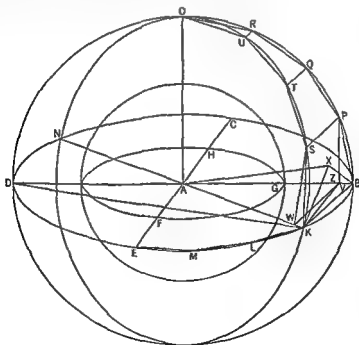
therefore LD DN are far from touching the circle $EFGH$

If then we fit into the circle $ABCD$ straight lines equal to the straight line LD and placed continuously, there will be inscribed in the circle $ABCD$ an equilateral polygon with an even number of sides which does not touch the lesser circle $EFGH$ Q E F

PROPOSITION 17

Given two spheres about the same centre, to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface

Let two spheres be conceived about the same centre A thus it is required to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface



Let the spheres be cut by any plane through the centre
then the sections will be circles

inasmuch as the sphere was produced by the diameter remaining fixed and the semicircle being carried round it, [xi Def 14]

hence in whatever position we conceive the semicircle to be the plane carried through it will produce a circle on the circumference of the sphere

And it is manifest that this circle is the greatest possible inasmuch as the diameter of the sphere which is of course the diameter both of the semicircle and of the circle is greater than all the straight lines drawn across in the circle or the sphere

Let then $BCDE$ be the circle in the greater sphere and FGH the circle in the lesser sphere let two diameters in them BD CE be drawn at right angles to one another, then given the two circles $BCDE$ FGH about the same centre, let there be inscribed in the greater circle $BCDE$ an equilateral polygon with an even number of sides which does not touch the lesser circle FGH

let BA AL LV VE be its sides in the quadrant BE ,

let AI be joined and carried through to N

let AO be set up from the point A at right angles to the plane of the circle $BCDE$ and let it meet the surface of the sphere at O and through IO and each of the straight lines BD AN let planes be carried they will then make greatest circles on the surface of the sphere for the reason stated

Let them make such

and in them let BOD AOV be the semicircles on BD AN

Now since OI is at right angles to the plane of the circle $BCDE$ therefore all the planes through OI are also at right angles to the plane of the circle $BCDE$ [xi 18]
hence the semicircles BOD AOV are also at right angles to the plane of the circle $BCDE$

And since the semicircles BED BOD AON are equal

for they are on the equal diameters BD AN

therefore the quadrants BL BO AO are also equal to one another

Therefore there are as many straight lines in the quadrants BO AO equal to the straight lines BA AL LV ME as there are sides of the polygon in the quadrant BE

Let them be inscribed and let them be BI PQ QR RO and AS ST TU LO

let SP TQ UR be joined

and from P Q let perpendiculars be drawn to the plane of the circle $BCDE$ [xi 11]

these will fall on BD AN the common sections of the planes inasmuch as the planes of BOD AOV are also at right angles to the plane of the circle $BCDE$ [cf xi Def 4]

Let them be PI SH

and let II be joined

Now since in the equal semicircles BOD AOV equal straight lines BP AS have been cut off

and the perpendiculars PI SH have been drawn

therefore BI is equal to SH and BI to AI [iii 27, 1^o]

But the whole BI is also equal to the whole AI

therefore the remainder VA is also equal to the remainder WA ,

therefore as BV is to VA so is KW to WA

therefore WV is parallel to KB [VI 2]

And since each of the straight lines PV , SW is at right angles to the plane of the circle $BCDF$,

therefore PV is parallel to SW [XI 6]

But it was also proved equal to it

therefore WV , SP are also equal and parallel [I 33]

And, since WV is parallel to SP ,

while WV is parallel to KB ,

therefore SP is also parallel to KB [XI 9]

And BP , KS join their extremities,

therefore the quadrilateral $KBPS$ is in one plane

inasmuch as if two straight lines be parallel, and points be taken at random on each of them, the straight line joining the points is in the same plane with the parallels [XI 7]

For the same reason

each of the quadrilaterals $SPQT$, $TQRU$ is also in one plane

But the triangle URO is also in one plane [XI 2]

If then we conceive straight lines joined from the points P , S , Q , T , R , U to A , there will be constructed a certain polyhedral solid figure between the circumferences BO , KO , consisting of pyramids of which the quadrilaterals $KBPS$, $SPQT$, $TQRU$ and the triangle URO are the bases and the point A the vertex

And if we make the same construction in the case of each of the sides AL , LM , ME as in the case of BA and further in the case of the remaining three quadrants

there will be constructed a certain polyhedral figure inscribed in the sphere and contained by pyramids, of which the said quadrilaterals and the triangle URO , and the others corresponding to them are the bases and the point A the vertex

I say that the said polyhedron will not touch the lesser sphere at the surface on which the circle FGH is

Let AX be drawn from the point A perpendicular to the plane of the quadrilateral $KBPS$ and let it meet the plane at the point X [XI 11]

let AB , XA be joined

Then since AX is at right angles to the plane of the quadrilateral $KBPS$, therefore it is also at right angles to all the straight lines which meet it and are in the plane of the quadrilateral [XI Def 3]

Therefore AX is at right angles to each of the straight lines BX , XA

And since AB is equal to AK

the square on AB is also equal to the square on AK

And the squares on AX , XB are equal to the square on AB

for the angle at X is right [I 47]

and the squares on AX , XK are equal to the square on AK [Id.]

Therefore the squares on AX , XB are equal to the squares on AX , XK

Let the square on AX be subtracted from each

therefore the remainder the square on BX is equal to the remainder the square on XK ,

therefore BX is equal to XK

Similarly we can prove that the straight lines joined from Λ to P & are equal to each of the straight lines $B\Lambda$ ΛK

Therefore the circle described with centre Λ and distance one of the straight lines ΛB , ΛA will pass through P & also

and ΛBPS will be a quadrilateral in a circle

Now, since ΛB is greater than $H V$

while $H V$ is equal to SP

therefore ΛB is greater than SP

But ΛB is equal to each of the straight lines ΛS BP

therefore each of the straight lines ΛS BP is greater than SP

And since ΛBPS is a quadrilateral in a circle

and KB BP , ΛS are equal and PS less

and $B\Lambda$ is the radius of the circle

therefore the square on KB is greater than double of the square on $B\Lambda$

Let ΛZ be drawn from Λ perpendicular to BV

Then since BD is less than double of DZ

and as BD is to DZ so is the rectangle DB BZ to the rectangle DZ ZB
if a square be described upon BZ and the parallelogram on ZD be completed
then the rectangle DB BZ is also less than double of the rectangle DZ ZB

And if ΛD be joined

the rectangle DB BZ is equal to the square on $B\Lambda$,

and the rectangle DZ ZB equal to the square on ΛZ [III 31 vi 8 and Por]

therefore the square on ΛB is less than double of the square on ΛZ

But the square on ΛB is greater than double of the square on $B\Lambda$

therefore the square on ΛZ is greater than the square on $B\Lambda$

And since $B\Lambda$ is equal to ΛI

the square on BA is equal to the square on ΛA

And the squares on $B\Lambda$ ΛA are equal to the square on BA

and the squares on ΛZ ZA equal to the square on ΛA , [I 47]

therefore the squares on $B\Lambda$ ΛA are equal to the squares on ΛZ ZA

and of these the square on ΛZ is greater than the square on $B\Lambda$

therefore the remainder the square on ZA is less than the square on ΛA

Therefore ΛA is greater than ΛZ

therefore $A\Lambda$ is much greater than AG

And $A\Lambda$ is the perpendicular on one base of the polyhedron

and AG on the surface of the lesser sphere

hence the polyhedron will not touch the lesser sphere on its surface

Therefore given two spheres about the same centre = polyhedral solid has been inscribed in the greater sphere which does not touch the lesser sphere at its surface

PROPOSITION But if in another sphere also a polyhedral solid be inscribed similar to the solid in the sphere $BCDE$

the polyhedral solid in the sphere $BCDE$ has to the polyhedral solid in the other sphere the ratio triplicate of that which the diameter of the sphere $BCDE$ has to the diameter of the other sphere

For the solids being divided into their pyramids similar in multitude and arrangement the pyramids will be similar

But similar pyramids are to one another in the triplicate ratio of their corresponding sides [XII 8 Por]

therefore the pyramid of which the quadrilateral $KBPS$ is the base and the point A the vertex has to the similarly arranged pyramid in the other sphere the ratio triplicate of that which the corresponding side has to the corresponding side that is of that which the radius AB of the sphere about A as centre has to the radius of the other sphere

Similarly also each pyramid of those in the sphere about A as centre has to each similarly arranged pyramid of those in the other sphere the ratio triplicate of that which AB has to the radius of the other sphere

And as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents, [v 12]

hence the whole polyhedral solid in the sphere about A as centre has to the whole polyhedral solid in the other sphere the ratio triplicate of that which AB has to the radius of the other sphere that is of that which the diameter BD has to the diameter of the other sphere Q E D

PROPOSITION 18

Spheres are to one another in the triplicate ratio of their respective diameters

Let the spheres ABC , DEF be conceived

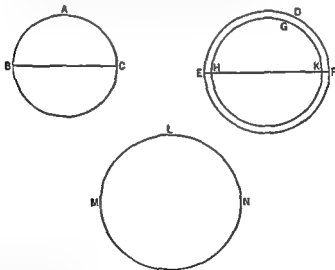
and let BC , EF be their diameters,

I say that the sphere ABC has to the sphere DEF the ratio triplicate of that which BC has to EF

For if the sphere ABC has not to the sphere DEF the ratio triplicate of that which BC has to EF

then the sphere ABC will have either to some less sphere than the sphere DEF , or to a greater the ratio triplicate of that which BC has to EF

First let it have that ratio to a less sphere GHA ,



let DEF be conceived about the same centre with GHA ,
let there be inscribed in the greater sphere DEF a polyhedral solid which does not touch the lesser sphere GHA at its surface [xii 17]

and let there also be inscribed in the sphere ABC a polyhedral solid similar to the polyhedral solid in the sphere DEF ,
therefore the polyhedral solid in ABC has to the polyhedral solid in DEF the ratio triplicate of that which BC has to EF [XII 17, Por]

But the sphere ABC also has to the sphere GHK the ratio triplicate of that which BC has to EF ,
therefore as the sphere ABC is to the sphere GHK , so is the polyhedral solid in the sphere ABC to the polyhedral solid in the sphere DEF ,
and alternately as the sphere ABC is to the polyhedron in it so is the sphere GHK to the polyhedral solid in the sphere DEF [v 16]

But the sphere ABC is greater than the polyhedron in it
therefore the sphere GHA is also greater than the polyhedron in the sphere DEF

But it is also less

for it is enclosed by it

Therefore the sphere ABC has not to a less sphere than the sphere DEF the ratio triplicate of that which the diameter BC has to EF

Similarly we can prove that neither has the sphere DEF to a less sphere than the sphere ABC the ratio triplicate of that which EF has to BC

I say next that neither has the sphere ABC to any greater sphere than the sphere DEF the ratio triplicate of that which BC has to EF

For if possible let it have that ratio to a greater LMN ,
therefore inversely the sphere LMN has to the sphere ABC the ratio triplicate of that which the diameter EF has to the diameter BC

But inasmuch as LMN is greater than DEF
therefore as the sphere LMN is to the sphere ABC so is the sphere DEF to some less sphere than the sphere ABC as was before proved [XII 2 Lemma]

Therefore the sphere DEF also has to some less sphere than the sphere ABC the ratio triplicate of that which EF has to BC

which was proved impossible

Therefore the sphere ABC has not to any sphere greater than the sphere DEF the ratio triplicate of that which BC has to EF

But it was proved that neither has it that ratio to a less sphere

Therefore the sphere ABC has to the sphere DEF the ratio triplicate of that which BC has to EF Q E D

BOOK THIRTEEN

PROPOSITIONS

PROPOSITION 1

If a straight line be cut in extreme and mean ratio the square on the greater segment added to the half of the whole is five times the square on the half

For let the straight line AB be cut in extreme and mean ratio at the point C ,
 --- and let AC be the greater segment,

let the straight line AD be produced in a straight line with CA
 and let AD be made half of AB ,

I say that the square on CD is five times the square on AD

For let the squares AE , DF be described on AB , DC

and let the figure in DF be drawn,
 let FC be carried through to G

Now, since AB has been cut in extreme and mean ratio at C ,
 therefore the rectangle AB BC is equal to the square on AC [vi Def 3 vi 17]

And CE is the rectangle AB , BC and FH the square on AC ,

therefore CE is equal to FH

And since BA is double of AD ,
 while BA is equal to KA and AD to AH ,
 therefore KA is also double of AH

But as KA is to AH so is CK to CH , [vi 1]
 therefore CK is double of CH

But LH , HC are also double of CH

Therefore AC is equal to LH HC

But CE was also proved equal to HF

therefore the whole square AE is equal to the gnomon MNO

And since BA is double of AD

the square on BA is quadruple of the square on AD ,
 that is, AE is quadruple of DH

But AE is equal to the gnomon MNO

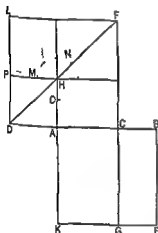
therefore the gnomon MNO is also quadruple of AP

therefore the whole DF is five times AP

And DF is the square on DC , and AP the square on DA ,

therefore the square on CD is five times the square on DA

Therefore etc



PROPOSITION 2

If the square on a straight line be five times the square on a segment of it then, when the double of the said segment is cut in extreme and mean ratio the greater segment is the remaining part of the original straight line

For let the square on the straight line AB be five times the square on the segment AC of it

and let CD be double of AC ,

I say that when CD is cut in extreme and mean ratio the greater segment is CB

Let the squares AF CG be described on AB CD respectively

let the figure in AF be drawn

and let BE be drawn through

Now since the square on BA is five times the square on AC

AF is five times AH

Therefore the gnomon MNO is quadruple of AH

And since DC is double of CA
therefore the square on DC is quadruple of the square on CA that is CG is quadruple of AH

But the gnomon MNO was also proved quadruple of AH

therefore the gnomon MNO is equal to CG

And since DC is double of CA

while DC is equal to CK and AC to CH

therefore KB is also double of BH

[VI 1]

But LH HB are also double of HB

therefore KB is equal to LH HB

But the whole gnomon MNO was also proved equal to the whole CG ,

therefore the remainder HF is equal to BG

And BG is the rectangle CD , DB

for CD is equal to DG ,

and HF is the square on CB

therefore the rectangle CD DB is equal to the square on CB

Therefore as DC is to CB so is CB to BD

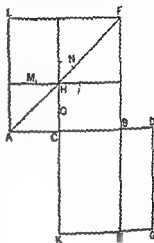
But DC is greater than CB

therefore CB is also greater than BD

Therefore when the straight line CD is cut in extreme and mean ratio CB is the greater segment

Therefore etc

Q.E.D.



LEMMA

That the double of AC is greater than BC is to be proved thus

If not let BC be if possible double of CA

Therefore the square on BC is quadruple of the square on CA

therefore the squares on BC CA are five times the square on CA

But by hypothesis the square on BA is also five times the square on CA ,

therefore the square on BA is equal to the squares on BC , CA
which is impossible

(II 4)

Therefore CB is not double of AC

Similarly we can prove that neither is a straight line less than CB double of CA ,

for the absurdity is much greater

Therefore the double of AC is greater than CB

Q E D

PROPOSITION 3

If a straight line be cut in extreme and mean ratio the square on the lesser segment added to the half of the greater segment is five times the square on the half of the greater segment

For let any straight line AB be cut in extreme and mean ratio at the point C ,

let AC be the greater segment

and let AC be bisected at D ,

I say that the square on BD is five times the square on DC

For let the square AE be described on AB
and let the figure be drawn double

Since AC is double of DC

therefore the square on AC is quadruple of the square on DC ,

that is RS is quadruple of FG

And since the rectangle AB , BC is equal to the square on AC

and CE is the rectangle AB , BC

therefore CE is equal to RS

But RS is quadruple of FG

therefore CE is also quadruple of FG

Again since AD is equal to DC

HK is also equal to AF

Hence the square GF is also equal to the square HL

Therefore GK is equal to KL that is MN to NE ,

hence MF is also equal to FE

But MF is equal to CG ,

therefore CG is also equal to FE

Let CA be added to each

therefore the gnomon OPQ is equal to CE

But CE was proved quadruple of GF ,

therefore the gnomon OPQ is also quadruple of the square FG

Therefore the gnomon OPQ and the square FG are five times FG

But the gnomon OPQ and the square FG are the square DN

And DN is the square on DB and GF the square on DC

Therefore the square on DB is five times the square on DC

Q E D

PROPOSITION 4

If a straight line be cut in extreme and mean ratio the square on the whole and the square on the lesser segment together are triple of the square on the greater segment

Let AB be a straight line,

and DH is equal to HC ,
therefore DH is also equal to HE

Therefore the whole DA is equal to the whole AE

And DK is the rectangle BD, DA ,

for AD is equal to DL ,

and AE is the square on AB ,

therefore the rectangle BD, DA is equal to the square on AB

Therefore as DB is to BA , so is BA to AD

[VI 17]

And DB is greater than BA ,

therefore BA is also greater than AD

[V 14]

Therefore DB has been cut in extreme and mean ratio at A and AB is the greater segment

Q E D

PROPOSITION 6

If a rational straight line be cut in extreme and mean ratio, each of the segments is the irrational straight line called apotome



Let AB be a rational straight line

let it be cut in extreme and mean ratio at C ,

and let AC be the greater segment

I say that each of the straight lines AC, CB
is the irrational straight line called apotome

For let BA be produced and let AD be made half of BA

Since then the straight line AB has been cut in extreme and mean ratio

and to the greater segment AC is added AD which is half of AB

therefore the square on CD is five times the square on DA [XIII 1]

Therefore the square on CD has to the square on DA the ratio which a number has to a number

therefore the square on CD is commensurable with the square on DA [X 6]

But the square on DA is rational

for DA is rational, being half of AB which is rational

therefore the square on CD is also rational,

[X Def 4]

therefore CD is also rational

And, since the square on CD has not to the square on DA the ratio which a square number has to a square number,

therefore CD is incommensurable in length with DA ,

[X 9]

therefore CD, DA are rational straight lines commensurable in square only

therefore AC is an apotome

[X 73]

Again since AB has been cut in extreme and mean ratio

and AC is the greater segment,

therefore the rectangle AB, BC is equal to the square on AC [VI Def 3 VI 17]

Therefore the square on the apotome AC if applied to the rational straight line AB produces BC as breadth

But the square on an apotome if applied to a rational straight line produces as breadth a first apotome

[X 97]

therefore CB is a first apotome

And CA was also proved to be an apotome

Therefore etc

Q E D

PROPOSITION 7

If three angles of an equilateral pentagon, taken either in order or not in order be equal, the pentagon will be equiangular

For in the equilateral pentagon $ABCDE$ let, first three angles taken in order those at A, B, C be equal to one another,

I say that the pentagon $ABCDE$ is equiangular

For let AC, BE, FD be joined

Now since the two sides CB, BA are equal to the two sides BA, AE respectively,

and the angle CBA is equal to the angle BAE

therefore the base AC is equal to the base BE ,

the triangle ABC is equal to the triangle ABE

and the remaining angles will be equal to the remaining

angles namely those which the equal sides subtend [1. 4]

that is the angle BCA to the angle BEA and the angle ABE to the angle CAB ,

hence the side AF is also equal to the side BF [1. 6]

But the whole AC was also proved equal to the whole BE

therefore the remainder FC is also equal to the remainder FE

But CD is also equal to DE

Therefore the two sides FC, CD are equal to the two sides FE, ED ,

and the base FD is common to them

therefore the angle FCD is equal to the angle FED [1. 8]

But the angle BCA was also proved equal to the angle AEB ,

therefore the whole angle BCD is also equal to the whole angle AED

But by hypothesis the angle BCD is equal to the angles at A, B

therefore the angle AED is also equal to the angles at A, B

Similarly we can prove that the angle CDE is also equal to the angles at A, B, C

therefore the pentagon $ABCDE$ is equiangular

Next let the given equal angles not be angles taken in order, but let the angles at the points A, C, D be equal

I say that in this case too the pentagon $ABCDE$ is equiangular

For let BD be joined

Then since the two sides BA, AE are equal to the two sides BC, CD

and they contain equal angles

therefore the base BE is equal to the base BD

the triangle ABE is equal to the triangle BCD

and the remaining angles will be equal to the remaining angles,

namely those which the equal sides subtend [1. 4]

therefore the angle AEB is equal to the angle CDB

But the angle BFD is also equal to the angle BDE [1. 5]

since the side BE is also equal to the side BD

Therefore the whole angle AED is equal to the whole angle CDE

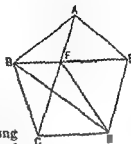
But the angle CDE is by hypothesis equal to the angles at A, C ,

therefore the angle AED is also equal to the angles at A, C

For the same reason

the angle ABC is also equal to the angles at A, C, D

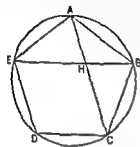
Therefore the pentagon $ABCDE$ is equiangular



PROPOSITION 8

If in an equilateral and equiangular pentagon straight lines subtend two angles taken in order, they cut one another in extreme and mean ratio and their greater segments are equal to the side of the pentagon

For in the equilateral and equiangular pentagon $ABCDE$ let the straight lines AC , BE , cutting one another at the point H subtend two angles taken in order, the angles at A B



I say that each of them has been cut in extreme and mean ratio at the point H and their greater segments are equal to the side of the pentagon

For let the circle $ABCDE$ be circumscribed about the pentagon $ABCDE$ [iv 14]

Then since the two straight lines EA , AB are equal to the two AB BC , and they contain equal angles

therefore the base BE is equal to the base AC

the triangle ABE is equal to the triangle ABC

and the remaining angles will be equal to the remaining angles respectively namely those which the equal sides subtend [i 4]

Therefore the angle BAC is equal to the angle ABE ,

therefore the angle AHE is double of the angle BAH [i 32]

But the angle EAC is also double of the angle BAC

inasmuch as the circumference EDC is also double of the circumference CB [iii 28 vi 33]

therefore the angle HAE is equal to the angle AHE

hence the straight line HE is also equal to EA that is to AB [i 6]

And since the straight line BA is equal to AE

the angle ABE is also equal to the angle AEB [i 5]

But the angle ABE was proved equal to the angle BAH

therefore the angle BEA is also equal to the angle BAH

And the angle ABE is common to the two triangles ABE and ABH , therefore the remaining angle BAE is equal to the remaining angle AHB [i 32]

therefore the triangle ABE is equiangular with the triangle ABH

therefore proportionally as EB is to BA so is AB to BH [vi 4]

But BA is equal to EH

therefore as BE is to EH so is EH to HB

And BE is greater than EH

therefore EH is also greater than HB [v 14]

Therefore BE has been cut in extreme and mean ratio at H and the greater segment HE is equal to the side of the pentagon

Similarly we can prove that AC has also been cut in extreme and mean ratio at H and its greater segment CH is equal to the side of the pentagon $Q.E.D.$

PROPOSITION 9

If the side of the hexagon and that of the decagon inscribed in the same circle be added together the whole straight line has been cut in extreme and mean ratio and its greater segment is the side of the hexagon

squares on the side of the hexagon and on that of the decagon inscribed in the circle $ABCDE$

For let the centre of the circle, the point F , be taken

let AF be joined and carried through to the point G ,

let FB be joined

let FH be drawn from F perpendicular to AB and be carried through to K ,

let AK, KB be joined

let FL be again drawn from F perpendicular to AK and be carried through to M ,

and let AN be joined

Since the circumference $ABCG$ is equal to the circumference $AEDG$

and in them ABC is equal to AED

therefore the remainder the circumference

CG is equal to the remainder GD

But CD belongs to a pentagon,

therefore CG belongs to a decagon

And since FA is equal to FB

and FH is perpendicular,

therefore the angle AFH is also equal to the angle KFB [I 5, I 26]

Hence the circumference AK is also equal to KB [III 26]

therefore the circumference AB is double of the circumference BA ,

therefore the straight line AK is a side of a decagon

For the same reason

AK is also double of KM

Now since the circumference AB is double of the circumference BA ,

while the circumference CD is equal to the circumference AB

therefore the circumference CD is also double of the circumference BK

But the circumference CD is also double of CG

therefore the circumference CG is equal to the circumference BK

But BA is double of KM since KA is so also

therefore CG is also double of KM

But, further the circumference CB is also double of the circumference BA ,

for the circumference CB is equal to BA

Therefore the whole circumference GB is also double of BM

hence the angle GFB is also double of the angle BFM [VI 33]

But the angle GFB is also double of the angle FAB

for the angle FAB is equal to the angle ABF

Therefore the angle BFN is also equal to the angle FAB

But the angle ABF is common to the two triangles ABF and BFN ,

therefore the remaining angle AFB is equal to the remaining angle BNF [I 32]

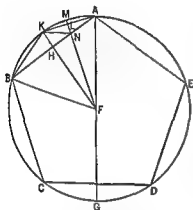
therefore the triangle ABF is equiangular with the triangle BFN

Therefore proportionally as the straight line AB is to BF so is FB to BN

[VI 4]

therefore the rectangle $AB BN$ is equal to the square on BF [VI 17]

Again since AL is equal to LH ,



while LA is common and at right angles

therefore the base KN is equal to the base AN [1 4]

therefore the angle LKN is also equal to the angle LAK

But the angle LAK is equal to the angle LKN ,

therefore the angle LKN is also equal to the angle LKN

And the angle at A is common to the two triangles LKN and LAK

Therefore the remaining angle LKN is equal to the remaining angle LAK [1 2]

therefore the triangle LKN is equiangular with the triangle LAK

Therefore proportionally as the straight line KN is to LK , so is LK to LA [1 4]

therefore the rectangle KN, LA is equal to the square on LK [1 17]

But the rectangle LK, BN was also proved equal to the square on BF

therefore the rectangle LK, BN together with the rectangle KN, LA that is the square on BA [1 2] is equal to the square on BF together with the square on LK

And BA is a side of the pentagon BF of the hexagon [iv 15 Por] and LA of the decagon

Therefore etc

Q E D

PROPOSITION 11

If in a circle which has its diameter rational an equilateral pentagon be inscribed the side of the pentagon is the irrational straight line called minor

For in the circle $ABCDE$ which has its diameter rational let the equilateral pentagon $ABCDE$ be inscribed

I say that the side of the pentagon is the irrational straight line called minor

For let the centre of the circle the point F be taken let AF, FB be joined and carried

through to the points G, H

let AC be joined

and let FK be made a fourth

part of AF

Now AF is rational

therefore FK is also rational

But BF is also rational

therefore the whole BA is rational

And since the circumference ACG is equal to the circumference ADG

and in them ABC is equal to AED

therefore the remainder CG is equal to the remainder GD

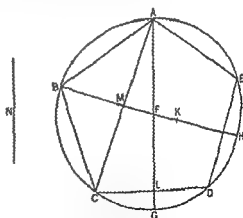
And if we join AD we conclude that the angles at L are right and CD is double of CL

For the same reason

the angles at M are also right

and AC is double of CM

Since then the angle ALL is equal to the angle AMF ,



and the angle LAC is common to the two triangles ACL and AMF
 therefore the remaining angle ACL is equal to the remaining angle MFA , [I 32]

therefore the triangle ACL is equiangular with the triangle AMF ,

therefore, proportionally as LC is to CA , so is MF to FA

And the doubles of the antecedents may be taken,

therefore as the double of LC is to CA , so is the double of MF to FA

But as the double of MF is to FA , so is MF to the half of FA ,

therefore also as the double of LC is to CA , so is MF to the half of FA

And the halves of the consequents may be taken,

therefore as the double of LC is to the half of CA , so is MF to the fourth of FA

And DC is double of LC CM is half of CA , and FK a fourth part of FA ,

therefore as DC is to CM so is MF to FK

Componendo also as the sum of DC CM is to CM , so is MA to KF , [V 18]

therefore also as the square on the sum of DC , CM is to the square on CM , so is the square on MA to the square on KF

And since when the straight line subtending two sides of the pentagon as AC is cut in extreme and mean ratio the greater segment is equal to the side of the pentagon that is to DC [XIII 8]

while the square on the greater segment added to the half of the whole is five times the square on the half of the whole [XIII 1]

and CM is half of the whole AC ,

therefore the square on DC CM taken as one straight line is five times the square on CM

But it was proved that as the square on DC CM taken as one straight line is to the square on CM so is the square on MA to the square on KF

therefore the square on MA is five times the square on KF

But the square on KF is rational

for the diameter is rational

therefore the square on MA is also rational

therefore MA is rational

And since BF is quadruple of FK

therefore BK is five times KF

therefore the square on BA is twenty five times the square on KF

But the square on MA is five times the square on KF

therefore the square on BA is five times the square on MA

therefore the square on BA has not to the square on MA the ratio which a square number has to a square number

therefore BA is incommensurable in length with MA [X 9]

And each of them is rational

Therefore BK MA are rational straight lines commensurable in square only

But if from a rational straight line there be subtracted a rational straight line which is commensurable with the whole in square only the remainder is irrational namely an apotome

therefore MB is an apotome and MA the annex to it [X 73]

I say next that MB is also a fourth apotome

Let the square on A be equal to that by which the square on BA is greater than the square on MA

therefore the square on BK is greater than the square on MA by the square on N

And, since AF is commensurable with FB ,

componendo also KB is commensurable with FB [x 15]

But BF is commensurable with BH ,

therefore BK is also commensurable with BH [x 12]

And, since the square on BK is five times the square on AM ,

therefore the square on BA has to the square on KM the ratio which 5 has to 1

Therefore, *convertendo* the square on BK has to the square on N the ratio which 5 has to 4 [v 19 Por], and this is not the ratio which is square number has to is square number

therefore BK is incommensurable with N , [x 9]

therefore the square on BA is greater than the square on AM by the square on a straight line incommensurable with BK

Since then the square on the whole BK is greater than the square on the annex AM by the square on a straight line incommensurable with BK , and the whole BA is commensurable with the rational straight line, BH , set out

therefore MB is a fourth apotome [x Def III 4]

But the rectangle contained by a rational straight line and a fourth apotome is irrational

and its square root is irrational and is called minor [x 94]

But the square on AB is equal to the rectangle HB BM , because when AH is joined the triangle ABH is equiangular with the triangle ABM and as HB is to BA so is AB to BM

Therefore the side AB of the pentagon is the irrational straight line called minor Q E D

PROPOSITION 12

If an equilateral triangle be inscribed in a circle the square on the side of the triangle is triple of the square on the radius of the circle

Let ABC be a circle

and let the equilateral triangle ABC be inscribed in it

I say that the square on one side of the triangle ABC is triple of the square on the radius of the circle

For let the centre D of the circle ABC be taken

let AD be joined and carried through to E

and let BE be joined

Then since the triangle ABC is equilateral therefore the circumference BEC is a third part of the circumference of the circle ABC

Therefore the circumference BE is a sixth part of the circumference of the circle

therefore the straight line BE belongs to a hexagon

therefore it is equal to the radius DE [iv 15 Por]

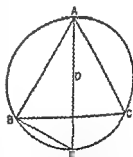
And since AE is double of DE

the square on AE is quadruple of the square on ED that is of the square on BE

But the square on AE is equal to the squares on AB BE [iii 31 i 47]

therefore the squares on AB BE are quadruple of the square on BE

Therefore *separando* the square on AB is triple of the square on BE



But BE is equal to DE ,

therefore the square on AB is triple of the square on DE

Therefore the square on the side of the triangle is triple of the square on the radius

Q E D

PROPOSITION 13

To construct a pyramid, to comprehend it in a given sphere and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid

Let the diameter AB of the given sphere be set out,

and let it be cut at the point C so that AC is double of CB ,

let the semicircle ADB be described on AB ,

let CD be drawn from the point C at right angles to AB ,

and let DA be joined,

let the circle EFG which has its radius equal to DC be set out,

let the equilateral triangle EFG be inscribed in the circle EFG , [IV 2]

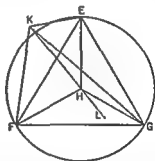
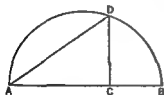
let the centre of the circle the point H , be taken [III 1]

let CH HF , HG be joined,

from the point H let HK be set up at right angles to the plane of the circle EFG , [XI 12]

let HK equal to the straight line AC be cut off from HK ,

and let KE , KF , KG be joined



Now since KH is at right angles to the plane of the circle EFG therefore it will also make right angles with all the straight lines which meet it and are in the plane of the circle EFG [XI Def 3]

But each of the straight lines HE HF HG meets it

therefore KH is at right angles to each of the straight lines HE , HF HG

And, since AC is equal to HK , and CD to HE

and they contain right angles

therefore the base DA is equal to the base KE [I 4]

For the same reason

each of the straight lines KF KG is also equal to DA

therefore the three straight lines KE KF , KG are equal to one another

And since AC is double of CB ,

therefore AB is triple of BC

But, as AB is to BC so is the square on AD to the square on DC , as will be proved afterwards

Therefore the square on AD is triple of the square on DC

But the square on FE is also triple of the square on EH , [XIII 12]
and DC is equal to EH ,

therefore DA is also equal to EF

But DA was proved equal to each of the straight lines KE , KF , KG
therefore each of the straight lines EF , FG , GE is also equal to each of the straight lines KE , KF , KG ,

therefore the four triangles EFG , KEF , KFG , KEG are equilateral

Therefore a pyramid has been constructed out of four equilateral triangles the triangle EFG being its base and the point K its vertex

It is next required to comprehend it in the given sphere and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid

For let the straight line HL be produced in a straight line with KH ,
and let HL be made equal to CB

Now, since as AC is to CD , so is CD to CB [VI 8 Por]

while AC is equal to KH , CD to HE , and CB to HL ,

therefore, as KH is to HE , so is EH to HL

therefore the rectangle KH , HL is equal to the square on EH [VI 17]

And each of the angles KHE EHL is right,

therefore the semicircle described on KL will pass through F also

[cf VI 8, III 31]

If then KL remaining fixed the semicircle be carried round and restored to the same position from which it began to be moved, it will also pass through the points F G ,

since if FL LG be joined the angles at F G similarly become right angles,

and the pyramid will be comprehended in the given sphere

For KL the diameter of the sphere is equal to the diameter AB of the given sphere, inasmuch as KH was made equal to AC and HL to CB

I say next that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid

For since AC is double of CB ,

therefore AB is triple of BC ,

and *converendo* BA is one and a half times AC

But as BA is to AC so is the square on BA to the square on AC

Therefore the square on BA is also one and a half times the square on AC

And BA is the diameter of the given sphere and AC is equal to the side of the pyramid

Therefore the square on the diameter of the sphere is one and a half times the square on the side of the pyramid Q E D

LEMMA

It is to be proved that as AB is to BC , so is the square on AD to the square on DC

For let the figure of the semicircle be set out

let DB be joined

let the square EC be described on AC ,

and let the parallelogram FB be completed

Since then because the triangle DAB is equiangular with the triangle DAC ,
as BA is to AD , so DA is to AC [vi 8 vi 4]
therefore the rectangle BA, AC is equal to
the square on AD [vi 17]

And since as AB is to BC , so is EB to BF
[vi 1]

and EB is the rectangle BA, AC , for EA is
equal to AC ,

and BF is the rectangle AC, CB ,
therefore as AB is to BC , so is the rectangle
 BA, AC to the rectangle AC, CB

And the rectangle BA, AC is equal to the
square on AD and the rectangle AC, CB to
the square on DC ,

for the perpendicular DC is a mean propor-
tional between the segments AC, CB of the
[vi 8 Por.]

base because the angle ADB is right

Therefore as AB is to BC , so is the square on AD to the square on DC

Q E D

PROPOSITION 14

To construct an octahedron and comprehend it in a sphere as in the preceding ca and to prove that the square on the diameter of the sphere is double of the square on the side of the octahedron

Let the diameter AB of the given sphere be set out

and let it be bisected at C

let the semicircle ADB be described on AB

let CD be drawn from C at right angles to AB

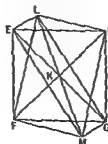
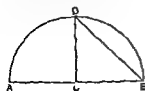
let DB be joined

let the square $EFGH$, having each of its sides equal to DB be set out

let HF, EG be joined

from the point K let the straight line KL be set up at right angles to the plane
of the square $EFGH$ [xi 12] and let it be carried through to the other side of
the plane as KM

from the straight lines KL, KM let KL, KM be respectively cut off equal to
one of the straight lines EK, FK, GK, HK



and let LE, LF, LG, LH
 ME, MF, MG, MH be
joined

Then, since KE is equal
to KH ,
and the angle EKH is right
therefore the square on HE
is double of the square on
 EK [i 47]

Again, since LK is equal
to KE ,

and the angle LKE is right

therefore the square on EL is double of the square on EK [14]

But the square on HE was also proved double of the square on EK ,
therefore the square on LE is equal to the square on EH ,
therefore LE is equal to EH

For the same reason

LH is also equal to HE ,

therefore the triangle LEH is equilateral

Similarly we can prove that each of the remaining triangles of which the sides of the square $FFGH$ are the bases, and the points L, M the vertices, is equilateral,
therefore an octahedron has been constructed which is contained by eight equilateral triangles

It is next required to comprehend it in the given sphere, and to prove that the square on the diameter of the sphere is double of the square on the side of the octahedron

For since the three straight lines LK, KM, KE are equal to one another,
therefore the semicircle described on LM will also pass through E

And for the same reason,

if LM remaining fixed the semicircle be carried round and restored to the same position from which it began to be moved

it will also pass through the points F, G, H ,

and the octahedron will have been comprehended in a sphere

I say next that it is also comprehended in the given sphere

For since LK is equal to KM ,

while KE is common

and they contain right angles

therefore the base LE is equal to the base EM [14]

And since the angle LKM is right for it is in a semicircle [III 31]

therefore the square on LM is double of the square on LE [14]

Again since AC is equal to CB ,

AB is double of BC

But as AB is to BC so is the square on AB to the square on BC ,

therefore the square on AB is double of the square on BC

But the square on LM was also proved double of the square on LE

And the square on DB is equal to the square on LE for EH was made equal to DB

Therefore the square on AB is also equal to the square on LM ,

therefore AB is equal to LM

And AB is the diameter of the given sphere

therefore LM is equal to the diameter of the given sphere

Therefore the octahedron has been comprehended in the given sphere and it has been demonstrated at the same time that the square on the diameter of the sphere is double of the square on the side of the octahedron Q E D

PROPOSITION 15

To construct a cube and comprehend it in a sphere like the pyramid and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube

Let the diameter AB of the given sphere be set out,

and let it be cut at C so that AC is double of CB ,

let the semicircle ADB be described on AB

let CD be drawn from C at right angles to AB ,

and let DB be joined,

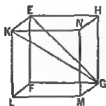
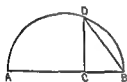
let the square $EFGH$ having its side equal to DB be set out

from $E F G H$ let EK, FL, GM, HN be drawn at right angles to the plane of the square $EFGH$

from EK, FL, GM, HN let EK, FL, GM, HN respectively be cut off equal to one of the straight lines EF, FG, GH, HE ,

and let KL, LM, MN, NK be joined

therefore the cube FN has been constructed which is contained by six equal squares



It is then required to comprehend it in the given sphere and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube

For let AG, EG be joined

Then since the angle KEG is right, because KE is also at right angles to the plane EG

[xi Def 3]

and of course to the straight line EG also,

therefore the semicircle described on AG will also pass through the point E

Again since GF is at right angles to each of the straight lines FL, FE

GF is also at right angles to the plane FA

hence also if we join FA GF will be at right angles to FA ,

and for this reason again the semicircle described on GA will also pass through F

Similarly it will also pass through the remaining angular points of the cube

If then AG remaining fixed the semicircle be carried round and restored to the same position from which it began to be moved,

the cube will be comprehended in a sphere

I say next that it is also comprehended in the given sphere

For since GF is equal to FE ,

and the angle at F is right

therefore the square on EG is double of the square on EF

But EF is equal to EA ,

therefore the square on EG is double of the square on EA

hence the squares on GE, EA that is the square on GA [i 47] is triple of the square on EA

And since AB is triple of BC

while as AB is to BC so is the square on AB to the square on BC

therefore the square on AB is triple of the square on BC

But the square on GA was also proved triple of the square on AE

And AE was made equal to DB

therefore AG is also equal to AB

And AB is the diameter of the given sphere

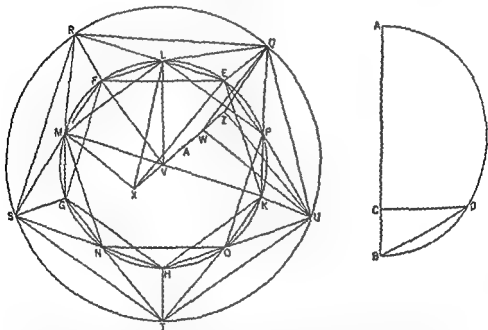
therefore AG is also equal to the diameter of the given sphere

Therefore the cube has been comprehended in the given sphere, and it has been demonstrated at the same time that the square on the diameter of the sphere is triple of the square on the side of the cube Q E D

PROPOSITION 16

To construct an icosahedron and comprehend it in a sphere, like the aforesaid figures and to prove that the side of the icosahedron is the irrational straight line called minor

Let the diameter AB of the given sphere be set out
 and let it be cut at C so that AC is quadruple of CB ,
 let the semicircle ADB be described on AB ,
 let the straight line CD be drawn from C at right angles to AB ,
 and let DB be joined



let the circle $EFGHA$ be set out and let its radius be equal to DB
 let the equilateral and equiangular pentagon $EFGHA$ be inscribed in the circle $EFGHA$
 let the circumferences EF , FG , GH , HA , AE be bisected at the points L , M ,
 N , O , P

and let LM , MN , NO , OP , PL , EP be joined

Therefore the pentagon $LMNOP$ is also equilateral

and the straight line EP belongs to a decagon

Now from the points E , F , G , H , A let the straight lines EQ , FR , GS , HT ,
 AL be set up at right angles to the plane of the circle and let them be equal
 to the radius of the circle $EFGHA$

let QR , RS , ST , TU , UQ , QL , LR , RM , MS , SN , NT , TO , OU , UP , PQ be
 joined

Now, since each of the straight lines EQ KU is at right angles to the same plane

therefore EQ is parallel to KU [XI 6]

But it is also equal to it,
and the straight lines joining those extremities of equal and parallel straight lines which are in the same direction are equal and parallel [I 33]

Therefore QU is equal and parallel to EK

But FK belongs to an equilateral pentagon,
therefore QU also belongs to the equilateral pentagon inscribed in the circle $EFGHK$

For the same reason
each of the straight lines QR , RS ST TU also belongs to the equilateral pentagon inscribed in the circle $EFGHK$,

therefore the pentagon $QRSTU$ is equilateral

And since QE belongs to a hexagon,

and EP to a decagon

and the angle QEP is right,

therefore QP belongs to a pentagon,

for the square on the side of the pentagon is equal to the square on the side of the hexagon and the square on the side of the decagon inscribed in the same circle [XIII 10]

For the same reason

PU is also a side of a pentagon

But QU also belongs to a pentagon,

therefore the triangle QPU is equilateral

For the same reason

each of the triangles QLR , RMV SNT , TOU is also equilateral

And since each of the straight lines QL QP was proved to belong to a pentagon

and LP also belongs to a pentagon,

therefore the triangle QLP is equilateral

For the same reason

each of the triangles LRM MSN NTO OUP is also equilateral

Let the centre of the circle $EFGHK$ the point V , be taken

from V let VZ be set up at right angles to the plane of the circle

let it be produced in the other direction as VX

let there be cut off VW the side of a hexagon, and each of the straight lines VX WZ being sides of a decagon

and let QZ QW UZ EV LV LX AM be joined

Now since each of the straight lines VW QV is at right angles to the plane of the circle

therefore VW is parallel to QE [XI 6]

But they are also equal

therefore EV QW are also equal and parallel [I 33]

But EV belongs to a hexagon

therefore QW also belongs to a hexagon

And, since QW belongs to a hexagon

and WZ to a decagon

and the angle QWZ is right

therefore QZ belongs to a pentagon

[xiii 10]

For the same reason

UZ also belongs to a pentagon,
inasmuch as, if we join VK , WU , they will be equal and opposite, and VK ,
being \equiv radius, belongs to a hexagon, [iv 15 Por]

therefore WU also belongs to a hexagon

But WZ belongs to a decagon

and the angle UVZ is right,

therefore UZ belongs to a pentagon

[xiii 10]

But QU also belongs to a pentagon,

therefore the triangle QUZ is equilateral

For the same reason

each of the remaining triangles of which the straight lines QR , RS , ST , TU are
the bases and the point Z the vertex is also equilateral

Again since VL belongs to a hexagon,

and $V\lambda$ to a decagon,

and the angle LVX is right,

therefore $L\lambda$ belongs to a pentagon

[xiii 10]

For the same reason

if we join MV , which belongs to a hexagon,

MY is also inferred to belong to a pentagon

But LM also belongs to a pentagon,

therefore the triangle LMY is equilateral

Similarly it can be proved that each of the remaining triangles of which MN ,
 NO , OP , PL are the bases and the point Y the vertex is also equilateral

Therefore an icosahedron has been constructed which \equiv contained by twenty
equilateral triangles

It is next required to comprehend it in the given sphere and to prove that
the side of the icosahedron is the irrational straight line called minor

For since VW belongs to a hexagon

and WZ to a decagon

therefore VZ has been cut in extreme and mean ratio at W ,

and VW is its greater segment,

[xiii 9]

therefore as ZV is to VW so is VW to WZ

But VW is equal to VE and WZ to VX

therefore as ZV is to VF so is EV to VX

And the angles ZVE , EVX are right,

therefore if we join the straight line EZ , the angle λEZ will be right because
of the similarity of the triangles XEZ , VEZ

For the same reason

since as ZV is to VW so is VW to WZ ,

and ZV is equal to VW and VW to WZ

therefore as VW is to WZ so is WZ to WZ

And for this reason again

if we join $Q\lambda$ the angle at Q will be right,

[vi 8]

therefore the semicircle described on λE will also pass through Q [iii 31]

And if λZ remaining fixed the semicircle be carried round and restored to
the same position from which it began to be moved, it will also pass through Q
and the remaining angular points of the icosahedron

and the icosahedron will have been comprehended in a sphere

I say next that it is also comprehended in the given sphere

For let VW be bisected at A'

Then, since the straight line VZ has been cut in extreme and mean ratio at W

and ZW is its lesser segment

therefore the square on ZW added to the half of the greater segment that is WA' , is five times the square on the half of the greater segment, [XIII 3]

therefore the square on ZA' is five times the square on $A'W$

And ZV is double of ZA' and VW double of $A'W$,

therefore the square on ZV is five times the square on VW

And, since AC is quadruple of CB ,

therefore AB is five times BC

But, as AB is to BC , so is the square on AB to the square on BD , [VI 8 v Def 9]

therefore the square on AB is five times the square on BD

But the square on ZV was also proved to be five times the square on VW

And DB is equal to VW ,

for each of them is equal to the radius of the circle $EFGHA$,

therefore AB is also equal to AZ

And AB is the diameter of the given sphere,

therefore AZ is also equal to the diameter of the given sphere

Therefore the icosahedron has been comprehended in the given sphere

I say next that the side of the icosahedron is the irrational straight line called minor

For, since the diameter of the sphere is rational

and the square on it is five times the square on the radius of the circle $EFGHA$,

therefore the radius of the circle $EFGHK$ is also rational

hence its diameter is also rational

But if an equilateral pentagon be inscribed in a circle which has its diameter rational the side of the pentagon is the irrational straight line called minor [XIII 11]

And the side of the pentagon $EFGHK$ is the side of the icosahedron

Therefore the side of the icosahedron is the irrational straight line called minor

PORISM From this it is manifest that the square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron has been described and that the diameter of the sphere is composed of the side of the hexagon and two of the sides of the decagon inscribed in the same circle Q E D

PROPOSITION 17

To construct a dodecahedron and comprehend it in a sphere, like the aforesaid figures and to prove that the side of the dodecahedron is the irrational straight line called apotome

Let $ABCD$ $CBEF$, two planes of the aforesaid cube at right angles to one another, be set out

let the sides AB BC , CD DA , EF , EB FC be bisected at G H , K , L M N , O respectively

let GA, HL, MH, VO be joined

let the straight lines NP, PO, HQ be cut in extreme and mean ratio at the points R, S, T respectively,

and let RP, PS, TQ be their greater segments,

from the points R, S, T let RU, SV, TW be set up at right angles to the planes of the cube towards the outside of the cube,

let them be made equal to RP, PS, TQ ,

and let UB, BV, VC, CV, VU be joined

I say that the pentagon $UBWCV$ is equilateral, and in one plane, and is further equiangular

For let RB, SB, VB be joined

Then since the straight line VP has been cut in extreme and mean ratio at R

and RP is the greater segment, therefore the squares on PN, NR are triple of the square on RP [xiii. 4]

But PN is equal to NB , and PR to RU ,

therefore the squares on BN, NR are triple of the square on RU

But the square on BR is equal to the squares on BN, NR , [i. 47]

therefore the square on BR is triple of the square on RU ,

hence the squares on BR, RU are quadruple of the square on RU

But the square on BU is equal to the squares on BR, RU ,

therefore the square on BU is quadruple of the square on RU ,

therefore BU is double of RU

But VU is also double of VR

inasmuch as SR is also double of PR , that is, of RU ,

therefore BU is equal to UV

Similarly it can be proved that each of the straight lines BW, WC, CV is also equal to each of the straight lines BU, UV

Therefore the pentagon $BUVCW$ is equilateral

I say next that it is also in one plane

For let $P\Lambda$ be drawn from P parallel to each of the straight lines RU, SV and towards the outside of the cube and let $\Lambda H, HW$ be joined,

I say that ΛHW is a straight line

For since HQ has been cut in extreme and mean ratio at T and QT is its greater segment

therefore as HQ is to QT , so is QT to TH

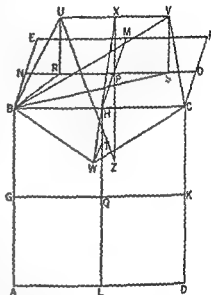
But HQ is equal to HP and QT to each of the straight lines $TH, P\Lambda$,

therefore as HP is to $P\Lambda$ so is HT to TH

And HP is parallel to TH

for each of them is at right angles to the plane BD ,

and TH is parallel to $P\Lambda$, [xi. 6]



for each of them is at right angles to the plane BF [id]

But if two triangles as $\triangle PH$, $\triangle TW$, which have two sides proportional to two sides be placed together at one angle so that their corresponding sides are also parallel

the remaining straight lines will be in a straight line, [vi 32]

therefore $\triangle H$ is in a straight line with HW

But every straight line is in one plane, [xi 1]

therefore the pentagon $UBWCV$ is in one plane

I say next that it is also equiangular

For since the straight line AP has been cut in extreme and mean ratio at R and PR is the greater segment

while PR is equal to PS

therefore NS has also been cut in extreme and mean ratio at P

and NP is the greater segment [xiii 5]

therefore the squares on NS SP are triple of the square on AP [xiii 4]

But NP is equal to NB and PS to SV ,

therefore the squares on NS , SV are triple of the square on NB ,
hence the squares on VS SN NB are quadruple of the square on NB

But the square on SB is equal to the squares on SN , NB ,
therefore the squares on BS SV , that is the square on BV —for the angle $\angle SBV$ is right—is quadruple of the square on NB ,

therefore BV is double of BN

But BC is also double of BN ,

therefore BV is equal to BC

And since the two sides BU , UV are equal to the two sides BW WC ,

and the base BV is equal to the base BC ,

therefore the angle BUV is equal to the angle BWC [i 8]

Similarly we can prove that the angle UVC is also equal to the angle BWC ,
therefore the three angles BWC , BUV , UVC are equal to one another

But if in an equilateral pentagon three angles are equal to one another the pentagon will be equiangular [xiii 7]

therefore the pentagon $BUVCH$ is equiangular

And it was also proved equilateral

therefore the pentagon $BUVCH$ is equilateral and equiangular and it is on one side BC of the cube

Therefore if we make the same construction in the case of each of the twelve sides of the cube

as solid figure will have been constructed which is contained by twelve equilateral and equiangular pentagons and which is called a dodecahedron

It is then required to comprehend it in the given sphere and to prove that the side of the dodecahedron is the irrational straight line called apotome

For let AP be produced and let the produced straight line be AZ
therefore PZ meets the diameter of the cube and they bisect one another
for this has been proved in the last theorem but one of the eleventh book

[xi 38]

Let them cut at Z

therefore Z is the centre of the sphere which comprehends the cube,

and ZP is half of the side of the cube

Let UZ be joined

Now, since the straight line NS has been cut in extreme and mean ratio at P ,

and NP = its greater segment

therefore the squares on NS , SP are triple of the square on NP [xiii 4]

But NS is equal to λZ

inasmuch as NP is also equal to PZ , and XP to PS

But further, PS is also equal to λU ,

since it is also equal to RP ,

therefore the squares on $Z\lambda$ XU are triple of the square on NP

But the square on UZ is equal to the squares on $Z\lambda$, λU ,

therefore the square on UZ is triple of the square on NP

But the square on the radius of the sphere which comprehends the cube is also triple of the square on the half of the side of the cube for it has previously been shown how to construct a cube and comprehend it in a sphere and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube [xiii 15]

But if whole = so related to whole so is half to half also,

and NP is half of the side of the cube,

therefore UZ is equal to the radius of the sphere which comprehends the cube

And Z is the centre of the sphere which comprehends the cube,

therefore the point U is on the surface of the sphere

Similarly we can prove that each of the remaining angles of the dodecahedron is also on the surface of the sphere,

therefore the dodecahedron has been comprehended in the given sphere

I say next that the side of the dodecahedron is the irrational straight line called apotome

For since when NP has been cut in extreme and mean ratio RP is the greater segment

and when PO has been cut in extreme and mean ratio PS is the greater segment

therefore when the whole NO is cut in extreme and mean ratio RS is the greater segment

[Thus since as NP is to PR so is PR to RN

the same is true of the doubles also

for parts have the same ratio as their equimultiples, [v 15]

therefore as NO is to RS so is RS to the sum of NR SO

But NO is greater than RS

therefore RS is also greater than the sum of NR SO ,

therefore NO has been cut in extreme and mean ratio

and RS is its greater segment]

But RS is equal to UV

therefore when NO is cut in extreme and mean ratio, UV is the greater segment

And since the diameter of the sphere = rational

and the square on it is triple of the square on the side of the cube

therefore NO being a side of the cube is rational

[But if a rational line be cut in extreme and mean ratio each of the segments is an irrational apotome]

Therefore UV being a side of the dodecahedron is an irrational apotome [xiii 6]

PORISM From this it is manifest that when the side of the cube is cut in extreme and mean ratio, the greater segment is the side of the dodecahedron

Q E D

PROPOSITION 18

To set out the sides of the five figures and to compare them with one another

Let AB , the diameter of the given sphere, be set out,
and let it be cut at C so that AC is equal to CB and at D so that AD is double of DB ,
let the semicircle AEB be described on AB ,
from C, D let CE, DF be drawn at right angles to AB

and let AF, FB, EB be joined

Then since AD is double of DB

therefore AB is triple of BD

Converting, therefore BA is one and a half times AD

But as BA is to AD so is the square on BA to the square on AD , [v Def 9 vi 8]
for the triangle AFB is equiangular with the triangle AFD ,

therefore the square on BA is one and a half times the square on AD

But the square on the diameter of the sphere is also one and a half times the square on the side of the pyramid [VIII 13]

And AB is the diameter of the sphere,

therefore AF is equal to the side of the pyramid

Again since AD is double of DB

therefore AB is triple of BD

But as AB is to BD so is the square on AB to the square on BD

[vi 8 v Def 9]

therefore the square on AB is triple of the square on BD

But the square on the diameter of the sphere is also triple of the square on the side of the cube [VIII 15]

And AB is the diameter of the sphere,

therefore BF is the side of the cube

And since AC is equal to CB ,

therefore AB is double of BC

But as AB is to BC so is the square on AB to the square on BC

therefore the square on AB is double of the square on BC

But the square on the diameter of the sphere is also double of the square on the side of the octahedron [VIII 14]

And AB is the diameter of the given sphere

therefore BE is the side of the octahedron

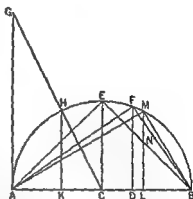
Next let AG be drawn from the point A at right angles to the straight line AB ,

let AG be made equal to AB

let GC be joined

and from H let HK be drawn perpendicular to AB

Then, since GA is double of AC ,



for GA is equal to AB ,
 and, as GA is to AC , so is HA to AC ,
 therefore HA is also double of AC

Therefore the square on HA is quadruple of the square on AC
 therefore the squares on HA , AC , that is the square on HC , is five times the square on AC

But HC is equal to CB ,
 therefore the square on BC is five times the square on CA
 And, since AB is double of CB ,
 and in them AD is double of DB ,

therefore the remainder BD is double of the remainder DC
 Therefore BC is triple of CD ,

therefore the square on BC is nine times the square on CD
 But the square on BC is five times the square on CA ,
 therefore the square on CA is greater than the square on CD ,
 therefore CA is greater than CD

Let CL be made equal to CA
 from L let LM be drawn at right angles to AB ,
 and let MB be joined

Now since the square on BC is five times the square on CA ,
 and AB is double of BC , and AL double of CA ,
 therefore the square on AB is five times the square on AL

But the square on the diameter of the sphere is also five times the square on the radius of the circle from which the icosahedron has been described
 [XIII 16 Por]

And AB is the diameter of the sphere
 therefore AL is the radius of the circle from which the icosahedron has been described

therefore AL is a side of the hexagon in the said circle [IV 15 Por]

And since the diameter of the sphere is made up of the side of the hexagon and two of the sides of the decagon inscribed in the same circle
 [XIII 16 Por]

and AB is the diameter of the sphere
 while AL is a side of the hexagon
 and AA is equal to LB

therefore each of the straight lines AA LB is a side of the decagon inscribed in the circle from which the icosahedron has been described

And since LB belongs to a decagon and MI to a hexagon
 for MI is equal to AI since it is also equal to HA being the same distance from the centre and each of the straight lines HA AL is double of AC

therefore MB belongs to a pentagon [XIII 10]

But the side of the pentagon is the side of the icosahedron, [XIII 16]

therefore MB belongs to the icosahedron

Now, since FB is a side of the cube

let it be cut in extreme and mean ratio at N ,

and let NB be the greater segment

therefore NB is a side of the dodecahedron [XIII 17 Por]

And since the square on the diameter of the sphere was proved to be one and a half times the square on the side AF of the pyramid double of the square on

the side BC of the octahedron and triple of the side FB of the cube therefore of parts of which the square on the diameter of the sphere contains six the square on the side of the pyramid contains four, the square on the side of the octahedron three and the square on the side of the cube two

Therefore the square on the side of the pyramid is four thirds of the square on the side of the octahedron and double of the square on the side of the cube and the square on the side of the octahedron is one and a half times the square on the side of the cube

The said sides therefore of the three figures I mean the pyramid, the octahedron and the cube, are to one another in rational ratios

But the remaining two, I mean the side of the icosahedron and the side of the dodecahedron are not in rational ratios either to one another or to the aforesaid sides

for they are irrational, the one being minor [XIII 16] and the other an apotome [XIII 17]

That the side MB of the icosahedron is greater than the side AB of the dodecahedron we can prove thus

For, since the triangle FDB is equiangular with the triangle FAB [VI 8] proportionally, as DB is to BF , so is BI to BA [VI 4]

And since the three straight lines are proportional, as the first is to the third so is the square on the first to the square on the second [V Def 9 VI 20, Por]

therefore as DB is to BA so is the square on DB to the square on BF , therefore inversely as AB is to BD , so is the square on FB to the square on BD

But AB is triple of BD

therefore the square on FB is triple of the square on BD

But the square on AD is also quadruple of the square on DB

for AD is double of DB ,

therefore the square on AD is greater than the square on FB ,

therefore AD is greater than FB ,

therefore AL is by far greater than FB

And when AL is cut in extreme and mean ratio

AL is the greater segment

inasmuch as LK belongs to a hexagon and LA to a decagon [XIII 9]

and when FB is cut in extreme and mean ratio NB is the greater segment

therefore AL is greater than NB

But AL is equal to LM ,

therefore LM is greater than NB

Therefore MB which is a side of the icosahedron is by far greater than AB which is a side of the dodecahedron

Q E D

I say next that no other figure besides the said five figures can be constructed which is contained by equilateral and equiangular figures equal to one another

For a solid angle cannot be constructed with two triangles or indeed planes

With three triangles the angle of the pyramid is constructed, with four the angle of the octahedron and with five the angle of the icosahedron

but a solid angle cannot be formed by six equilateral and equiangular triangles placed together at one point,
 for the angle of the equilateral triangle being two-thirds of a right angle, the six will be equal to four right angles
 which is impossible, for any solid angle is contained by angles less than four right angles [xi 21]

For the same reason, neither can a solid angle be constructed by more than six plane angles

By three squares the angle of the cube is contained but by four it is impossible for a solid angle to be contained,

for they will again be four right angles

By three equilateral and equiangular pentagons the angle of the dodecahedron is contained,

but by four such it is impossible for any solid angle to be contained for the angle of the equilateral pentagon being a right angle and a fifth the four angles will be greater than four right angles

which is impossible

Neither again will a solid angle be contained by other polygonal figures by reason of the same absurdity

Therefore etc

Q E D

LEMMA

But that the angle of the equilateral and equiangular pentagon is a right angle and a fifth we must prove thus

Let $ABCDF$ be an equilateral and equiangular pentagon,
 let the circle $ABCDE$ be circumscribed
 about it,

let its centre F be taken

and let FA, FB, FC, FD, FE be joined

Therefore they bisect the angles of the pentagon at A, B, C, D, E

And since the angles at F are equal to four right angles and are equal

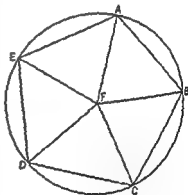
therefore one of them as the angle AFB is one right angle less a fifth,

therefore the remaining angles FAB, ABF consist of one right angle and a fifth

But the angle FAB is equal to the angle FBC ,

therefore the whole angle ABC of the pentagon consists of one right angle and a fifth

Q E D



**THE WORKS OF ARCHIMEDES
INCLUDING THE METHOD**

BIOGRAPHICAL NOTE

ARCHIMEDES, ■ 287-212 B C

ARCHIMEDES was ■ citizen of Syracuse in Sicily, where he was born a-round the year 287 B C. He was intimate with Hiero King of Syracuse and with his son, Gelo and Plutarch says that he was related to them. In his *Sand Reckoner*, which was dedicated to Gelo Archimedes speaks of his father Phidias as an astronomer who investigated the sizes and distances of the sun and moon.

As a young man Archimedes seems to have spent some time in Egypt where he invented the water-screw as a means of drawing water out of the Nile for irrigating the fields though it is also said that he invented this machine to drain bilge water from a huge ship built for King Hiero. He may have studied with the pupils of Euclid in Alexandria. It was probably there that he made the friendship of Conon of Samos and Eratosthenes. To Conon he was in the habit of communicating his discoveries before their publication and it was for Eratosthenes that he wrote the *Method* and through him that he addressed the famous *Cattle Problem* to the mathematicians of Alexandria—if the tradition is to be credited that associates Archimedes with this problem. After the death of Conon Archimedes sent his discoveries to Conon's friend and pupil, Dositheus of Pelusium to whom four of the extant treatises are dedicated.

His mechanical inventions won great fame for Archimedes and figure largely in the traditions about him. After discovering the solution of the problem *To move a given weight by a given force* he boasted to King Hiero: 'Give me ■ place to stand on and I can move the earth.' Asked for a practical demonstration he contrived a machine by which with the use of only one arm he drew out of the dock a large ship laden with passengers and goods which the combined strength of the Syracusans could scarcely move. From that day Hiero ordered that

Archimedes was to be believed in everything he might say. At the King's request Archimedes then made for him catapults, battering rams, cranes and many other engines of war which were later used with such success in the defense of Syracuse against the Romans that they were unable to take the city except by treachery. There ■ also a story in Lucian that Archimedes set fire to the Roman ships by an arrangement of burning glasses.

Although Archimedes acquired by his mechanical inventions 'the renown of more than human sagacity' according to Plutarch he would not deign to leave behind him any commentary or writing on such subjects since he considered them "sordid and ignoble." He did however write a description now lost of an apparatus composed of concentric glass spheres moved by water power representing the Eudoxian system of the world. This astronomical machine which survived to be seen and described by Cicero in his *Republic* was sufficiently accurate to show the eclipses of the sun and the moon. Except for this lost work *On Sphere making* Archimedes wrote only on strictly mathematical subjects. He took all the mathematical sciences for his province: arithmetic, geometry, astronomy, mechanics and hydrostatics. Unlike Euclid and Apol-

lonius he wrote no textbooks. Of his writings, although some have been lost the most important have survived.

The absorption of Archimedes in his mathematical investigations was so great that he forgot his food and neglected his person, and when carried by force to the bath Plutarch records 'he used to trace geometrical figures in the ashes of the fire and diagrams in the oil on his body'. Asked by Hiero to discover whether a goldsmith had alloyed with silver the gold of his crown Archimedes found the answer while bathing by considering the water displaced by his body whereupon he is reported to have run home in his excitement without his clothes, shouting, Eureka (I have found it).

Archimedes' preoccupation with mathematics is even said to have been the cause of his death. In the general massacre which followed the capture of Syracuse by Marcellus in 212 B.C., Archimedes was so intent upon a mathematical diagram that he took no notice, and when ordered by a soldier to attend the victorious general he refused until he should have solved his problem whereupon he was slain by the enraged soldier. No blame attaches to the Roman general, Marcellus, since he had given orders to spare the house and person of the mathematician and in the midst of his triumph he lamented the death of Archimedes, provided him with an honorable burial and befriended his surviving relatives. In accordance with the expressed desire of Archimedes, his family and friends inscribed on his tomb the figure of his favorite theorem on the sphere and the circumscribed cylinder and the ratio of the containing solid to the contained. When Cicero was in Sicily as quaestor in 75 B.C., he discovered the neglected and forgotten tomb of Archimedes near the Agrigentine Gate and piously restored it.

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ON THE SPHERE AND CYLINDER

BOOK ONE

ARCHIMEDES to DOSITHEUS greeting

"On a former occasion I sent you the investigations which I had up to that time completed including the proofs showing that any segment bounded by a straight line and a section of a right angled cone [a parabola] is four thirds of the triangle which has the same base with the segment and equal height. Since then certain theorems not hitherto demonstrated have occurred to me, and I have worked out the proofs of them. They are these: first that the surface of any sphere is four times its greatest circle, next that the surface of any segment of a sphere is equal to a circle whose radius is equal to the straight line drawn from the vertex of the segment to the circumference of the circle which is the base of the segment, and, further that any cylinder having its base equal to the greatest circle of those in the sphere, and height equal to the diameter of the sphere is itself [i.e. in content] half as large again as the sphere and its surface also [including its bases] is half as large again as the surface of the sphere. Now these properties were all along naturally inherent in the figures referred to but remained unknown to those who were before my time engaged in the study of geometry. Having however, now discovered that the properties are true of these figures I cannot feel any hesitation in setting them side by side both with my former investigations and with those of the theorems of Eudoxus on solids which are held to be most irrefragably established namely, that any pyramid is one third part of the prism which has the same base with the pyramid and equal height, and that any cone is one third part of the cylinder which has the same base with the cone and equal height. For though these properties also were naturally inherent in the figures all along yet they were in fact unknown to all the many able geometers who lived before Eudoxus and had not been observed by any one. Now however it will be open to those who possess the requisite ability to examine these discoveries of mine. They ought to have been published while Conon was still alive for I should conceive that he would best have been able to grasp them and to pronounce upon them the appropriate verdict but as I judge it well to communicate them to those who are conversant with mathematics I send them to you with the proofs written out which it will be open to mathematicians to examine. Farewell.

'I first set out the axioms and the assumptions which I have used for the proofs of my propositions

DEFINITIONS

1 "There are in a plane certain terminated bent lines which either lie wholly on the same side of the straight lines joining their extremities, or have no part of them on the other side

2 "I apply the term *concave in the same direction* to a line such that, if any two points on it are taken either all the straight lines connecting the points fall on the same side of the line, or some fall on one and the same side while others fall on the line itself but none on the other side "

3 "Similarly also there are certain terminated surfaces not themselves being in a plane but having their extremities in a plane and such that they will either be wholly on the same side of the plane containing their extremities or have no part of them on the other side

4 "I apply the term *concave in the same direction* to surfaces such that, if any two points on them are taken the straight lines connecting the points either all fall on the same side of the surface, or some fall on one and the same side of it while some fall upon it but none on the other side

5 "I use the term *solid sector* when a cone cuts a sphere, and has its apex at the centre of the sphere to denote the figure comprehended by the surface of the cone and the surface of the sphere included within the cone "

6 "I apply the term *solid rhombus* when two cones with the same base have their apices on opposite sides of the plane of the base in such a position that their axes lie in a straight line to denote the solid figure made up of both the cones "

ASSUMPTIONS

1 *Of all lines which have the same extremities the straight line is the least "*

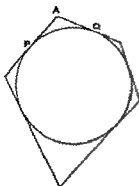
2 "Of other lines in a plane and having the same extremities, [any two] such are unequal whenever both are concave in the same direction and one of them is either wholly included between the other and the straight line which has the same extremities with it or is partly included by, and is partly common with, the other and that [line] which is included is the lesser [of the two]

3 *Similarly of surfaces which have the same extremities if those extremities are in a plane the plane is the least [in area]*

4 "Of other surfaces with the same extremities the extremities being in a plane [any two] such are unequal whenever both are concave in the same direction and one surface is either wholly included between the other and the plane which has the same extremities with it or is partly included by and partly common with the other and that [surface] which is included is the lesser [of the two in area]

5 Further of unequal lines unequal surfaces and unequal solids the greater exceed the less by such a magnitude as when added to itself can be made to exceed any assigned magnitude among those which are comparable with it and with one another

"These things being premised if a polygon be inscribed in a circle it is plain that the perimeter of the inscribed polygon is less than the circumference of the circle for each of the sides of the polygon is less than that part of the circumference of the circle which is cut off by it



PROPOSITION 1

If a polygon be circumscribed about a circle the perimeter of the circumscribed polygon is greater than the perimeter of the circle

Let any two adjacent sides meeting in A touch the circle at P, Q respectively

Then [Assumptions 2]

$$PA + AQ > (\text{arc } PQ)$$

A similar inequality holds for each angle of the polygon and by addition, the required result follows

PROPOSITION 2

Given two unequal magnitudes it is possible to find two unequal straight lines such that the greater straight line has to the less a ratio less than the greater magnitude has to the less

Let AB, D represent the two unequal magnitudes AB being the greater

Suppose BC measured along BA equal to D and let GH be any straight line

Then if CA be added to itself a sufficient number of times the sum will exceed D . Let AF be this sum and take E on GH produced such that GH is the same multiple of HE that AF is of AC

$$\text{Thus } EH : HG = AC : AF$$

But since $AF > D$ (or CB)

$$AC : AF < AC : CB$$

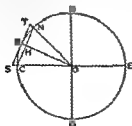
Therefore componendo

$$EG : GH < AB : D$$

Hence EG, GH are two lines satisfying the given condition

PROPOSITION 3

Given two unequal magnitudes and a circle it is possible to inscribe a polygon in the circle and to describe another about it so that the side of the circumscribed polygon may have to the side of the inscribed polygon a ratio less than that of the greater magnitude to the less



Let A, B represent the given magnitudes A being the greater

Find [Prop 2] two straight lines F, AL of which F is the greater such that

$$F : AL < A : B$$

(1)

Draw LM perpendicular to LA and of such length that $KM = F$

In the given circle let CE DG be two diameters at right angles. Then, bisecting the angle DOC , bisecting the half again and so on we shall arrive ultimately at an angle (as NOC) less than twice the angle LKM

Join NC which (by the construction) will be the side of a regular polygon inscribed in the circle. Let OP be the radius of the circle bisecting the angle NOC (and therefore bisecting NC at right angles in H , say), and let the tangent at P meet OC , ON produced in S T respectively

Now since $\angle COV < 2\angle LKM$,
 $\angle HOC < \angle LKM$,

and the angles at H , L are right

therefore $MK/LA > OC/OH$
 $> OP/OH$

Hence $ST/CN < MK/LK$
 $< F/LA$,

therefore *a fortiori*, by (1)

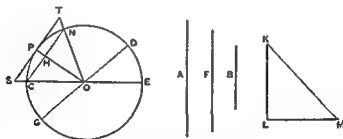
$ST/CN < A/B$

Thus two polygons are found satisfying the given condition

PROPOSITION 4

Again, given two unequal magnitudes and a sector it is possible to describe a polygon about the sector and to inscribe another in it so that the side of the circumscribed polygon may have to the side of the inscribed polygon a ratio less than the greater magnitude has to the less

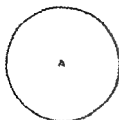
[The "inscribed polygon" found in this proposition is one which has for two sides the two radii bounding the sector, while the remaining sides (the number of which is by construction some power of 2) subtend equal parts of the arc of the sector. The circumscribed polygon is formed by the tangents parallel to the sides of the inscribed polygon and by the two bounding radii produced.]



In this case we make the same construction as in the last proposition except that we bisect the angle COD of the sector, instead of the right angle between two diameters. Then bisect the half again, and so on. The proof is exactly similar to the preceding one.

PROPOSITION 5

Given a circle and two unequal magnitudes, to describe a polygon about the circle and inscribe another in it so that the circumscribed polygon may have to the inscribed a ratio less than the greater magnitude has to the less



B

C

D

E

F

Let A be the given circle and B C the given magnitudes B being the greater

Take two unequal straight lines D E of which D is the greater, such that $D : E < B : C$ [Prop 2] and let F be a mean proportional between D E so that D is also greater than F

Describe (in the manner of Prop 3) one polygon about the circle and inscribe another in it, so that the side of the former has to the side

of the latter a ratio less than the ratio $D : F$

Thus the duplicate ratio of the side of the former polygon to the side of the latter is h^2 \equiv than the ratio $D^2 : F^2$

But the said duplicate ratio of the sides is equal to the ratio of the areas of the polygons since they are similar,

therefore the area of the circumscribed polygon has to the area of the inscribed polygon a ratio less than the ratio $D^2 : F^2$ or $D : E$, and a fortiori less than the ratio $B : C$

PROPOSITION 6

Similarly we can show that given two unequal magnitudes and a sector it is possible to circumscribe a polygon about the sector and inscribe in it another similar one so that the circumscribed may have to the inscribed a ratio less than the greater magnitude has to the less

And it is likewise clear that if a circle or a sector as well as a certain area be given it is possible by inscribing regular polygons in the circle or sector and by continually inscribing such in the remaining segments to leave segments of the circle or sector which are [together] less than the given area For this \equiv proved in the *Elements* [Eucl xiv 2]

But it \equiv yet to be proved that given a circle or sector and an area it is possible to describe a polygon about the circle or sector such that the area remaining between the circumference and the circumscribed figure is less than the given area

The proof for the circle (which as Archimedes says can be equally applied to a sector) is as follows

Let A be the given circle and B the given area

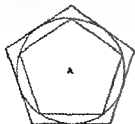
Now there being two unequal magnitudes $A + B$ and A let \equiv polygon (C) be circumscribed about the circle and a polygon (I) inscribed in it [as in Prop 5]

so that

$$C : I < A + B : A$$

The circumscribed polygon (C) shall be that required

(1)



For the circle (A) is greater than the inscribed polygon (I)

Therefore, from (1), *a fortiori*

$$\begin{aligned} C & A < A+B & A, \\ & C < A+B, \\ \text{or} & C-A < B \end{aligned}$$

whence
or

PROPOSITION 7

If in an isosceles cone [i.e. a right circular cone] a pyramid be inscribed having an equilateral base, the surface of the pyramid excluding the base is equal to a triangle having its base equal to the perimeter of the base of the pyramid and its height equal to the perpendicular drawn from the apex on one side of the base

Since the sides of the base of the pyramid are equal it follows that the perpendiculars from the apex to all the sides of the base are equal, and the proof of the proposition is obvious

PROPOSITION 8

If a pyramid be circumscribed about an isosceles cone, the surface of the pyramid excluding its base is equal to a triangle having its base equal to the perimeter of the base of the pyramid and its height equal to the side [i.e. a generator] of the cone

The base of the pyramid is a polygon circumscribed about the circular base of the cone and the line joining the apex of the cone or pyramid to the point of contact of any side of the polygon is perpendicular to that side. Also all these perpendiculars being generators of the cone, are equal whence the proposition follows immediately

PROPOSITION 9

If in the circular base of an isosceles cone a chord be placed and from its extremities straight lines be drawn to the apex of the cone the triangle so formed will be less than the portion of the surface of the cone intercepted between the lines drawn to the apex

Let ABC be the circular base of the cone, and O its apex

Draw a chord AB in the circle and join O to OB . Bisect the arc ACB in C and join AC , BC , OC

Then $\triangle OAC + \triangle OBC > \triangle OAB$

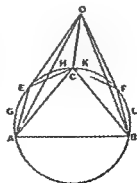
Let the excess of the sum of the first two triangles over the third be equal to the area D

Then D is either less than the sum of the segments AEC , CFB or not less

I Let D be not less than the sum of the segments referred to

We have now two surfaces
(1) that consisting of the portion OAC of the surface of the cone together with the segment AEC and

(2) the triangle OAC



and, since the two surfaces have the same extremities (the perimeter of the triangle OAC), the former surface is greater than the latter, which is included by it [Assumptions III or 4]

Hence (surface $OAEC$) + (segment AEC) > $\triangle OAC$

Similarly (surface $OCFB$) + (segment CFB) > $\triangle OBC$

Therefore, since D is not less than the sum of the segments we have, by addition,

$$\begin{aligned} (\text{surface } OAECFB) + D &> \triangle OAC + \triangle OBC \\ &> \triangle OAB + D, \text{ by hypothesis} \end{aligned}$$

Taking away the common part D , we have the required result

II Let D be less than the sum of the segments AEC CFB

If now we bisect the arcs AC , CB , then bisect the halves and so on we shall ultimately leave segments which are together less than D [Prop 6]

Let AGE , EHC , CKF , FLB be those segments, and join OE OF

Then as before,

$$(\text{surface } OAGE) + (\text{segment } AGE) > \triangle OAE$$

$$\text{and } (\text{surface } OEHC) + (\text{segment } EHC) > \triangle OEC$$

$$\begin{aligned} \text{Therefore } (\text{surface } OAGHC) + (\text{segments } AGE, EHC) \\ &> \triangle OAE + \triangle OEC \\ &> \triangle OAC, \text{ a fortiori} \end{aligned}$$

Similarly for the part of the surface of the cone bounded by OC OB and the arc CFB

Hence by addition

$$\begin{aligned} (\text{surface } OAGEHCKFLB) + (\text{segments } AGE, EHC, CKF, FLB) \\ &> \triangle OAC + \triangle OBC \\ &> \triangle OAB + D, \text{ by hypothesis} \end{aligned}$$

But the sum of the segments is less than D and the required result follows

PROPOSITION 10

If in the plane of the circular base of an isosceles cone two tangents be drawn to the circle meeting in a point and the points of contact and the point of concurrence of the tangents be respectively joined to the apex of the cone the sum of the two triangles formed by the joining lines and the two tangents are together greater than the included portion of the surface of the cone

Let ABC be the circular base of the cone, O its apex AD BD the two tangents to the circle meeting in D Join OA OB , OD

Let ECF be drawn touching the circle at C , the middle point of the arc ACB , and therefore parallel to AB Join OE OF

$$\text{Then } ED + DF > EF,$$

and, adding $AE + FB$ to each side

$$AD + DB > AE + EF + FB$$

Now OA OC , OB being generators of the cone, are equal and they are respectively perpendicular to the tangents at A , C , B

It follows that

$$\triangle OAD + \triangle ODB > \triangle OAE + \triangle OEF + \triangle OFB$$

Let the area G be equal to the excess of the first sum over the second

G is then either less or not less than the sum of the spaces $EAHC$ $FCKB$ remaining between the circle and the tangents which sum we will call L

I Let G be not less than L

We have now two surfaces

(1) that of the pyramid with apex O and base $AEFB$, excluding the face OAB ,

(2) that consisting of the part $OACB$ of the surface of the cone together with the segment ACB

These two surfaces have the same extremities, viz the perimeter of the triangle OAB , and, since the former includes the latter, the former is the greater [Assumptions, 4]

That is the surface of the pyramid exclusive of the face OAB is greater than the sum of the surface $OACB$ and the segment ACB

Taking away the segment from each sum, we have

$$\triangle OAE + \triangle OEF + \triangle OFB + L > \text{the surface } OAHCKB$$

And G is not less than L

It follows that

$$\triangle OAE + \triangle OEF + \triangle OFB + G,$$

which is by hypothesis equal to $\triangle OAD + \triangle ODB$ is greater than the same surface

II Let G be less than L

If we bisect the arcs AC , CB and draw tangents at their middle points, then bisect the halves and draw tangents, and so on, we shall lastly arrive at a polygon such that the sum of the parts remaining between the sides of the polygon and the circumference of the segment is less than G

Let the remainders be those between the segment and the polygon $APQRSB$ and let their sum be M

Join OP OQ etc

Then as before

$$\triangle OAE + \triangle OEF + \triangle OFB > \triangle OAP + \triangle OPQ + \dots + \triangle OSB$$

Also as before

(surface of pyramid $OAPQRSB$ excluding the face OAB) $>$ the part $OACB$ of the surface of the cone together with the segment ACB

Taking away the segment from each sum

$$\triangle OAP + \triangle OPQ + \dots + M > \text{the part } OACB \text{ of the surface of the cone}$$

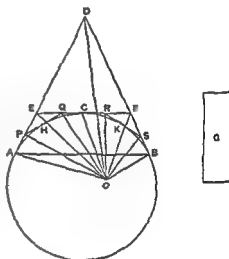
Hence is fortiori

$$\triangle OAP + \triangle OEF + \triangle OFB + G$$

which is by hypothesis equal to

$$\triangle OAD + \triangle ODB$$

is greater than the part $OACB$ of the surface of the cone



PROPOSITION 11

If a plane parallel to the axis of a right cylinder cut the cylinder the part of the surface of the cylinder cut off by the plane is greater than the area of the parallelogram in which the plane cuts it

PROPOSITION 12

If at the extremities of two generators of any right cylinder tangents be drawn to the circular bases in the planes of those bases respectively and if the pairs of tangents meet, the parallelograms formed by each generator and the two corresponding tangents respectively are together greater than the included portion of the surface of the cylinder between the two generators

[The proofs of these two propositions follow exactly the methods of Props 9 10 respectively, and it is therefore unnecessary to reproduce them]

"From the properties thus proved it is clear (1) that, if a pyramid be inscribed in an isosceles cone the surface of the pyramid excluding the base is less than the surface of the cone [excluding the base], and (2) that, if a pyramid be circumscribed about an isosceles cone, the surface of the pyramid excluding the base is greater than the surface of the cone excluding the base

It is also clear from what has been proved both (1) that if a prism be inscribed in a right cylinder the surface of the prism made up of its parallelograms [excluding its bases] is less than the surface of the cylinder excluding its bases and (2) that if a prism be circumscribed about a right cylinder the surface of the prism made up of its parallelograms is greater than the surface of the cylinder excluding its bases'

PROPOSITION 13

The surface of any right cylinder excluding the bases is equal to a circle whose radius is a mean proportional between the side [i.e. a generator] of the cylinder and the diameter of its base

Let the base of the cylinder be the circle A and make CD equal to the diameter of this circle and EF equal to the height of the cylinder

Let H be a mean proportional between CD EF , and B a circle with radius equal to H

Then the circle B shall be equal to the surface of the cylinder (excluding the bases), which we will call S

For, if not B must be either greater or less than S

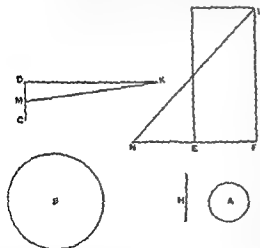
I Suppose $B < S$

Then it is possible to circumscribe a regular polygon about B and to inscribe another in it such that the ratio of the former to the latter is less than the ratio $S : B$

Suppose this done and circumscribe about A a polygon

similar to that described about B then erect on the polygon about A a prism of the same height as the cylinder The prism will therefore be circumscribed to the cylinder

Let AD perpendicular to CD and FL perpendicular to EF be each equal to the perimeter of the polygon about A Bisect CD in M , and join MA



Then $\triangle KDM$ = the polygon about A

Also $\square EL$ = surface of prism (excluding bases)

Produce FE to N so that $FE = EN$, and join NL

Now the polygons about A, B , being similar, are in the duplicate ratio of the radii of A, B

Thus

$$\begin{aligned}\triangle KDM \text{ (polygon about } B) &= MD \cdot H^2 \\ &= MD^2 \cdot CD \cdot EF \\ &= MD \cdot NF \\ &= \triangle KDM \cdot \triangle LFN \quad (\text{since } DK = FL)\end{aligned}$$

$$\begin{aligned}\text{Therefore (polygon about } B) &= \triangle LFN \\ &= \square EL \\ &= (\text{surface of prism about } A),\end{aligned}$$

from above

But (polygon about B) (polygon in B) $< S \cdot B$

Therefore (surface of prism about A) (polygon in B) $< S \cdot B$,

and alternately,

(surface of prism about A) $S < (\text{polygon in } B) \cdot B$,
which is impossible since the surface of the prism is greater than S , while the polygon inscribed in B is less than B

Therefore $B < S$

II Suppose $B > S$

Let a regular polygon be circumscribed about B and another inscribed in it so that

$$(\text{polygon about } B) \cdot (\text{polygon in } B) < B \cdot S$$

Inscribe in A a polygon similar to that inscribed in B and erect a prism on the polygon inscribed in A of the same height as the cylinder

Again let DK, FL drawn as before be each equal to the perimeter of the polygon inscribed in A

Then, in this case

$$\triangle KDM > (\text{polygon inscribed in } A)$$

(since the perpendicular from the centre on a side of the polygon is less than the radius of A)

Also $\triangle LFN = \square EL$ = surface of prism (excluding bases)

Now

$$\begin{aligned}(\text{polygon in } A) \cdot (\text{polygon in } B) &= MD^2 \cdot H^2, \\ &= \triangle KDM \cdot \triangle LFN, \text{ as before}\end{aligned}$$

And $\triangle KDM > (\text{polygon in } A)$

Therefore

$$\triangle LFN \text{ or (surface of prism)} > (\text{polygon in } B)$$

But this is impossible because

$$\begin{aligned}(\text{polygon about } B) \cdot (\text{polygon in } B) &< B \cdot S \\ &< (\text{polygon about } B) \cdot S \text{ a fortiori,}\end{aligned}$$

so that $(\text{polygon in } B) > S$

$$> (\text{surface of prism}) \text{ a fortiori}$$

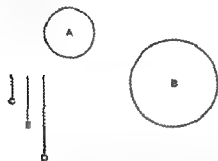
Hence B is neither greater nor less than S , and therefore

$$B = S$$

PROPOSITION 14

The surface of any isosceles cone excluding the base is equal to a circle whose radius is a mean proportional between the side of the cone [a generator] and the radius of the circle which is the base of the cone

Let the circle A be the base of the cone draw C equal to the radius of the circle, and D equal to the side of the cone and let E be a mean proportional between C, D



Draw a circle B with radius equal to E

Then shall B be equal to the surface of the cone (excluding the base) which we will call S

If not, B must be either greater or less than S

I Suppose $B < S$

Let a regular polygon be described about B and a similar one inscribed in it such that the former has to the latter a ratio less than the ratio $S : B$

Describe about A another similar polygon and on it set up a pyramid with apex the same as that of the cone

Then (polygon about A) (polygon about B)

$$= C^2 : E^2$$

$$= C : D$$

$$= (\text{polygon about } A) : (\text{surface of pyramid excluding base})$$

Therefore

$$(\text{surface of pyramid}) = (\text{polygon about } B)$$

$$\text{Now } (\text{polygon about } B) : (\text{polygon in } B) < S : B$$

Therefore

$$(\text{surface of pyramid}) : (\text{polygon in } B) < S : B$$

which is impossible (because the surface of the pyramid is greater than S , while the polygon in B is less than B)

$$\text{Hence } B < S$$

II Suppose $B > S$

Take regular polygons circumscribed and inscribed to B such that the ratio of the former to the latter is less than the ratio $B : S$

Inscribe in A a similar polygon to that inscribed in B and erect a pyramid on the polygon inscribed in A with apex the same as that of the cone

In this case

$$(\text{polygon in } A) : (\text{polygon in } B) = C^2 : E^2$$

$$= C : D$$

$$> (\text{polygon in } A) : (\text{surface of pyramid excluding base})$$

This is clear because the ratio of C to D is greater than the ratio of the perpendicular from the centre of A on a side of the polygon to the perpendicular from the apex of the cone on the same side

Therefore

$$(\text{surface of pyramid}) > (\text{polygon in } B)$$

But

$$(\text{polygon about } B) : (\text{polygon in } B) < B : S$$

Therefore, *a fortiori*,

(polygon about B) (surface of pyramid) $< B \ S$,

which is impossible

Since therefore B is neither greater nor less than S ,

$$B = S$$

PROPOSITION 15

The surface of any right circular cone has the same ratio to its base as the side of the cone has to the radius of the base

By Prop 14, the surface of the cone is equal to a circle whose radius is a mean proportional between the side of the cone and the radius of the base

Hence since circles are to one another as the squares of their radii, the proposition follows

PROPOSITION 16

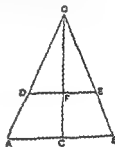
If an oblique circular cone be cut by a plane parallel to the base the portion of the surface of the cone between the parallel planes is equal to a circle whose radius is a mean proportional between (1) the portion of the side of the cone intercepted by the parallel planes and (2) the line which is equal to the sum of the radii of the circles in the parallel planes

Let OAB be a triangle through the axis of a cone, DE its intersection with the plane cutting off the frustum and OFC the axis of the cone

Then the surface of the cone OAB is equal to a circle whose radius is equal to $\sqrt{OA \cdot AC}$ [Prop 14]

Similarly the surface of the cone ODE is equal to a circle whose radius is equal to $\sqrt{OD \cdot DF}$

And the surface of the frustum is equal to the difference between the two circles



Now

$$OA \cdot AC - OD \cdot DF = DA \cdot AC + OD \cdot AC - OD \cdot DF$$

But

$$OD \cdot AC = OD \cdot DF$$

since

$$OA \cdot AC = OD \cdot DF$$

Hence

$$OA \cdot AC - OD \cdot DF = DA \cdot AC + DA \cdot DF \\ = DA \cdot (AC + DF)$$

And since circles are to one another as the squares of their radii it follows that the difference between the circles whose radii are $\sqrt{OA \cdot AC}$, $\sqrt{OD \cdot DF}$ respectively is equal to a circle whose radius is $\sqrt{DA \cdot (AC + DF)}$

Therefore the surface of the frustum is equal to this circle

LEMMAS

"1. Cones having equal height have the same ratio as their bases and those having equal bases have the same ratio as their heights"

"2. If a cylinder be cut by a plane parallel to the base then, as the cylinder is to the cylinder so is the axis to the axis"

"Euclid XII 11 Cones and cylinders of equal height are to one another as their bases"

"Euclid XII 14 Cones and cylinders on equal bases are to one another as their heights"

"Euclid XII 13 If a cylinder be cut by a plane parallel to the opposite planes [the bases], then, as the cylinder is to the cylinder so will the axis be to the axis"

3 The cones which have the same bases as the cylinders [and equal height] are in the same ratio as the cylinders

4 Also the bases of equal cones are reciprocally proportional to their heights and those cones whose bases are reciprocally proportional to their heights are equal¹

5 Also the cones the diameters of whose bases have the same ratio as their axes are to one another in the triplicate ratio of the diameters of the bases²

And all these propositions have been proved by earlier geometers.

PROPOSITION 17

If there be two isosceles cones, and the surface of one cone be equal to the base of the other while the perpendicular from the centre of the base [of the first cone] on the side of that cone is equal to the height [of the second] the cones will be equal

Let OAB , DEF be triangles through the axes of two cones respectively, C , G the centres of the respective bases GH the perpendicular from G on FD and suppose that the base of the cone OAB is equal to the surface of the cone DEF , and that $OC = GH$

Then since the base of OAB is equal to the surface of DEF
 (base of cone OAB) (base of cone DEF)

$$\begin{aligned} &= (\text{surface of } DEF) \quad (\text{base of } DEF) \\ &= DF \cdot FG \quad [\text{Prop 15}] \\ &= DG \cdot GH \text{ by similar triangles,} \\ &= DG \cdot OC \end{aligned}$$

Therefore the bases of the cones are reciprocally proportional to their heights whence the cones are equal [Lemma 4]

PROPOSITION 18

Any solid rhombus consisting of isosceles cones is equal to the cone which has its base equal to the surface of one of the cones composing the rhombus and its height equal to the perpendicular drawn from the apex of the second cone to one side of the first cone

Let the rhombus be $OABD$ consisting of two cones with apices O , D and with a common base (the circle about AB as diameter)

Let FHK be another cone with base equal to the surface of the cone OAB and height FG equal to DE the perpendicular from D on OB

Then shall the cone FHK be equal to the rhombus

Construct a third cone LMA with base (the circle about MN) equal to the base of OAB and height LP equal to OD

¹Euclid vii 15 The bases of equal cones and cylinders are reciprocally proportional to their heights and those cones and cylinders whose bases are reciprocally proportional to their heights are equal.

²Euclid xii 12 Similar cones and cylinders are to one another in the triplicate ratio of the diameters of their bases

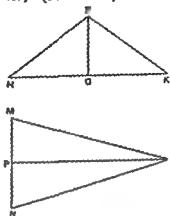
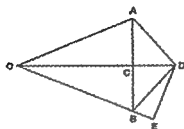
Then, since

$$IP = OD,$$

$$LP \cdot CD = OD \cdot CD$$

But [Lemma 1] $OD \cdot CD = (\text{rhombus } OADB) \text{ (cone } DAB),$
and $LP \cdot CD = (\text{cone } LMN) \text{ (cone } DAB)$

It follows that



$$(\text{rhombus } OADB) = (\text{cone } LMN) \quad (1)$$

Again, since $AB = MN$, and

$$(\text{surface of } OAB) = (\text{base of } FHK),$$

$$\begin{aligned} (\text{base of } FHK) \cdot (\text{base of } LMN) &= (\text{surface of } OAB) \cdot (\text{base of } OAB) \\ &= OB \cdot BC \quad [\text{Prop 15}] \\ &= OD \cdot DE, \text{ by similar triangles,} \\ &= LP \cdot FG, \text{ by hypothesis} \end{aligned}$$

Thus, in the cones FHK , LMN the bases are reciprocally proportional to the heights

Therefore the cones FHK , LMN are equal

and hence, by (1) the cone FHK is equal to the given solid rhombus

PROPOSITION 19

If an isosceles cone be cut by a plane parallel to the base and on the resulting circular section a cone be described having as its apex the centre of the base [of the first cone] and if the rhombus so formed be taken away from the whole cone, the part remaining will be equal to the cone with base equal to the surface of the portion of the first cone between the parallel planes and with height equal to the perpendicular drawn from the centre of the base of the first cone on one side of that cone

Let the cone OAB be cut by a plane parallel to the base in the circle on DF as diameter. Let C be the centre of the base of the cone and with C as apex and the circle about DE as base describe a cone making with the cone ODF the rhombus $ODCE$

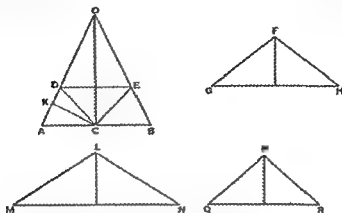
Take a cone FGH with base equal to the surface of the frustum $DABF$ and height equal to the perpendicular (CH) from C on AO

Then shall the cone FGH be equal to the difference between the cone OAB and the rhombus $ODCE$

Take (1) a cone LMN with base equal to the surface of the cone OAB and height equal to CH

(2) a cone PQR with base equal to the surface of the cone ODE and height equal to CH

Now, since the surface of the cone OAB is equal to the surface of the cone ODE together with that of the frustum $DABE$, we have, by the construction,



$$(\text{base of } LMN) = (\text{base of } FGH) + (\text{base of } PQR)$$

and since the heights of the three cones are equal

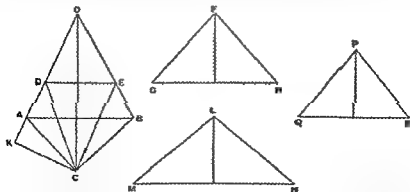
$$(\text{cone } LMN) = (\text{cone } FGH) + (\text{cone } PQR)$$

But the cone LMN is equal to the cone OAB [Prop. 17] and the cone PQR is equal to the rhombus $ODCE$ [Prop. 18]

Therefore $(\text{cone } OAB) = (\text{cone } FGH) + (\text{rhombus } ODCE)$ and the proposition is proved

PROPOSITION 20

If one of the two isosceles cones forming a rhombus be cut by a plane parallel to the base and on the resulting circular section a cone be described having the same apex as the second cone and if the resulting rhombus be taken from the whole rhombus the remainder will be equal to the cone with base equal to the surface of the portion of the cone between the parallel planes and with height equal to the perpendicular drawn from the apex of the second cone to the side of the first cone



Let the rhombus be $OACB$ and let the cone OAB be cut by a plane parallel to its base in the circle about DE as diameter. With this circle as base and C

as apex describe a cone which therefore with ODE forms the rhombus $ODCE$

Take a cone FGH with base equal to the surface of the frustum $DABE$ and height equal to the perpendicular (CK) from C on OA

The cone FGH shall be equal to the difference between the rhombi $OACB$, $ODCE$

For take (1) a cone LMN with base equal to the surface of OAB and height equal to CK ,

(2) a cone PQR , with base equal to the surface of ODE , and height equal to CK

Then, since the surface of OAB is equal to the surface of ODE together with that of the frustum $DABE$ we have, by construction

$$(\text{base of } LMN) = (\text{base of } PQR) + (\text{base of } FGH),$$

and the three cones are of equal height

therefore $(\text{cone } LMN) = (\text{cone } PQR) + (\text{cone } FGH)$

But the cone LMN is equal to the rhombus $OACB$, and the cone PQR is equal to the rhombus $ODCE$ [Prop 18]

Hence the cone FGH is equal to the difference between the two rhombi $OACB$ $ODCE$

PROPOSITION 21

A regular polygon of an even number of sides being inscribed in a circle, as $ABC \dots A' \dots C'B A$ so that AA' is a diameter if two angular points next but one to each other as B, B' be joined, and the other lines parallel to BB' and joining pairs of angular points be drawn, as CC', DD' , then

$(BB' + CC' + \dots) AA' = A'B BA$

Let $BB' CC' DD'$, meet AA' in F, G, H , and let $CB' DC'$, be joined meeting AA' in K, L , respectively

Then clearly CB, DC' are parallel to one another and to AB

Hence by similar triangles,

$$BF FA = BF FK$$

$$= CG GK$$

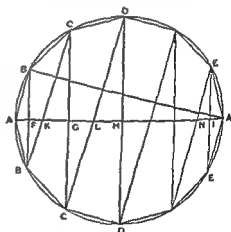
$$= CG GL$$

$$= E'I IA'$$

and summing the antecedents and consequents respectively we have

$$(BB' + CC' + \dots) AA' = BF FA$$

$$= A'B BA$$

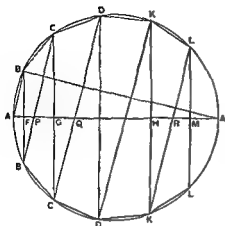


PROPOSITION 22

If a polygon be inscribed in a segment of a circle LAL' so that all its sides excluding the base are equal and their number even as $LK \dots A \dots K'L'$, A being the middle point of the segment and if the lines $BB' CC'$, parallel to the base LL' and joining pairs of angular points be drawn then

$$(BB' + CC' + \dots + LM) AM = A'B BA$$

where M is the middle point of LL' and AA is the diameter through M



Joining CB, DC', LK' , as in the last proposition, and supposing that they meet AM in P, Q, R while BB', CC', KK' meet AM in F, G, H we have, by similar triangles

$$\begin{aligned} BF : FA &= BF : FP \\ &= CG : PG \\ &= CH : HQ \end{aligned}$$

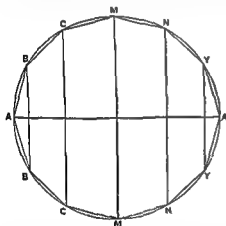
$$= LM : RM,$$

and, summing the antecedents and consequents we obtain

$$\begin{aligned} (BF' + CC' + LM) : AM &= BF : FA \\ &= A'B : BA \end{aligned}$$

PROPOSITION 23

Take a great circle ABC of a sphere and inscribe in it a regular polygon whose sides are a multiple of four in number. Let AA', MM' be diameters at right angles and joining opposite angular points of the polygon



Then, if the polygon and great circle revolve together about the diameter AA' , the angular points of the polygon, except A, A' will describe circles on the surface of the sphere at right angles to the diameter AA' . Also the sides of the polygon will describe portions of conical surfaces e.g. BC will describe a surface forming part of a cone whose base is a circle about CC' as diameter and whose apex is the point in which CB CB' produced meet each other and the diameter AA' .

Comparing the hemisphere VAM' and that half of the figure described by the revolution of the polygon

which is included in the hemisphere we see that the surface of the hemisphere and the surface of the inscribed figure have the same boundaries in one plane (viz the circle on MM' as diameter) the former surface entirely includes the latter and they are both concave in the same direction

Therefore [Assumptions 4] the surface of the hemisphere is greater than that of the inscribed figure and the same is true of the other halves of the figures

Hence the surface of the sphere is greater than the surface described by the revolution of the polygon inscribed in the great circle about the diameter of the great circle

PROPOSITION 24

If a regular polygon $AB \dots A' \dots B'A$, the number of whose sides is a multiple of four, be inscribed in a great circle of a sphere, and if BB' subtending two sides be joined, and all the other lines parallel to BB' and joining pairs of angular points be drawn, then the surface of the figure inscribed in the sphere by the revolution of the polygon about the diameter AA' is equal to a circle the square of whose radius is equal to the rectangle

$$BA(BB + CC' + \dots)$$

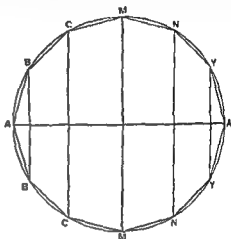
The surface of the figure is made up of the surfaces of parts of different cones

Now the surface of the cone ABB' is equal to a circle whose radius is $\sqrt{BA \frac{1}{2} BB}$ [Prop 14]

The surface of the frustum $BB'CC'$ is equal to a circle of radius $\sqrt{BC \frac{1}{2}(BB + CC')}$, [Prop 16] and so on

It follows since $BA = BC = \dots$, that the whole surface is equal to a circle whose radius is equal to

$$\sqrt{BA(BB + CC + \dots + MM + \dots + YY')}$$



PROPOSITION 25

The surface of the figure inscribed in a sphere as in the last propositions, consisting of portions of conical surfaces, is less than four times the greatest circle in the sphere

Let $AB \dots A' \dots B'A$ be a regular polygon inscribed in a great circle, the number of its sides being a multiple of four

As before let BB' be drawn subtending two sides, and CC' , YY' parallel to BB'

Let R be a circle such that the square of its radius is equal to

$$1B(BB + CC + \dots + YY'),$$

so that the surface of the figure inscribed in the sphere is equal to R

[Prop 24]

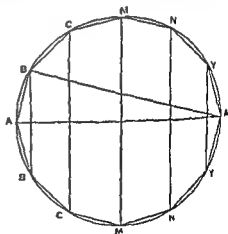
Now

$$(BB + CC + \dots + YY') \cdot AA' = AB \cdot AB \quad [\text{Prop 21}]$$

$$\text{whence } AB(BB + CC + \dots + YY') = AA' \cdot AB$$

$$\text{Hence } (\text{radius of } R)^2 = A \frac{1}{2} A'B < A \frac{1}{2} A^2$$

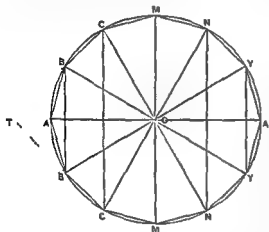
Therefore the surface of the inscribed figure or the circle R is less than four times the circle $\frac{1}{4} AA'$



PROPOSITION 26

The figure inscribed as above in a sphere is equal [in volume] to a cone whose base is a circle equal to the surface of the figure inscribed in the sphere and whose height is equal to the perpendicular drawn from the centre of the sphere to one side of the polygon.

Suppose, as before that $AB A' B'A$ is the regular polygon inscribed in a great circle, and let BB', CC', \dots be joined



With apex O construct cones whose bases are the circles on BB', CC', \dots as diameters in planes perpendicular to AA'

Then $OBAB$ is a solid rhombus, and its volume is equal to a cone whose base is equal to the surface of the cone ABB' and whose height is equal to the perpendicular from O on AB [Prop 18] Let the length of the perpendicular be p

Again, if $CB C'B$ produced meet in T , the portion of the solid figure which is described by the revolution of the triangle

angle BOC about AA' is equal to the difference between the rhombi $OCTC$ and $OBTB$ i.e. to a cone whose base is equal to the surface of the frustum $BB'CC'$ and whose height is p [Prop 20]

Proceeding in this manner, and adding we prove that since cones of equal height are to one another as their bases the volume of the solid of revolution is equal to a cone with height p and base equal to the sum of the surfaces of the cone BAB' , the frustum $BB'CC'$ etc. i.e. a cone with height p and base equal to the surface of the solid

PROPOSITION 27

The figure inscribed in the sphere as before is less than four times the cone whose base is equal to a great circle of the sphere and whose height is equal to the radius of the sphere

By Prop 26 the volume of the solid figure is equal to a cone whose base is equal to the surface of the solid and whose height is p the perpendicular from O on any side of the polygon Let R be such a cone

Take also a cone S with base equal to the great circle and height equal to the radius of the sphere

Now, since the surface of the inscribed solid is less than four times the great circle [Prop 25] the base of the cone R is less than four times the base of the cone S

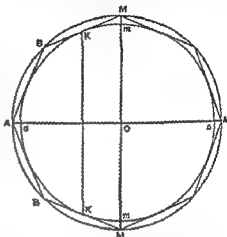
Also the height (p) of R is less than the height of S

Therefore the volume of R is less than four times that of S and the proposition is proved

PROPOSITION 28

Let a regular polygon, whose sides are a multiple of four in number, be circumscribed about a great circle of a given sphere, as $AB A' B'A$, and about the polygon describe another circle, which will therefore have the same centre as the great circle of the sphere. Let AA' bisect the polygon and cut the sphere in a, a'

If the great circle and the circumscribed polygon revolve together about AA' the great circle will describe the surface of a sphere the angular points of the polygon except A, A' will move round the surface of a larger sphere, the points of contact of the sides of the polygon with the great circle of the inner sphere will describe circles on that sphere in planes perpendicular to AA' , and the sides of the polygon themselves will describe portions of conical surfaces. The circumscribed figure will thus be greater than the sphere itself



Let any side as BM touch the

inner circle in K , and let A' be the point of contact of the circle with $B M'$

Then the circle described by the revolution of $KA A'$ about AA' is the boundary in one plane of two surfaces

- (1) the surface formed by the revolution of the circular segment $KA A'$, and
- (2) the surface formed by the revolution of the part $KB A B'A'$ of the polygon

Now the second surface entirely includes the first, and they are both concave in the same direction

therefore [Assumptions 4] the second surface is greater than the first

The same is true of the portion of the surface on the opposite side of the circle on KA as diameter

Hence adding we see that the surface of the figure circumscribed to the given sphere is greater than that of the sphere itself

PROPOSITION 29

In a figure circumscribed to a sphere in the manner shown in the previous proposition the surface is equal to a circle the square on whose radius is equal to

$$AB(BB + CC + \dots)$$

For the figure circumscribed to the sphere is inscribed in a larger sphere, and the proof of Prop 24 applies

PROPOSITION 30

The surface of a figure circumscribed as before about a sphere is greater than four times the great circle of the sphere

Let $AB A' B'A$ be the regular polygon of $4n$ sides which by its revolu

tion about AA' describes the figure circumscribing the sphere of which $ama m'$ is a great circle. Suppose aa', AA' to be in one straight line.

Let R be a circle equal to the surface of the circumscribed solid.

Now

$$(BB' + CC' + \dots) AA' = A'B \cdot BA \quad [\text{as in Prop 21}]$$

so that

$$\begin{aligned} AB(BB' + CC' + \dots) &= AA' \cdot AB \\ \text{Hence (radius of } R) &= \sqrt{AA' \cdot AB} \quad [\text{Prop 29}] \\ &> AB \end{aligned}$$

But $AB = 2OP$ where P is the point in which AB touches the circle $ama m'$.

Therefore (radius of R) > (diameter of circle $ama m'$)

hence R , and therefore the surface of the circumscribed solid is greater than four times the great circle of the given sphere.

PROPOSITION 31

The solid of revolution circumscribed as before about a sphere is equal to a cone whose base is equal to the surface of the solid and whose height is equal to the radius of the sphere.

The solid is as before a solid inscribed in a larger sphere and, since the perpendicular on any side of the revolving polygon is equal to the radius of the inner sphere, the proposition is identical with Prop 26.

Con. The solid circumscribed about the smaller sphere is greater than four times the cone whose base is a great circle of the sphere and whose height is equal to the radius of the sphere.

For since the surface of the solid is greater than four times the great circle of the inner sphere [Prop 30] the cone whose base is equal to the surface of the solid and whose height is the radius of the sphere is greater than four times the cone of the same height which has the great circle for base [Lemma 1].

Hence by the proposition the volume of the solid is greater than four times the latter cone.

PROPOSITION 32

If a regular polygon with $4n$ sides be inscribed in a great circle of a sphere as $ab a' b a'$ and a similar polygon $AB A' B A'$ be described about the great circle and if the polygons revolve with the great circle about the diameters $aa' AA'$ respectively so that they describe the surfaces of solid figures inscribed in and circumscribed to the sphere respectively then

(1) the surfaces of the circumscribed and inscribed figures are to one another in the duplicate ratio of their sides and

(2) the figures themselves [i.e. their volumes] are in the triplicate ratio of their sides.

(1) Let AA', aa' be in the same straight line, and let $MmOm'M'$ be a diameter at right angles to them

Join $BB', CC',$ and $bb', cc',$ which will all be parallel to one another and MM'

Suppose R, S to be circles such that

$$R = (\text{surface of circumscribed solid}),$$

$$S = (\text{surface of inscribed solid})$$

Then $(\text{radius of } R)^2 = AB(BB' + CC' + \dots)$ [Prop 29]

$(\text{radius of } S)^2 = ab(bb' + cc' + \dots)$ [Prop 24]

And, since the polygons are similar the rectangles in these two equations are similar and are therefore in the ratio of

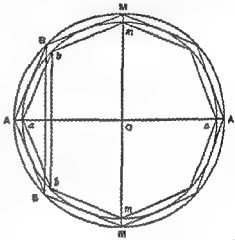
$$AB^2 : ab^2$$

Hence

$(\text{surface of circumscribed solid}) : (\text{surface of inscribed solid}) = AB^2 : ab^2$

(2) Take a cone V whose base is the circle R and whose height is equal to Oa and a cone W whose base is the circle S and whose height is equal to the perpendicular from O on ab which we will call p

Then V, W are respectively equal to the volumes of the circumscribed and inscribed figures



[Props 31, 26]

Now since the polygons are similar

$$AB : ab = Oa : p$$

$$= (\text{height of cone } V) : (\text{height of cone } W),$$

and, as shown above the bases of the cones (the circles R, S) are in the ratio of AB^2 to ab^2

Therefore

$$V : W = AB^2 : ab^2$$

PROPOSITION 33

The surface of any sphere is equal to four times the greatest circle in it

Let C be a circle equal to four times the great circle

Then if C is not equal to the surface of the sphere it must either be less or greater

I Suppose C less than the surface of the sphere

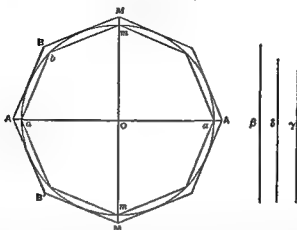
It is then possible to find two lines β, γ of which β is the greater, such that $\beta, \gamma < (\text{surface of sphere}) : C$ [Prop 2]

Take such lines and let δ be a mean proportional between them

Suppose similar regular polygons with $4n$ sides circumscribed about and inscribed in a great circle such that the ratio of their sides is less than the ratio $\beta : \delta$ [Prop 3]

Let the polygons with the circle revolve together about a diameter common to all, describing solids of revolution as before

Then (surface of outer solid) (surface of inner solid)
 $= (\text{side of outer})^2 (\text{side of inner})^2$ [Prop 32]
 $< \beta^2 \delta^2$, or $\beta \gamma$
 $< (\text{surface of sphere}) \ C$, *a fortiori*



But this is impossible since the surface of the circumscribed solid is greater than that of the sphere [Prop 28], while the surface of the inscribed solid is less than C [Prop 25]

Therefore C is not less than the surface of the sphere

II Suppose C greater than the surface of the sphere

Take lines β, γ , of which β is the greater, such that

$$\beta \gamma < C \text{ (surface of sphere)}$$

Circumscribe and inscribe to the great circle similar regular polygons, as before such that their sides are in a ratio less than that of β to δ , and suppose solids of revolution generated in the usual manner

Then, in this case,

$$\begin{aligned} &(\text{surface of circumscribed solid}) > (\text{surface of inscribed solid}) \\ &< C \text{ (surface of sphere)} \end{aligned}$$

But this is impossible, because the surface of the circumscribed solid is greater than C [Prop 30] while the surface of the inscribed solid is less than that of the sphere [Prop 23]

Thus C is not greater than the surface of the sphere

Therefore since it is neither greater nor less C is equal to the surface of the sphere

PROPOSITION 34

Any sphere is equal to four times the cone which has its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere

Let the sphere be that of which $ama m$ is a great circle

If now the sphere is not equal to four times the cone described it is either greater or less

I If possible let the sphere be greater than four times the cone

Suppose V to be a cone whose base is equal to four times the great circle and whose height is equal to the radius of the sphere

Then by hypothesis the sphere is greater than V , and two lines β, γ can be found (of which β is the greater) such that

$$\beta \cdot \gamma < (\text{volume of sphere}) \cdot V$$

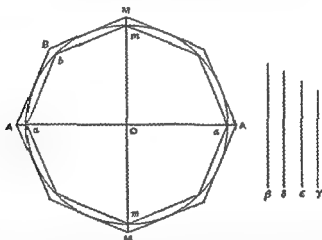
Between β and γ place two arithmetic means δ, ϵ

As before let similar regular polygons with sides $4n$ in number be circumscribed about and inscribed in the great circle, such that their sides are in a ratio less than $\beta \cdot \delta$

Imagine the diameter aa' of the circle to be in the same straight line with a diameter of both polygons and imagine the latter to revolve with the circle about aa' describing the surfaces of two solids of revolution. The volumes of these solids are therefore in the triplicate ratio of their sides [Prop 32]

Thus

| | |
|---|-----------------------------|
| (vol of outer solid) | (vol of inscribed solid) |
| $< \beta^3$ | δ^3 , by hypothesis, |
| $< \beta \cdot \gamma$, a fortiori (since $\beta > \gamma > \beta^2 \cdot \delta^3$), | |
| $< (\text{volume of sphere}) \cdot V$, a fortiori | |



But this is impossible since the volume of the circumscribed solid is greater than that of the sphere [Prop 28] while the volume of the inscribed solid is less than V [Prop 27]

Hence the sphere is not greater than V or four times the cone described in the enunciation

II If possible let the sphere be less than V

In this case we take $\beta \cdot \gamma$ (β being the greater) such that

$$\beta \cdot \gamma < V \cdot (\text{volume of sphere})$$

The rest of the construction and proof proceeding as before we have finally

| | |
|-------------------------|-----------------------------|
| (volume of outer solid) | (volume of inscribed solid) |
| $< V$ | (volume of sphere) |

But this is impossible because the volume of the outer solid is greater than V [Prop 31 Cor.] and the volume of the inscribed solid is less than the volume of the sphere

Hence the sphere is not less than V

Since then the sphere is neither less nor greater than V , it is equal to V , or to four times the cone described in the enunciation

COR From what has been proved it follows that every cylinder whose base is the greatest circle in a sphere and whose height is equal to the diameter of the sphere is $\frac{1}{2}$ of the sphere, and its surface together with its bases is $\frac{3}{2}$ of the surface of the sphere

For the cylinder is three times the cone with the same base and height [Eucl XII 10] i.e. six times the cone with the same base and with height equal to the radius of the sphere.

But the sphere is four times the latter cone [Prop 34] Therefore the cylinder is $\frac{3}{2}$ of the sphere

Again the surface of a cylinder (excluding the bases) is equal to a circle whose radius is a mean proportion between the height of the cylinder and the diameter of its base [Prop 13]

In this case the height is equal to the diameter of the base and therefore the circle is that whose radius is the diameter of the sphere or a circle equal to four times the great circle of the sphere

Therefore the surface of the cylinder with the bases is equal to $\frac{3}{2}$ times the great circle

And the surface of the sphere is four times the great circle [Prop 33] whence
(surface of cylinder with bases) = $\frac{3}{4}$ (surface of sphere)

PROPOSITION 35

If in a segment of a circle LAL (where A is the middle point of the arc) a polygon $LHAK'L$ be inscribed of which LL' is one side while the other sides are $2n$ in number and all equal, and if the polygon revolve with the segment about the diameter AM generating a solid figure inscribed in a segment of a sphere then the surface of the inscribed solid is equal to a circle the square on whose radius is equal to the rectangle

$$AB \left(BB' + CC' + \dots + KK' + \frac{LL'}{2} \right)$$

The surface of the inscribed figure is made up of portions of surfaces of cones

If we take these successively the surface of the cone BAB' is equal to a circle whose radius is

$$\sqrt{AB \cdot \frac{BB'}{2}} \quad [\text{Prop 14}]$$

The surface of the frustum of a cone $BCC'B'$ is equal to a circle whose radius is

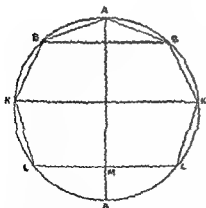
$$\sqrt{AB \cdot \frac{BB' + CC'}{2}}, \quad [\text{Prop 16}]$$

and so on

Proceeding in this way and adding we find since circles are to one another as the squares of their radii that the

surface of the inscribed figure is equal to a circle whose radius is

$$\sqrt{AB \left(BB' + CC' + \dots + KK' + \frac{LL'}{2} \right)}$$



PROPOSITION 36

The surface of the figure inscribed as before in the segment of a sphere is less than that of the segment of the sphere

This is clear because the circular base of the segment is a common boundary of each of two surfaces, of which one, the segment, includes the other, the solid, while both are concave in the same direction [Assumptions, 4]

PROPOSITION 37

The surface of the solid figure inscribed in the segment of the sphere by the revolution of Lh A $h'L'$ about AM is less than a circle with radius equal to AL

Let the diameter AM meet the circle of which LAL' is a segment again in A . Join $A'B$.

As in Prop 35, the surface of the inscribed solid is equal to a circle the square on whose radius is

$$AB(BB' + CC' + \dots + h'h' + LM)$$

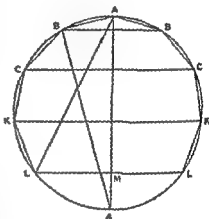
But this rectangle

$$= A'B \cdot AM \quad [\text{Prop 22}]$$

$$< A'A \cdot AM$$

$$< AL^2$$

Hence the surface of the inscribed solid is less than the circle whose radius is AL .



PROPOSITION 38

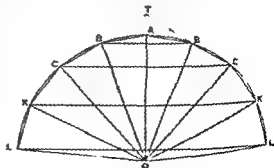
The solid figure described as before in a segment of a sphere less than a hemisphere, together with the cone whose base is the base of the segment and whose apex is the centre of the sphere is equal to a cone whose base is equal to the surface of the inscribed solid and whose height is equal to the perpendicular from the centre of the sphere on any side of the polygon

Let O be the centre of the sphere, and p the length of the perpendicular from O on AB .

Suppose cones described with O as apex and with the circles on BB' CC' as diameters as bases.

Then the rhombus $OBAB$ is equal to a cone whose base is equal to the surface of the cone BAB and whose height is p . [Prop 18]

Again if CB CB' meet in T the solid described by the triangle BOC as the polygon revolves about AO is the difference between the rhombi $OCTC'$ and $OBTB$ and is therefore equal to a cone whose base is equal to the surface of



the frustum $BCCB$ and whose height is p

[Prop 20]

Similarly for the part of the solid described by the triangle COD as the poly-
gon revolves and so on

Hence, by addition the solid figure inscribed in the segment together with
the cone OLL' is equal to a cone whose base is the surface of the inscribed solid
and whose height is p

COR The cone whose base is a circle with radius equal to AL and whose height
is equal to the radius of the sphere is greater than the sum of the inscribed solid and
the cone OLL'

For by the proposition, the inscribed solid together with the cone OLL' is
equal to a cone with base equal to the surface of the solid and with height p

This latter cone is less than a cone with height equal to OA and with base
equal to the circle whose radius is AL because the height p is less than OA
while the surface of the solid is less than a circle with radius AL [Prop 37]

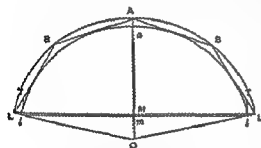
PROPOSITION 39

Let lal be a segment of a great circle of a sphere being less than a semicircle
Let O be the centre of the sphere and join Ol , Ol' Suppose a polygon circum-
scribed about the sector $Olal$ such that its sides excluding the two radii are $2n$
in number and all equal as LK , BA , AB , KL and let OA be that
radius of the great circle which bisects the segment lal

The circle circumscribing the polygon will then have the same centre O as
the given great circle

Now suppose the polygon and the two circles to revolve together about OA
The two circles will describe spheres the angular points except A will describe

circles on the outer sphere with
diameters BB etc the points
of contact of the sides with the
inner segment will describe
circles on the inner sphere the
sides themselves will describe
the surfaces of cones or frusta
of cones and the whole figure
circumscribed to the segment of
the inner sphere by the revolu-
tion of the equal sides of the
polygon will have for its base
the circle on LL as diameter



The surface of the solid figure so circumscribed about the sector of the sphere
[excluding its base] will be greater than that of the segment of the sphere whose base
is the circle on LL as diameter

I or draw the tangents IT , lT to the inner segment at l , l' These with the
sides of the polygon will describe by their revolution a solid whose surface is
greater than that of the segment [Assumptions 4]

But the surface described by the revolution of IT is less than that described
by the revolution of LlT since the angle TLl is a right angle and therefore
 $LlT > IT$

Hence a fortiori the surface described by LK , A , K , L is greater than
that of the segment

Cor. The surface of the figure so described about the sector of the sphere is equal to a circle the square on whose radius is equal to the rectangle

$$AB (BB' + CC' + \dots + KK' + \frac{1}{2}LL')$$

For the circumscribed figure is inscribed in the outer sphere, and the proof of Prop. 35 therefore applies

PROPOSITION 40

The surface of the figure circumscribed to the sector as before is greater than a circle whose radius is equal to al

Let the diameter AaO meet the great circle and the circle circumscribing the revolving polygon again in a' , A' . Join $A'B$, and let ON be drawn to N , the point of contact of AB with the inner circle

Now by Prop. 39 Cor., the surface of the solid figure circumscribed to the sector $OlAl$ is equal to a circle the square on whose radius is equal to the rectangle

$$AB \left(BB' + CC' + \dots + KK' + \frac{LL'}{2} \right)$$

But this rectangle is equal to $A'B \cdot AM$ [as in Prop. 22]

Next since $AL' \cdot al$ are parallel the triangles AML' and aml are similar. And $AL' > al$ therefore $AM > am$

$$\text{Also } A'B = 2ON = aa'$$

$$\text{Therefore } A'B \cdot AM > am \cdot aa' > al^2$$

Hence the surface of the solid figure circumscribed to the sector is greater than a circle whose radius is equal to al or al

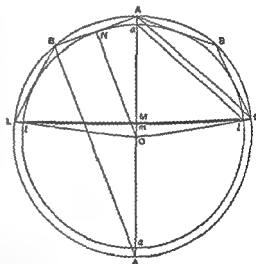
Cor. 1 The volume of the figure circumscribed about the sector together with the cone whose apex is O and base the circle on LL' as diameter is equal to the volume of a cone whose base is equal to the surface of the circumscribed figure and whose height is ON

For the figure is inscribed in the outer sphere which has the same centre as the inner. Hence the proof of Prop. 38 applies

Cor. 2 The volume of the circumscribed figure with the cone OLL' is greater than the cone whose base is a circle with radius equal to al and whose height is equal to the radius (Oa) of the inner sphere

For the volume of the figure with the cone OLL' is equal to a cone whose base is equal to the surface of the figure and whose height is equal to OV

And the surface of the figure is greater than a circle with radius equal to al [Prop. 40] while the heights Oa , OV are equal



PROPOSITION 41

Let lal be a segment of a great circle of a sphere which is less than a semicircle

Suppose a polygon inscribed in the sector $Olal$ such that the sides ll' , ba , ab , ll' are $2n$ in number and all equal. Let a similar polygon be circumscribed about the sector so that its sides are parallel to those of the first polygon and draw the circle circumscribing the outer polygon.

Now let the polygons and circles revolve together about OaA the radius bisecting the segment lal .

Then (1) the surfaces of the outer and inner solids of revolution so described are in the ratio of AB^2 to ab^2 , and (2) their volumes together with the corresponding cones with the same base and with apex O in each case are as AA^3 to ab^3 .

(1) For the surfaces are equal to circles the squares on whose radii are equal respectively to

$$AB \left(BB + CC' + \dots + AA' + \frac{LL'}{2} \right), \quad [\text{Prop 39 Cor}]$$

$$\text{and} \quad ab \left(bb + cc + \dots + kk + \frac{ll}{2} \right) \quad [\text{Prop 35}]$$

But these rectangles are in the ratio of AB^2 to ab^2 . Therefore so are the surfaces.

(2) Let OnN be drawn perpendicular to ab and AB , and suppose the circles which are equal to the surfaces of the outer and inner solids of revolution to be denoted by S , s respectively.

Now the volume of the circumscribed solid together with the cone OLL is equal to a

cone whose base is S and whose height is OV [Prop 40 Cor 1]

And the volume of the inscribed figure with the cone Oll is equal to a cone with base s and height On [Prop 38]

$$\text{But} \quad S : s = AB^2 : ab^2$$

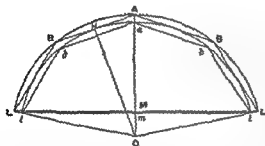
$$\text{and} \quad ON : On = AB : ab$$

Therefore the volume of the circumscribed solid together with the cone OLL is to the volume of the inscribed solid together with the cone Oll as AB^3 is to ab^3 [Lemma 5]

PROPOSITION 42

If lal be a segment of a sphere less than a hemisphere and Oa the radius perpendicular to the base of the segment the surface of the segment is equal to a circle whose radius is equal to al .

Let R be a circle whose radius is equal to al . Then the surface of the segment which we will call S must if it be not equal to R be either greater or less than R .

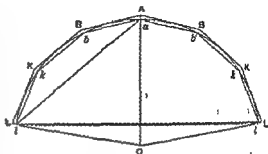


I Suppose, if possible, $S > R$

Let lal' be a segment of a great circle which is less than a semicircle Join Ol , and let similar polygons with $2n$ equal sides be circumscribed and inscribed to the sector, as in the previous propositions, but such that (circumscribed polygon) (inscribed polygon) $< S$ R

[Prop 8]

Let the polygons now revolve with the segment about OaA , generating solids of revolution circumscribed and inscribed to the segment of the sphere



Then

(surface of outer solid) (surface of inner solid)

$= AB^2 ab^2$

[Prop 41]

$=$ (circumscribed polygon) (inscribed polygon)

$< S$ R by hypothesis

But the surface of the outer solid is greater than S [Prop 39]

Therefore the surface of the inner solid is greater than R , which is impossible, by Prop 37

II Suppose if possible $S < R$

In this case we circumscribe and inscribe polygons such that their ratio is less than R S and we arrive at the result that

(surface of outer solid) (surface of inner solid)

$< R$ S

But the surface of the outer solid is greater than R [Prop 40] Therefore the surface of the inner solid is greater than S which is impossible [Prop 36]

Hence since S is neither greater nor less than R ,

$S = R$

PROPOSITION 43

Even if the segment of the sphere is greater than a hemisphere, its surface is still equal to a circle whose radius is equal to al

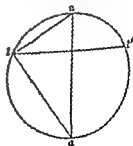
For let lal' be a great circle of the sphere aa' being the diameter perpendicular to ll' and let $la'l'$ be a segment less than a semicircle

Then by Prop 42 the surface of the segment $la'l'$ of the sphere is equal to a circle with radius equal to al

Also the surface of the whole sphere is equal to a circle with radius equal to aa' [Prop 33]

But $aa'^2 - a'l'^2 = al^2$ and circles are to one another as the squares on their radii

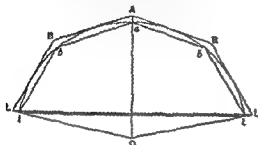
Therefore the surface of the segment $la'l'$ being the difference between the surfaces of the sphere and of $la'l$ is equal to a circle with radius equal to al



PROPOSITION 44

The volume of any sector of a sphere is equal to a cone whose base is equal to the surface of the segment of the sphere included in the sector, and whose height is equal to the radius of the sphere

Let R be a cone whose base is equal to the surface of the segment lal of a sphere and whose height is equal to the radius of the sphere, and let S be the volume of the sector Oal



Then if S is not equal to R it must be either greater or less

I Suppose, if possible, that $S > R$

Find two straight lines β , γ , of which β is the greater, such that

$$\beta : \gamma < S : R$$

and let δ , ϵ be two arithmetic means between β , γ

Let lal be a segment of a great circle of the sphere Join Ol , Ol' and let similar polygons with $2n$ equal sides be circumscribed and inscribed to

the sector of the circle as before but such that their sides are in a ratio less than $\beta : \delta$ [Prop 4]

Then let the two polygons revolve with the segment about OaA generating two solids of revolution

Denoting the volumes of these solids by V & respectively we have

$$(V + \text{cone } OLL') : (v + \text{cone } Oll) = AB^3 : ab^3 \quad [\text{Prop 41}]$$

$$< \beta^3 : \delta^3$$

$$< \beta : \gamma, \text{ a fortiori,}$$

$$< S : R \text{ by hypothesis}$$

Now

$$(V + \text{cone } OLL') > S$$

Therefore also

$$(v + \text{cone } Oll) > R$$

But this is impossible by Prop 38 Cor combined with Props 42 43

Hence

$$S > R$$

II Suppose if possible that $S < R$

In this case we take β , γ such that

$$\beta : \gamma < R : S$$

and the rest of the construction proceeds as before

We thus obtain the relation

$$(V + \text{cone } OLL') : (v + \text{cone } Oll) < R : S$$

Now

$$(v + \text{cone } Oll) < S$$

Therefore

$$(V + \text{cone } OLL') < R$$

which is impossible by Prop 40 Cor 2 combined with Props 42 43

Since then S is neither greater nor less than R

$$S = R$$

ON THE SPHERE AND CYLINDER

BOOK TWO

ARCHIMEDES to Dositheus greeting

'On a former occasion you asked me to write out the proofs of the problems the enunciations of which I had myself sent to Conon. In point of fact they depend for the most part on the theorems of which I have already sent you the demonstrations, namely (1) that the surface of any sphere is four times the greatest circle in the sphere (2) that the surface of any segment of a sphere is equal to a circle whose radius is equal to the straight line drawn from the vertex of the segment to the circumference of its base (3) that the cylinder whose base is the greatest circle in any sphere and whose height is equal to the diameter of the sphere is itself in magnitude half as large again as the sphere while its surface [including the two bases] is half as large again as the surface of the sphere, and (4) that any solid sector is equal to a cone whose base is the circle which is equal to the surface of the segment of the sphere included in the sector and whose height is equal to the radius of the sphere. Such then of the theorems and problems as depend on these theorems I have written out in the book which I send herewith, those which are discovered by means of a different sort of investigation, those namely which relate to spirals and the conoids. I will endeavour to send you soon

The first of the problems was as follows *Given a sphere to find a plane area equal to the surface of the sphere*

The solution of this is obvious from the theorems aforesaid. For four times the greatest circle in the sphere is both a plane area and equal to the surface of the sphere.

The second problem was the following

PROPOSITION 1 (PROBLEM)

Given a cone or a cylinder to find a sphere equal to the cone or to the cylinder

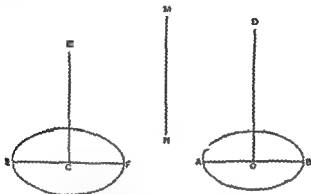
If Γ be the given cone or cylinder we can make a cylinder equal to Γ . Let this cylinder be the cylinder whose base is the circle on AB as diameter and whose height is OD .

Now if we could make another cylinder equal to the cylinder (OD) but such that its height is equal to the diameter of its base the problem would be solved because this latter cylinder would be equal to Γ and the sphere whose diameter is equal to the height (or to the diameter of the base) of the same cylinder would then be the sphere required [I. 31 Cor.]

Suppose the problem solved and let the cylinder (CG) be equal to the cylinder (OD) , while EF the diameter of the base is equal to the height CG .

Then since in equal cylinders the heights and bases are reciprocally proportional,

$$\begin{aligned} AB^2 \cdot EF^2 &= CG \cdot OD \\ &= EF \cdot OD \end{aligned} \quad (1)$$



Suppose MN to be such a line that

$$EF = AB \cdot MN \quad (2)$$

Hence

$$AB \cdot EF = EF \cdot MN,$$

and combining (1) and (2), we have

$$AB \cdot MN = EF \cdot OD$$

or

$$AB \cdot EF = MN \cdot OD$$

Therefore

$$AB \cdot EF = EF \cdot MN = MN \cdot OD$$

and $EF \cdot MN$ are two mean proportionals between $AB \cdot OD$

The synopsis of the problem is therefore as follows. Take two mean proportionals EF, MN between AB and OD and describe a cylinder whose base is a circle on EF as diameter and whose height CG is equal to EF

Then since

$$AB \cdot EF = EF \cdot MN = MN \cdot OD$$

$$EF^2 = AB \cdot MN$$

and therefore

$$AB^2 \cdot EF^2 = AB \cdot MN$$

$$= EF \cdot OD$$

$$= CG \cdot OD$$

whence the bases of the two cylinders (OD) (CG) are reciprocally proportional to their heights

Therefore the cylinders are equal and it follows that

$$\text{cylinder } (CG) = \frac{2}{3}V$$

The sphere on EF as diameter is therefore the sphere required being equal to $\frac{1}{3}V$

PROPOSITION 2

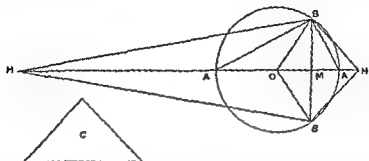
If BAB be a segment of a sphere BB a diameter of the base of the segment and O the centre of the sphere and if AA be the diameter of the sphere bisecting BB in M then the volume of the segment is equal to that of a cone whose base is the same as that of the segment and whose height is h where

$$h \cdot AM = OA + AM \cdot AM$$

Measure MH along MA equal to h and MH along MA equal to h where

$$h \cdot AM = OA + AM \cdot AM$$

Suppose the three cones constructed which have O, H, H' for their apices and the base (BB') of the segment for their common base. Join $AB, A'B$



Let C be a cone whose base is equal to the surface of the segment BAB' of the sphere i.e. to a circle with radius equal to AB [I 42] and whose height is equal to OA

Then the cone C is equal to the solid sector $OBAB'$ [I 44]

Now since $HM : NA = OA' : A'M$,
 dividendo $HA : AM = OA : A'M$,
 and, alternately $HA : AO = AM : MA'$,
 so that

$$\begin{aligned} HO : OA &= AA' : A'M \\ &= AB^2 : BM^2 \\ &= (\text{base of cone } C) : (\text{circle on } BB' \text{ as diameter}) \end{aligned}$$

But OA is equal to the height of the cone C , therefore since cones are equal if their bases and heights are reciprocally proportional it follows that the cone C (or the solid sector $OBAB'$) is equal to a cone whose base is the circle on BB' as diameter and whose height is equal to OH

And this latter cone is equal to the sum of two others having the same base and with heights OM, MH i.e. to the solid rhombus $OBHB'$

Hence the sector $OBAB'$ is equal to the rhombus $OBHB'$

Taking away the common part, the cone OBB' ,
 the segment BAB = the cone HBB'

Similarly by the same method we can prove that
 the segment $BA'B'$ = the cone HBB'

Alternative proof of the latter property

Suppose D to be a cone whose base is equal to the surface of the whole sphere and whose height is equal to OA

Thus D is equal to the volume of the sphere

[I 33 34]

Now since $OA + AM : M = HM : MA$
 dividendo and alternando, as before,

$$OA : AH = A'M : MA$$

Again since $HM : MA' = OA + AM : AM$,

$$HA' : OA = AM : MA$$

$$= OA : AH, \text{ from above}$$

Componendo, $HO : OA = OH : HA$

Alternately $HO : OH = OA : AH$

(1)

(2)

and *componendo*,

$$\begin{aligned} HH' \quad HO &= OH \quad HA, \\ &= H'O \quad OA \text{ from (1),} \end{aligned}$$

whence

$$HH \quad OA = HO \quad OH \quad (3)$$

Next since

$$\begin{aligned} HO \quad OH &= OA \quad AH \text{ by (2)} \\ &= A'M \quad MA \end{aligned}$$

$$(H'O + OH)^2 \quad HO \quad OH = (A'M + MA)^2 \quad A'M \quad MA,$$

whence, by means of (3),

$$HH \quad HH \quad OA = AA' \quad A'M \quad MA$$

or

$$HH \quad OA = AA'^2 \quad BM^2$$

Now the cone D which is equal to the sphere has for its base a circle whose radius is equal to AA' , and for its height a line equal to OA

Hence this cone D is equal to a cone whose base is the circle on BB as diameter and whose height is equal to HH

therefore

the cone D = the rhombus $HBH B$,

or

the rhombus $HBH B$ = the sphere

But

the segment BAB = the cone HBB

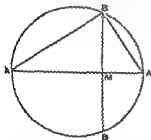
therefore the remaining segment $BA B$ = the cone HBB

COR The segment BAB is to a cone with the same base and equal height in the ratio of $OA + A'M$ to $A'M$

PROPOSITION 3 (PROBLEM)

To cut a given sphere by a plane so that the surfaces of the segments may have to one another a given ratio

Suppose the problem solved Let AA' be a diameter of a great circle of the sphere and suppose that a plane perpendicular to AA cuts the plane of the great circle in the straight line BB and AA in M , and that it divides the sphere so that the surface of the segment BAB has to the surface of the segment $BA B$ the given ratio



Now these surfaces are respectively equal to circles with radii equal to AB , AB [I 42 43]

Hence the ratio $AB^2 : AB$ is equal to the given ratio i.e. AM is to MA in the given ratio

Accordingly the synthesis proceeds as follows

If $H : K$ be the given ratio divide AA in M so that

$$AM : MA = H : K$$

Then $AM : MA = AB : AB^2$

$$= (\text{circle with radius } AB) : (\text{circle with radius } A'B)$$

$$= (\text{surface of segment } BAB) : (\text{surface of segment } BAB)$$

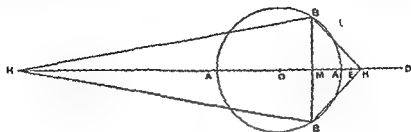
Thus the ratio of the surfaces of the segments is equal to the ratio $H : K$

PROPOSITION 4 (PROBLEM)

To cut a given sphere by a plane so that the volumes of the segments are to one another in a given ratio

Suppose the problem solved and let the required plane cut the great circle

ABA' at right angles in the line BB' let $A A'$ be that diameter of the great circle which bisects BB' at right angles (in M), and let O be the centre of the sphere



Take H on OA produced, and H' on OA' produced such that

$$OA + A'M : A'M = HM : MA, \quad (1)$$

and
$$OA + AM : AM = H'M : MA' \quad (2)$$

Join $BH, B'H, BH', B'H'$

Then the cones $HBB, H'B'B$ are respectively equal to the segments $BAB', BA'B'$ of the sphere [Prop 2]

Hence the ratio of the cones and therefore of their altitudes, is given, i.e.

$$HM : H'M = \text{the given ratio} \quad (3)$$

We have now three equations (1) (2) (3) in which there appear three as yet undetermined points M, H, H' , and it is first necessary to find, by means of them, another equation in which only one of these points (M) appears, i.e. we have so to speak to eliminate H, H'

Now, from (3), it is clear that $HH' : H'M$ is also a given ratio, and Archimedes' method of elimination is *first*, to find values for each of the ratios $A'H : H'M$ and $HH : H'A'$ which are alike independent of H, H' , and then *secondly*, to equate the ratio compounded of these two ratios to the known value of the ratio $HH' : H'M$

(a) To find such a value for $A'H : H'M$

It is at once clear from equation (2) above that

$$A'H : H'M = OA : OA' + AM \quad (4)$$

(b) To find such a value for $HH : H'A'$

From (1) we derive

$$\begin{aligned} AM : MA &= OA' + A'M : HM \\ &= OA' : AH \end{aligned} \quad (5)$$

and from (2)
$$A'M : MA = H'M : OA' + AM = H'M : OA' \quad (6)$$

Thus $HA : AO = OA' : AH$,

whence $OH : OA' = OH' : A'H'$,

or $OH : OH = OA' : A'H'$

It follows that

$$HH : OH = OH' : A'H',$$

or $HH : H'A' = OH'^2$

Therefore $HH : HA = OH^2 : HA^2$
 $= AA'^2 : A'M$ by means of (6)

(c) To express the ratios $A'H : H'M$ and $HH' : H'M$ more simply we make

the following construction Produce OA to D so that $OA = AD$ (D will lie beyond H , for $A'V > MA$ and therefore by (5) $OA > AH$)

$$\begin{aligned} \text{Then} \quad AH \cdot H'M &= OA \cdot OA + AM \\ &= AD \cdot DM \end{aligned} \quad (7)$$

Now divide AD at E so that

$$HH \cdot H'A' = AD \cdot DE \quad (8)$$

Thus using equations (8) (7) and the value of $HH \cdot H'A'$ above found we have

$$\begin{aligned} AD \cdot DE &= HH \cdot H'M \\ &= (HH \cdot HA') (AH \cdot HM) \\ &= (AA^2 - AM^2) (AD \cdot DM) \end{aligned}$$

$$\begin{aligned} \text{But} \quad AD \cdot DE &= (DV \cdot DE) (AD \cdot DV) \\ \text{Therefore} \quad MD \cdot DE &= AA^2 - A'M^2 \end{aligned} \quad (9)$$

And D is given since $AD = OA$. Also $AD \cdot DE$ (being equal to $HH \cdot H'M$) is a given ratio. Therefore DE is given.

Hence the problem reduces itself to the problem of dividing $A'D$ into two parts at M so that

$$MD \text{ (a given length)} = \text{(a given area)} \cdot A'M$$

Archimedes adds 'If the problem is propounded in this general form it requires a *διαπισμός* [i.e. it is necessary to investigate the limits of possibility] but if there be added the conditions subsisting in the present case it does not require a *διαπισμός*'

In the present case the problem is

Given a straight line $A'A$ produced to D so that $AA = 2AD$ and given a point E on AD to cut AA in a point M so that

$$AA^2 - A'M^2 = MD \cdot DE$$

And the analysis and synthesis of both problems will be given at the end.

The synthesis of the main problem will be as follows. Let $R \cdot S$ be the given ratio R being less than S . AA being a diameter of a great circle and O the centre produce OA to D so that $OA = AD$ and divide AD in E so that

$$AE \cdot ED = R \cdot S$$

Then cut AA in M so that

$$MD \cdot DE = AA \cdot A'M^2$$

Through M erect a plane perpendicular to AA this plane will then divide the sphere into segments which will be to one another as R to S .

Take H on AA produced and H on AA produced so that

$$OA + AM \cdot AM = HM \cdot MA \quad (1)$$

$$OA + VM \cdot AM = HM \cdot MA \quad (2)$$

We have then to show that

$$HM \cdot MH = R \cdot S \text{ or } AE \cdot ED$$

(a) We first find the value of $HH \cdot HA$ as follows

As was shown in the analysis (b)

$$HH \cdot HA = OH^2$$

$$\text{or} \quad HH \cdot HA = OH^2 - H'V^2$$

$$= AA^2 - A'M^2$$

$$= MD \cdot DE \text{ by construction}$$

¹As Archimedes commentator Eutocius notes we do not find the promise kept in any of the copies. Sir Thomas Heath's translation of Eutocius' note on the matter along with the solutions of Dionysodorus and Dioctes is omitted from this edition.—Ed

(8) Next we have

$$\begin{aligned} H'A' \cdot H'M &= OA \cdot OA + AM \\ &= AD \cdot DM \end{aligned}$$

$$\begin{aligned} \text{Therefore } HH' \cdot HM &= (HH' \cdot H'A') (H'A' \cdot H'M) \\ &= (MD \cdot DE) (AD \cdot DM) \\ &= AD \cdot DE \end{aligned}$$

$$\begin{aligned} \text{whence } HM \cdot MH' &= AE \cdot ED \\ &= R \cdot S \end{aligned}$$

Q.E.D.

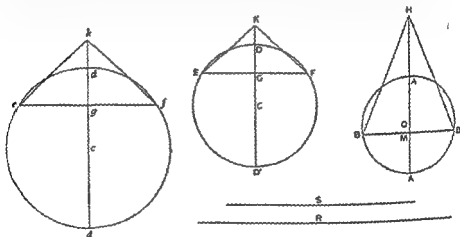
PROPOSITION 5 (PROBLEM)

To construct a segment of a sphere similar to one segment and equal in volume to another

Let ABP' be one segment whose vertex is A and whose base is the circle on BB' as diameter and let DEF be another segment whose vertex is D and whose base is the circle on FF' as diameter. Let AA' , DD' be diameters of the great circles passing through BB' , FF' respectively, and let O , C be the respective centres of the spheres.

Suppose it required to draw a segment similar to DEF and equal in volume to ABB' .

Analysis. Suppose the problem solved, and let def be the required segment, d being the vertex and ef the diameter of the base. Let dd' be the diameter of the sphere which bisects ef at right angles, c the centre of the sphere.



Let M , C , g be the points where BB' , FF' , ef are bisected at right angles by AA' , DD' , dd' respectively and produce OA , CD , cd respectively to H , K , l so that

$$\left. \begin{aligned} OA + A'M \cdot A'M &= HM \cdot MA \\ CD + DG \cdot DG &= KG \cdot GD \\ cd + dg \cdot dg &= lg \cdot gd \end{aligned} \right\},$$

and suppose cones formed with vertices H , K , l and with the same bases as the respective segments. The cones will then be equal to the segments respectively [Prop. 2]

Therefore by hypothesis,

the cone HBB = the cone λef

Hence

(circle on diameter BB) (circle on diameter ef) = λg HM ,

so that $BB^2 \cdot ef^2 = \lambda g \cdot HM$ (1)

But since the segments DEF def are similar so are the cones λEF λef

Therefore $\lambda C \cdot EF = \lambda g \cdot ef$

And the ratio $\lambda G \cdot EF$ is given Therefore the ratio $\lambda g \cdot ef$ is given

Suppose a length R taken such that

$$\lambda g \cdot ef = HM \cdot R \quad (2)$$

Thus R is given

Again since $\lambda g \cdot HM = BB'^2 \cdot ef^2 = ef^2 \cdot R$, by (1) and (2), suppose a length S taken such that

$$ef^2 = BB' \cdot S$$

$$\text{or } BB'^2 \cdot ef^2 = BB' \cdot S$$

$$\text{Thus } BB' \cdot ef = ef \cdot S = S \cdot R,$$

and ef , S are two mean proportionals in continued proportion between BB , R

Synthesis Let ABB DEF be great circles AA' , DD' the diameters bisecting BB' , EF at right angles in M , G respectively, and O , C the centres

Take H , λ in the same way as before and construct the cones HBB λEF , which are therefore equal to the respective segments ABB' , DEI

Let R be a straight line such that

$$\lambda G \cdot EF = HM \cdot R$$

and between BB' R take two mean proportionals ef , S

On ef as base describe a segment of a circle with vertex d and similar to the segment of a circle DEF Complete the circle, and let dd' be the diameter through d and c the centre Conceive a sphere constructed of which def is a great circle and through ef draw a plane at right angles to dd'

Then shall def be the required segment of a sphere

For the segments DEF def of the spheres are similar, like the circular segments DEF def

Produce cd to λ so that

$$cd + d'g \cdot dg = \lambda g \cdot gd$$

The cones λEF , λef are then similar

$$\text{Therefore } \lambda g \cdot ef = \lambda G \cdot EF = HM \cdot R,$$

$$\text{whence } \lambda g \cdot HM = ef \cdot R$$

But since BB ef S λ are in continued proportion

$$BB \cdot ef = BB' \cdot S$$

$$= ef \cdot R$$

$$= \lambda g \cdot HM$$

Thus the bases of the cones HBB , λef are reciprocally proportional to their heights The cones are therefore equal and def is the segment required being equal in volume to the cone λef [Prop 2]

PROPOSITION 6 (PROBLEM)

Given two segments of spheres to find a third segment of a sphere similar to one of the given segments and having its surface equal to that of the other

Let ABB be the segment to whose surface the surface of the required segment is to be equal AB B the great circle whose plane cuts the plane of the

base of the segment ABB' at right angles in BB' . Let AA' be the diameter which bisects BB' at right angles.

Let DEF be the segment to which the required segment is to be similar, $DED'F$ the great circle cutting the base of the segment at right angles in EF . Let DD' be the diameter bisecting EF at right angles in G .

Suppose the problem solved def being a segment similar to DEF and having its surface equal to that of ABB' and complete the figure for def as for DEF , corresponding points being denoted by small and capital letters respectively.

Join AB, DF, df .

Now, since the surfaces of the segments def, ABB' are equal, so are the circles on df, AB as diameters.

[I 42 43]

that is $df = AB$.

From the similarity of the segments DEF, def we obtain

$$d d \quad dg = D D \quad DG,$$

and $dg \quad df = DG \quad DF$,

whence $d'd \quad df = D'D \quad DF$,

or $d'd \quad AB = D'D \quad DF$.

But $AB, D'D, DF$ are all given.

therefore $d'd$ is given.

Accordingly the synthesis is as follows.

Take $d d$ such that

$$d d \quad AB = D D \quad DF \quad (1)$$

Describe a circle on $d d$ as diameter, and conceive a sphere constructed of which this circle is a great circle.

Divide $d d$ at g so that

$$d'g \quad gd = DG \quad GD$$

and draw through g a plane perpendicular to $d d$ cutting off the segment def of the sphere and intersecting the plane of the great circle in ef . The segments def, DEF are thus similar and

$$dg \quad df = DG \quad DF$$

But from above *componendo*

$$d'd \quad dg = D'D \quad DG$$

$$d'd \quad df = D'D \quad DF$$

Therefore *ex aequali*

$$d'd \quad df = D'D \quad DF$$

whence by (1) $df = AB$.

Therefore the segment def has its surface equal to the surface of the segment ABB' [I 42 43], while it is also similar to the segment DEF .

PROPOSITION 7 (PROBLEM)

From a given sphere to cut off a segment by a plane so that the segment may have a given ratio to the cone which has the same base as the segment and equal height.

Let AA' be the diameter of a great circle of the sphere. It is required to draw a plane at right angles to AA' cutting off a segment ABB' , such that the segment ABB' has to the cone ABB' a given ratio.

Analysis

Suppose the problem solved and let the plane of section cut the plane of the great circle in BB' and the diameter AA' in V . Let O be the centre of the sphere.

Produce OA' to H so that

$$OA' + V'A' = A'V = HV \quad (1)$$

Thus the cone HBB' is equal to the segment ABB' .

[Prop 2]

Therefore the given ratio must be equal to the ratio of the cone HBB' to the cone ABB' , i.e. to the ratio $HV/V'A'$.

Hence the ratio $OA' + V'A' : V'A' =$ given and therefore $V'A'$ is given.

διορισμός

Now so that

$$\begin{aligned} OA' + V'A' &> OA' + A'A \\ OA' + V'A' &> OA' + A'A \\ &> 3 \cdot 2 \end{aligned}$$

Thus in order that a solution may be possible it is a necessary condition that the given ratio must be greater than $3 \cdot 2$.

The synthesis proceeds thus.

Let AA' be a diameter of a great circle of the sphere O the centre.

Take a line DE and a point F on it, such that $DE : EF$ is equal to the given ratio being greater than $3 \cdot 2$.

Now since

$$\begin{aligned} OA' + V'A' &= 3 \cdot 2 \\ DE : EF &> OA' + A'A : V'A' \\ DF : FE &> OA' : V'A' \end{aligned}$$

so that

Hence a point M can be found on AA' such that

$$DF : FE = OM : M'A' \quad (2)$$

Through M draw a plane at right angles to AA' intersecting the plane of the great circle in BB' and cutting off from the sphere the segment ABB' .

As before take H on OA' produced such that

$$OA' + AM = AM = HM = VA'$$

Therefore $HM : MA = DE : EF$ by means of (2).

It follows that the cone HBB' or the segment ABB' is to the cone ABB' in the given ratio $DE : EF$.

PROPOSITION 8

If a sphere be cut by a plane not passing through the centre into two segments ABB' $A'B'B'$, of which $A'B'B'$ is the greater, then the ratio (segment ABB')

$$\begin{aligned} &< (\text{surface of } A'B'B')^2 : (\text{surface of } ABB') \\ &\text{but} > (\text{surface of } A'B'B')^3 : (\text{surface of } ABB')^3 \end{aligned}$$

Let the plane of section cut a great circle $ABAB'$ at right angles in BB' , and let AA' be the diameter bisecting BB' at right angles in M .

Let O be the centre of the sphere.

R will fall between O and M

Also, since $AB^2 = DE$, $AR = CD$

Produce OA' to K so that $OA' = A'K$, and produce $A'A$ to H so that

$$A'K : A'M = HA : AM,$$

or *componendo* $A'K + A'M : A'M = HM : MA$ (1)

Thus the cone HBB' is equal to the segment ABB' [Prop 2]

Again, produce CD to F so that $CD = DF$, and the cone FEE' will be equal to the hemisphere DEE' [Prop 2]

Now $AR : RA' > AM : MA'$,

and $AR^2 = \frac{1}{2}AB^2 = \frac{1}{2}AM \cdot AA' = AM \cdot A'K$

Hence

$$AR : RA' + RA^2 > AM : MA' + AM \cdot A'K,$$

or $AA' : AR > AM : MK$

$$> HM : A'M, \text{ by (1)}$$

Therefore $AA' : A'M > HM : AR$,

or $AB^2 : BM^2 > HM : AR$,

i.e. $AR^2 : BM^2 > HM : 2AR$ since $AB^2 = 2AR$,

$$> HM : CF$$

Thus since $AR = CD$, or CE ,

$$(\text{circle on diam } EE) (\text{circle on diam } BB') > HM : CF$$

It follows that

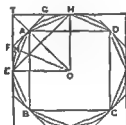
$$(\text{the cone } FEE') > (\text{the cone } HBB'),$$

and therefore the hemisphere DEE' is greater in volume than the segment ABB'

MEASUREMENT OF A CIRCLE

PROPOSITION 1

The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle



Let $ABCD$ be the given circle K the triangle described

Then if the circle is not equal to K it must be either greater or less

I If possible let the circle be greater than K

Inscribe a square $ABCD$ bisect the arcs AB BC CD DA then bisect (if necessary) the halves and so on until the sides of the inscribed polygon whose angular points are the points of division subtend segments whose sum is less than the excess of the area of the circle over K

Thus the area of the polygon is greater than K

Let AE be any side of it and ON the perpendicular on AE from the centre O

Then ON is less than the radius of the circle and therefore less than one of the sides about the right angle in K Also the perimeter of the polygon is less than the circumference of the circle and less than the other side about the right angle in K

Therefore the area of the polygon is less than K which is inconsistent with the hypothesis

Thus the area of the circle is not greater than K

II If possible let the circle be less than K

Circumscribe a square and let two adjacent sides touching the circle in E H meet in T Bisect the arcs between adjacent points of contact and draw the

tangents at the points of bisection. Let A be the middle point of the arc EH , and FAG the tangent at A .

Then the angle TAG is a right angle.

Therefore $\begin{aligned}
TG &> GA \\
&> GH
\end{aligned}$

It follows that the triangle FTG is greater than half the area $TEAH$.

Similarly, if the arc AH be bisected and the tangent at the point of bisection be drawn, it will cut off from the area GAH more than one-half.

Thus, by continuing the process we shall ultimately arrive at a circumscribed polygon such that the spaces intercepted between it and the circle are together less than the excess of A over the area of the circle.

Thus the area of the polygon will be less than A .

Now, since the perpendicular from O on any side of the polygon is equal to the radius of the circle while the perimeter of the polygon is greater than the circumference of the circle it follows that the area of the polygon is greater than the triangle A , which is impossible.

Therefore the area of the circle is not less than A .

Since then the area of the circle is neither greater nor less than A , it is equal to it.

PROPOSITION 2

The area of a circle is to the square on its diameter as 11 to 14¹.

PROPOSITION 3

The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{4}$ but greater than $3\frac{1}{5}$.

I let AB be the diameter of any circle O its centre AC the tangent at A and let the angle AOC be one-third of a right angle.

Then $OA : AC [= \sqrt{3} : 1] > 265 : 153$ (1)

and $OC : CA [= 2 : 1] = 306 : 153$ (2)

First draw OD bisecting the angle COA and meeting AC in D .

Now $CO : OI = CD : DI$ [Eucl VI 3]

so that $[CO + OI : OI = CA : DI]$ or

$$CO + OI : CA = OA : AD$$

Therefore [by (1) and (2)]

$$OA : AD > 571 : 153 \quad (3)$$

The text of this proposition is not satisfactory and Archimedes cannot have placed it before Proposition 3 as the approximation depends upon the result of that proposition.

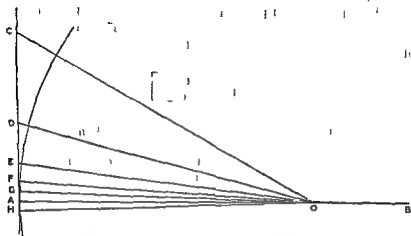
In view of the interesting questions arising out of the arithmetical content of this proposition of Archimedes it is necessary in reproducing it to distinguish carefully the actual steps set out in the text as we have it from the intermediate steps (mostly supplied by Eutocius) which it is convenient to put in for the purpose of making the proof easier to follow. Accordingly all the steps not actually appearing in the text have been enclosed in square brackets in order that it may be clearly seen how far Archimedes omits actual calculations and only gives results. It will be observed that he gives two fractional approximations to $\sqrt{3}$ (one being less and the other greater than the real value) without any explanation as to how he arrived at them and in like manner approximations to the square roots of several large numbers which are not complete squares are merely stated.

Hence

$$\begin{aligned} OD^2 AD &= (OA + AD) AD^2 \\ &> (571 + 153^2) 153 \\ &> 349150 \quad 23409, \end{aligned}$$

so that

$$OD DA > 591\frac{1}{2} 153 \quad (4)$$



Secondly let OE bisect the angle AOD, meeting AD in E

[Then $DO OA = DE EA$]

so that $DO + OA = DE + EA$

Therefore $OA AE [> (591\frac{1}{2} + 571) 153, \text{ by (3) and (4)}]$
 $> 1162\frac{1}{2} 153$

(5)

[It follows that

$$\begin{aligned} OE^2 EA^2 &> \{ (1162\frac{1}{2})^2 + 153^2 \} 153^2 \\ &> (135034\frac{1}{4} + 23409) 23409 \\ &> 1373943\frac{1}{4} 23409 \end{aligned}$$

Thus $OE EA > 1172\frac{1}{2} 153$

(6)

Thirdly let OF bisect the angle AOE and meet AE in F

We thus obtain the result [corresponding to (3) and (5) above] that

$$\begin{aligned} OA AF [> (1162\frac{1}{2} + 1172\frac{1}{2}) 153] \\ > 2334\frac{1}{2} 153 \end{aligned}$$

(7)

[Therefore $OF^2 FA^2 > \{ (2334\frac{1}{2})^2 + 153^2 \} 153^2$
 $> 5472132\frac{1}{4} 23409]$

Thus $OF FA > 2339\frac{1}{2} 153$

(8)

Fourthly let OG bisect the angle AOF meeting AF in G

We have then

$$\begin{aligned} OA AG [> (2334\frac{1}{2} + 2339\frac{1}{2}) 153 \text{ by means of (7) and (8)}] \\ > 4673\frac{1}{2} 153 \end{aligned}$$

Now the angle AOC which is one third of a right angle has been bisected four times and it follows that

$$\angle AOG = \frac{1}{16} \text{ (a right angle)}$$

Make the angle AOH on the other side of O equal to the angle AOC and let GA produced meet OH in H

Then $\angle GOH = \frac{1}{16} \text{ (a right angle)}$

Thus GH is one side of a regular polygon of 96 sides circumscribed to the given circle

And, since $OA \ AG > 4673\frac{1}{2} \ 153$,
while $AB = 2OA$, $GH = 2AG$,¹¹

it follows that

$$AB \text{ (perimeter of polygon of 96 sides)} [> 4673\frac{1}{2} \ 153 \times 96] \\ > 4673\frac{1}{2} \ 14688$$

But

$$\frac{14688}{4673\frac{1}{2}} = 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}} \\ \left[< 3 + \frac{667\frac{1}{2}}{4672\frac{1}{2}} \right] \\ < 3\frac{1}{2}$$

Therefore the circumference of the circle (being less than the perimeter of the polygon) is *a fortiori* less than $3\frac{1}{2}$ times the diameter AB

II Next let AB be the diameter of a circle, and let AC , meeting the circle in C make the angle CAB equal to one-third of a right angle Join BC

Then $AC \ CB [= \sqrt{3} \ 1] < 1351 \ 780$

First let AD bisect the angle BAC and meet BC in d and the circle in D
Join BD

Then

$$\angle BAD = \angle DAC \\ = \angle CBD$$

and the angles at D , C are both right angles

It follows that the triangles ADB $\{ACd\}$ BDd are similar

$$\text{Therefore} \quad AD \ DB = BD \ Dd \\ [= AC \ Cd] \\ = AB \ Bd \\ = AB + AC \ Bd + Cd \\ = AB + AC \ BC$$

[Eucl vi 3]

or $BA + AC \ BC = AD \ DB$
{But $AC \ CB < 1351 \ 780$, from above,
while $Bd \ BC = 2 \ 1$

$$= 1500 \ 780 \} \quad (1)$$

$$\text{Therefore} \quad AD \ DB < 2911 \ 780 \\ \text{Hence} \quad AB^2 \ BD^2 < (2911^2 + 780^2) \ 780 \\ < 9082321 \ 608100 \}$$

$$\text{Thus} \quad AB \ BD < 3013\frac{1}{2} \ 780 \quad (2)$$

Secondly let BE bisect the angle BAD , meeting the circle in E and let BE be joined

Then we prove in the same way as before that

$$1E \ EB [= BA + AD \ BD] \\ < (3013\frac{1}{2} + 2911) \ 780 \text{ by (1) and (2)} \\ < 5924\frac{1}{2} \ 780 \\ < 5924\frac{1}{2} \times \frac{1}{2} \ 780 \times \frac{1}{2} \\ < 1823 \ 210 \quad (3)$$

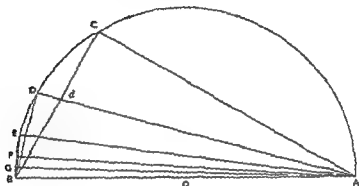
$$\text{Hence} \quad AB^2 \ BE^2 < (1823 + 210) \ 210^2 \\ < 3380929 \ 57600 \} \quad (4)$$

Therefore $1B \ BE < 1838\frac{1}{2} \ 210$

Thirdly, let AF bisect the angle BAE meeting the circle in F

Thus

$$\begin{aligned}
 AF \cdot FB & [= BA + AE \cdot BE] \\
 & < 3661 \frac{1}{11} \times 240 \text{ by (3) and (4)} \\
 & < 3661 \frac{1}{11} \times 1 \frac{1}{11} \cdot 240 \times 1 \frac{1}{11} \\
 & < 1007 \cdot 66
 \end{aligned}
 \tag{5}$$



[It follows that

$$\begin{aligned}
 AB^2 \cdot BF^2 & < (1007^2 + 66^2) \cdot 66^2 \\
 & < 1018405 \cdot 4356]
 \end{aligned}
 \tag{6}$$

Therefore

$$AB \cdot BF < 1009 \frac{1}{2} \cdot 66$$

Fourthly, let the angle BAF be bisected by AG meeting the circle in G

Then

$$\begin{aligned}
 AG \cdot GB & [= BA + AF \cdot BF] \\
 & < 2016 \frac{1}{2} \cdot 66, \text{ by (5) and (6)}
 \end{aligned}$$

[And

$$\begin{aligned}
 AB^2 \cdot BG^2 & < \{(2016 \frac{1}{2})^2 + 66^2\} \cdot 66^2 \\
 & < 4069234 \frac{1}{4} \cdot 4356]
 \end{aligned}$$

Therefore

$$AB \cdot BG < 2017 \frac{1}{4} \cdot 66$$

whence

$$BG \cdot AB > 66 \cdot 2017 \frac{1}{4} \tag{7}$$

[Now the angle BAG which is the result of the fourth bisection of the angle BAC or of one-third of a right angle is equal to one-forty eighth of a right angle

Thus the angle subtended by BG at the centre is

$$\frac{1}{48} \text{ (a right angle)}]$$

Therefore BG is a side of a regular inscribed polygon of 96 sides

It follows from (7) that

$$\begin{aligned}
 (\text{perimeter of polygon}) \cdot AB & [> 96 \times 66 \cdot 2017 \frac{1}{4}] \\
 & > 6336 \cdot 2017 \frac{1}{4}
 \end{aligned}$$

And

$$\frac{6336}{2017 \frac{1}{4}} > 3 \frac{1}{4}$$

Much more then is the circumference of the circle greater than $3 \frac{1}{4}$ times the diameter

Thus the ratio of the circumference to the diameter

$$< 3 \frac{1}{2} \text{ but } > 3 \frac{1}{4}$$

ON CONOIDS AND SPHEROIDS

INTRODUCTION¹

"ARCHIMEDES to Dositheus greeting

"In this book I have set forth and send you the proofs of the remaining theorems not included in what I sent you before and also of some others discovered later which, though I had often tried to investigate them previously, I had failed to arrive at because I found their discovery attended with some difficulty. And this is why even the propositions themselves were not published with the rest. But afterwards, when I had studied them with greater care, I discovered what I had failed in before.

"Now the remainder of the earlier theorems were propositions concerning the right angled conoid [paraboloid of revolution] but the discoveries which I have now added relate to an obtuse-angled conoid [hyperboloid of revolution] and to spheroidal figures some of which I call *oblong* and others *flat*."

I Concerning the *right-angled conoid* it was laid down that, if a section of a right-angled cone [a parabola] be made to revolve about the diameter [axis] which remains fixed and return to the position from which it started, the figure comprehended by the section of the right angled cone is called a *right-angled conoid* and the diameter which has remained fixed is called its *axis*, while its *vertex* is the point in which the axis meets the surface of the conoid. And if a plane touch the right-angled conoid, and another plane drawn parallel to the tangent plane cut off a segment of the conoid the *base* of the segment cut off is defined as the portion intercepted by the section of the conoid on the cutting plane the *vertex* [of the segment] as the point in which the first plane touches the conoid and the *axis* [of the segment] as the portion cut off within the segment from the line drawn through the vertex of the segment parallel to the axis of the conoid.

'The questions propounded for consideration were'

- (1) why if a segment of the right angled conoid be cut off by a plane at right angles to the axis will the segment so cut off be half as large again as the cone which has the same base as the segment and the same axis, and
- (2) 'why if two segments be cut off from the right angled conoid by planes drawn in any manner will the segments so cut off have to one another the duplicate ratio of their axes

II "Respecting the *obtuse-angled conoid* we lay down the following premises. If there be in a plane a section of an obtuse-angled cone [a hyperbola], its

¹The whole of this introductory matter including the definitions is translated literally from the Greek text in order that the terminology of Archimedes may be faithfully represented. When this has once been set out nothing will be lost by returning to modern phraseology and notation. These will accordingly be employed as usual, when we come to the actual propositions of the treatise.

diameter [axis], and the nearest lines to the section of the obtuse-angled cone [i.e. the asymptotes of the hyperbola], and if, the diameter [axis] remaining fixed, the plane containing the aforesaid lines be made to revolve about it and return to the position from which it started the nearest lines to the section of the obtuse-angled cone [the asymptotes] will clearly comprehend an isosceles cone whose vertex will be the point of concurrence of the nearest lines and whose axis will be the diameter [axis] which has remained fixed. The figure comprehended by the section of the obtuse-angled cone is called an *obtuse-angled conoid* [hyperboloid of revolution] its *axis* is the diameter which has remained fixed, and its *vertex* the point in which the axis meets the surface of the conoid. The cone comprehended by the nearest lines to the section of the obtuse-angled cone is called [the cone] *enveloping the conoid* and the straight line between the vertex of the conoid and the vertex of the cone enveloping the conoid is called [the line] *adjacent to the axis*. And if a plane touch the obtuse-angled conoid and another plane drawn parallel to the tangent plane cut off a segment of the conoid, the base of the segment so cut off is defined as the portion intercepted by the section of the conoid on the cutting plane the *vertex* [of the segment] is the point of contact of the plane which touches the conoid the *axis* [of the segment] as the portion cut off within the segment from the line drawn through the vertex of the segment and the vertex of the cone enveloping the conoid, and the straight line between the said vertices is called *adjacent to the axis*.

"Right angled conoids are all similar, but of obtuse-angled conoids let those be called similar in which the cones enveloping the conoids are similar

The following questions are propounded for consideration"

(1) 'why, if a segment be cut off from the obtuse-angled conoid by a plane at right angles to the axis, the segment so cut off has to the cone which has the same base as the segment and the same axis the ratio which the line equal to the sum of the axis of the segment and three times the line adjacent to the axis bears to the line equal to the sum of the axis of the segment and twice the line adjacent to the axis and'

(2) 'why, if a segment of the obtuse-angled conoid be cut off by a plane not at right angles to the axis, the segment so cut off will bear to the figure which has the same base as the segment and the same axis, being a segment of a cone, the ratio which the line equal to the sum of the axis of the segment and three times the line adjacent to the axis bears to the line equal to the sum of the axis of the segment and twice the line adjacent to the axis

III Concerning spheroidal figures we lay down the following premisses. If a section of an acute-angled cone [ellipse] be made to revolve about the greater diameter [major axis] which remains fixed and return to the position from which it started, the figure comprehended by the section of the acute-angled cone is called an *oblong spheroid*. But if the section of the acute-angled cone revolve about the lesser diameter [minor axis] which remains fixed and return to the position from which it started the figure comprehended by the section of the acute-angled cone is called a *flat spheroid*. In either of the spheroids the *axis* is defined as the diameter [axis] which has remained fixed the *vertex* as the point in which the axis meets the surface of the spheroid the *centre* as the middle point of the axis and the *diameter* as the line drawn through the centre at right angles to the axis. And if parallel planes touch without cutting, either

of the spheroidal figures, and if another plane be drawn parallel to the tangent planes and cutting the spheroid, the base of the resulting segments = defined as the portion intercepted by the section of the spheroid on the cutting plane, their *vertices* as the points in which the parallel planes touch the spheroid and their *axes* as the portions cut off within the segments from the straight line joining their vertices. And that the planes touching the spheroid meet its surface at one point only and that the straight line joining the points of contact passes through the centre of the spheroid we shall prove. Those spheroidal figures are called *similar* in which the axes have the same ratio to the 'diameters'. And let segments of spheroidal figures and conoids be called *similar* if they are cut off from similar figures and have their bases similar, while their axes, being either at right angles to the planes of the bases or making equal angles with the corresponding diameters [axes] of the bases have the same ratio to one another as the corresponding diameters [axes] of the bases.

"The following questions about spheroids are propounded for consideration

(1) 'why if one of the spheroidal figures be cut by a plane through the centre at right angles to the axis each of the resulting segments will be double of the cone having the same base as the segment and the same axis, while if the plane of section be at right angles to the axis without passing through the centre (a) the greater of the resulting segments will bear to the cone which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the straight line which is the axis of the spheroid and the axis of the lesser segment bears to the axis of the lesser segment and (b) the lesser segment bears to the cone which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the straight line which is the axis of the spheroid and the axis of the greater segment bears to the axis of the greater segment

(2) why if one of the spheroids be cut by a plane passing through the centre but not at right angles to the axis each of the resulting segments will be double of the figure having the same base as the segment and the same axis and consisting of a segment of a cone

(3) But if the plane cutting the spheroid be neither through the centre nor at right angles to the axis (a) the greater of the resulting segments will have to the figure which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the line joining the vertices of the segments and the axis of the lesser segment bears to the axis of the lesser segment and (b) the lesser segment will have to the figure with the same base as the segment and the same axis the ratio which the line equal to the sum of half the line joining the vertices of the segments and the axis of the greater segment bears to the axis of the greater segment. And the figure referred to is in these cases also a segment of a cone

'When the aforesaid theorems are proved there are discovered by means of them many theorems and problems

'Such for example are the theorems

(1) 'that similar spheroids and similar segments both of spheroidal figures and conoids have to one another the triplicate ratio of their axes and

(2) 'that in equal spheroidal figures the squares on the diameters' are reciprocally proportional to the axes and if in spheroidal figures the squares on

the 'diameters' are reciprocally proportional to the axes, the spheroids are equal

Such also is the problem, From a given spheroidal figure or conoid to cut off a segment by a plane drawn parallel to a given plane so that the segment cut off is equal to a given cone or cylinder or to a given sphere

"After prefixing therefore the theorems and directions which are necessary for the proof of them, I will then proceed to expound the propositions themselves to you Farewell"

DEFINITIONS

'If a cone be cut by a plane meeting all the sides [generators] of the cone the section will be either a circle or a section of an acute-angled cone [an ellipse] If then the section be a circle, it is clear that the segment cut off from the cone towards the same parts as the vertex of the cone will be a cone But if the section be a section of an acute-angled cone [an ellipse] let the figure cut off from the cone towards the same parts as the vertex of the cone be called a *segment of a cone* Let the base of the segment be defined as the plane comprehended by the section of the acute-angled cone its *vertex* as the point which is also the vertex of the cone, and its *axis* as the straight line joining the vertex of the cone to the centre of the section of the acute-angled cone

'And if a cylinder be cut by two parallel planes meeting all the sides [generators] of the cylinder the sections will be either circles or sections of acute-angled cones [ellipses] equal and similar to one another If then the sections be circles it is clear that the figure cut off from the cylinder between the parallel planes will be a cylinder But, if the sections be sections of acute-angled cones [ellipses] let the figure cut off from the cylinder between the parallel planes be called a *frustum of a cylinder* And let the bases of the frustum be defined as the planes comprehended by the sections of the acute-angled cones [ellipses] and the *axis* as the straight line joining the centres of the sections of the acute-angled cones so that the axis will be in the same straight line with the axis of the cylinder'

LEMMA

If in an ascending arithmetical progression consisting of the magnitudes A_1, A_2, \dots, A_n the common difference be equal to the least term A_1 , then

$$n A_n < 2(A_1 + A_2 + \dots + A_n),$$

$$\text{and} \quad > 2(A_1 + A_2 + \dots + A_{n-1})$$

[The proof of this is given incidentally in the treatise *On Spirals* Prop. 11 By placing lines side by side to represent the terms of the progression and then producing each so as to make it equal to the greatest term Archimedes gives the equivalent of the following proof

$$\begin{aligned} \text{If} \quad S_n &= A_1 + A_2 + \dots + A_{n-1} + A_n \\ \text{we have also} \quad S &= A_n + A_{n-1} + A_{n-2} + \dots + A_1 \\ \text{And} \quad A_1 + A_{n-1} &= A_2 + A_{n-2} = \dots = A_n \end{aligned}$$

$$\begin{aligned} \text{Therefore} \quad 2S &= (n+1)A_n, \\ \text{whence} \quad n A_n &< 2S \\ \text{and} \quad n A_1 &> 2S_{n-1} \end{aligned}$$

Thus if the progression is a $2a, na$

$$S = \frac{n(n+1)}{2}a,$$

and
but

$$n^2a < 2S \\ > 2S_{n-1}]$$

PROPOSITION I

If $A_1, B_1, C_1, \dots, K_1$ and $A_2, B_2, C_2, \dots, K_2$ be two series of magnitudes such that

$$\left. \begin{aligned} A_1, B_1 &= A, B_2 \\ B_1, C_1 &= B, C_2 \text{ and so on} \end{aligned} \right\} \quad (\alpha)$$

and if $A_3, B_3, C_3, \dots, K_3$ and $A_4, B_4, C_4, \dots, K_4$ be two other series such that

$$\left. \begin{aligned} A_3, A_4 &= A_2, A_4 \\ B_3, B_4 &= B_2, B_4 \text{ and so on} \end{aligned} \right\} \quad (\beta)$$

$$\text{then} \quad \begin{aligned} (A_1 + B_1 + C_1 + \dots + K_1) & (A_2 + B_2 + C_2 + \dots + K_2) \\ &= (A_3 + B_3 + C_3 + \dots + K_3) (A_4 + B_4 + \dots + K_4) \end{aligned}$$

The proof is as follows

$$\begin{aligned} \text{Since} \quad & 1_2, A_1 = A_4, A_2 \\ \text{and} \quad & A_1, B_1 = A_2, B_2 \\ \text{while} \quad & B_1, B_2 = B_2, B_4 \\ \text{we have ex aequali} \quad & A_2, B_2 = A_4, B_4 \\ \text{Similarly} \quad & B_3, C_3 = B_4, C_4 \text{ and so on} \end{aligned} \quad (\gamma)$$

Again it follows from equations (α) that

$$A_1, A_2 = B_1, B_2 = C_1, C_2 = \dots$$

Therefore

$$\begin{aligned} A_1, A_4 &= (A_1 + B_1 + C_1 + \dots + K_1) (A_2 + B_2 + \dots + K_2), \\ \text{or} \quad & (1_1 + B_1 + C_1 + \dots + K_1) A_1 = (A_2 + B_2 + C_2 + \dots + K_2) A_4 \\ \text{and} \quad & A_1, 1_2 = A_2, A_4 \end{aligned}$$

while from equations (γ) it follows in like manner that

$$A_2 (A_3 + B_3 + C_3 + \dots + K_3) = A_4 (A_4 + B_4 + C_4 + \dots + K_4)$$

By the last three equations ex aequali

$$\begin{aligned} (1_1 + B_1 + C_1 + \dots + K_1) (A_2 + B_2 + C_2 + \dots + K_2) \\ = (1_2 + B_2 + C_2 + \dots + K_2) (A_4 + B_4 + C_4 + \dots + K_4) \end{aligned}$$

Con. If any terms in the third and fourth series corresponding to terms in the first and second be left out the result is the same. For example if the last terms A_3, A_4 are absent

$$\begin{aligned} (1_1 + B_1 + C_1 + \dots + K_1) (A_2 + B_2 + C_2 + \dots + I_2) \\ = (A_2 + B_2 + C_2 + \dots + K_2) (A_4 + B_4 + C_4 + \dots + I_4) \end{aligned}$$

where I immediately precedes A in each series

LEMMA TO PROPOSITION 2

[On Spirals Prop 10]

If $1, 4, A_1, \dots, 1$ be n lines forming an ascending arithmetical progression in which the common difference is equal to the least term A_1 then

$$(n+1)1^2 + 1_1(1_1 + A_2 + 1_2 + \dots + A) = 3(A_1^2 + A_2^2 + A_3^2 + \dots + A^2)$$

Let the lines $1, 1_{n-1}, 1, \dots, A_1$ be placed in a row from left to right. Produce $1, 1_{n-1}, 1, \dots, 1$ until they are each equal to A so that the parts produced are respectively equal to $A_1, 1_2, \dots, A_{n-1}$.

Taking each line successively we have

$$21^2 = 2A^2$$

$$(1_1 + 1_{n-1})^2 = 1_1^2 + A^2_{n-1} + 21_1 A_{n-1}$$

$$(1 + 1_{n-2})^2 = 1^2 + A^2_{n-2} + 2A_2 1_{n-2}$$

$$(A_{-1} + A_1)^2 = A_{-1}^2 + A_1^2 + 2A_{-1}A_1$$

And, by addition

$$\begin{aligned} (n+1)A^2 &= 2(A_1^2 + A_2^2 + \dots + A_n^2) \\ &+ 2A_1A_{-1} + 2A_2A_{-2} + \dots + 2A_{n-1}A_1 \\ \text{Therefore in order to obtain the re-} \\ \text{quired result, we have to prove that} \\ 2(A_1A_{-1} + A_2A_{-2} + \dots + A_{n-1}A_1) &+ \\ A_1(A_1 + A_2 + A_3 + \dots + A_n) &= \\ = A_1^2 + A_2^2 + \dots + A_n^2 \quad (\alpha) \end{aligned}$$

Now

$$\begin{aligned} 2A_2A_{-2} &= A_1A_{-1} \text{ because } A_2 = 2A_1 \\ 2A_3A_{-3} &= A_1A_{-1} \text{ because } A_3 = 3A_1 \end{aligned}$$

$$2A_{-1}A_1 = A_1 \cdot 2(n-1)A_1$$

It follows that

$$\begin{aligned} 2(A_1A_{-1} + A_2A_{-2} + \dots + A_{n-1}A_1) &+ \\ A_1(A_1 + A_2 + \dots + A_n) &= A_1 \{ A_{-1} + 3A_{-2} + \dots \\ &+ 5A_{-3} + \dots + (2n-1)A_1 \} \end{aligned}$$

And this last expression can be proved to be equal to

$$A_1^2 + A_2^2 + \dots + A_n^2$$

For

$$\begin{aligned} A_{-1} &= A_1(n-1) \\ &= A_1 \{ A_{-1} + (n-1)A_1 \} \\ &= A_1 \{ A_{-1} + 2(A_{-2} + A_{-3} + \dots + A_1) \} \\ &\text{because } (n-1)A_1 = A_{-1} + A_1 \\ &\quad + A_{-2} + A_2 \\ &\quad + \dots \\ &\quad + A_1 + A_{-1} \end{aligned}$$

$$\text{Similarly } A_{-2} = A_1 \{ A_{-2} + 2(A_{-3} + A_{-4} + \dots + A_1) \}$$

$$A_2^2 = A_1^2(A_2 + 2A_1)$$

$$A_1^2 = A_1^2A_1$$

whence by addition

$$A_1^2 + A_2^2 + A_3^2 + \dots + A_n^2 = A_1^2 \{ A_{-1} + 3A_{-2} + 5A_{-3} + \dots + (2n-1)A_1 \}$$

Thus the equation marked (α) above is true and it follows that

$$(n+1)A^2 + A_1(A_1 + A_2 + \dots + A_n) = 3(A_1^2 + A_2^2 + \dots + A_n^2)$$

Cor 1 From this it is evident that

$$n-1 < 3(A_1^2 + A_2^2 + \dots + A_n^2) \quad (1)$$

Also $A_{-1} = A_1 \{ A_{-1} + 2(A_{-2} + A_{-3} + \dots + A_1) \}$ as above

so that $A_{-1} > A_1(A_{-2} + A_{-3} + \dots + A_1)$

and therefore

$$A_{-1} + A_1(A_2 + A_3 + \dots + A_n) < 2A_{-1}^2$$

It follows from the proposition that

$$n-1 > 3(A_1^2 + A_2^2 + \dots + A_n^2) \quad (2)$$

Cor 2 All these results will hold if we substitute similar figures for squares on all the lines for similar figures are in the duplicate ratio of their sides

PROPOSITION 2

If A_1, A_2, \dots, A_n be any number of areas such that

$$A_1 = ax + x$$

$$A_2 = a \cdot 2x + (2x)^2,$$

$$A_3 = a \cdot 3x + (3x)^2$$

$$A_n = a \cdot nx + (nx)^2$$

then $n A_1 (A_1 + A_2 + \dots + A_n) < (a + nx) \left(\frac{a}{2} + \frac{nx}{3} \right),$

and $n A_n (A_1 + A_2 + \dots + A_{n-1}) > (a + nx) \left(\frac{a}{2} + \frac{nx}{3} \right)$

For by the Lemma immediately preceding Prop 1,

$$n a n x < (a x + a \cdot 2x + \dots + a \cdot n x) \\ > 2(a x + a \cdot 2x + \dots + a \cdot (n-1)x)$$

and

Also by the Lemma preceding this proposition

$$n (nx)^2 < 3 \{x^2 + (2x)^2 + (3x)^2 + \dots + (nx)^2\} \\ > 3 \{x + (2x)^2 + \dots + (n-1)x^2\}$$

and

Hence

$$\frac{a n^2 x}{2} + \frac{n(n x)}{3} < [(a x + x^2) + \{a \cdot 2x + (2x)^2\} + \dots + \{a \cdot nx + (nx)^2\}],$$

and

$$> [(a x + x) + \{a \cdot 2x + (2x)^2\} + \dots + \{a \cdot (n-1)x + (n-1)x^2\}],$$

or

$$\frac{a n^2 x}{2} + \frac{n(n x)}{3} < A_1 + A_2 + \dots + A_n$$

and

$$> A_1 + A_2 + \dots + A_{n-1}$$

It follows that

$$n-1 (A_1 + A_2 + \dots + A_n) < n \{a \cdot nx + (nx)^2\} \left\{ \frac{a n x}{2} + \frac{n(n x)}{3} \right\}$$

or

$$n-1 (A_1 + A_2 + \dots + A_n) < (a + nx) \left(\frac{a}{2} + \frac{nx}{3} \right)$$

also

$$n-1 (A_1 + A_2 + \dots + A_{n-1}) > (a + nx) \left(\frac{a}{2} + \frac{nx}{3} \right)$$

PROPOSITION 3

(1) If TP, TP be two tangents to any conic meeting in T and if Qq, Qq be any two chords parallel respectively to TP, TP and meeting in O then

$$QO \cdot Oq \cdot QO \cdot Oq = TP \cdot TP^2$$

And this is proved in the elements of conics ²

(2) If QQ be a chord of a parabola bisected in I by the diameter PI , and if PI be of constant length then the areas of the triangle PQQ and of the segment PQQ are both constant whatever be the direction of QQ

Let IBB be the particular segment of the parabola whose vertex is I so that BB is bisected perpendicularly by the axis at the point H where $4HI = PI$

Draw QD perpendicular to PI

²In the treatise on conics by Aristaeus and Euclid

Let p be the parameter of the principal ordinates and let p be another line of such length that

$$QV \cdot QD = p \cdot p,$$

it will then follow that p is equal to the parameter of the ordinates to the diameter PV i.e. those which are parallel to QV .

"For this is proved in the conics."

$$\text{Thus } QV^2 = p \cdot PV$$

$$\text{And } BH^2 = p \cdot AH \text{ while } AH = PV$$

$$\text{Therefore } QV^2 \cdot BH = p \cdot p$$

$$\text{But } QV^2 \cdot QD = p \cdot p,$$

$$\text{hence } BH = QD$$

$$\text{Thus } BH \cdot AH = QD \cdot PV$$

$$\text{and therefore } \triangle ABB = \triangle PQQ,$$

that is the area of the triangle PQQ is constant so long as PV is of constant length

Hence also the area of the segment PQQ is constant under the same conditions for the segment is equal to $\frac{1}{2} \triangle PQQ$ [*Quadrature of the Parabola, Prop 17 or 24*]

PROPOSITION 4

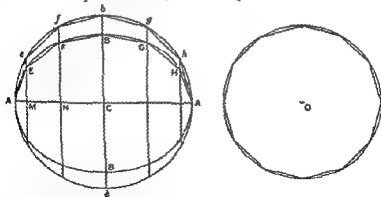
The area of any ellipse is to that of the auxiliary circle as the minor axis to the major

Let AA' be the major and BB' the minor axis of the ellipse and let BB' meet the auxiliary circle in b, b'

Suppose O to be such a circle that

$$(\text{circle } A b A' b') \quad O = CA \cdot CB$$

Then shall O be equal to the area of the ellipse



For if not O must be either greater or less than the ellipse

I. If possible let O be greater than the ellipse

We can then inscribe in the circle O an equilateral polygon of $4n$ sides such that its area is greater than that of the ellipse [cf. *On the Sphere and Cylinder* I. 6]

The theorem which is here assumed by Archimedes as known is easily deduced from Apollonius I. 49

Let this be done and inscribe in the auxiliary circle of the ellipse the polygon $AefbghA'$ similar to that inscribed in O . Let the perpendiculars eM, fN , on AA' meet the ellipse in E, F , respectively. Join AF, EF, FB .

Suppose that P denotes the area of the polygon inscribed in the auxiliary circle and P' that of the polygon inscribed in the ellipse.

Then, since all the lines eM, fN , are cut in the same proportions at E, F ,

$$eM : EM = fN : FN = bC : BC$$

the pairs of triangles, as eAM, EAM and the pairs of trapeziums, as $eMNf, EMNf$, are all in the same ratio to one another as bC to BC , or as CA to CB .

Therefore, by addition

$$P' : P = CA : CB$$

Now P (polygon inscribed in O)

$$= (\text{circle } AbA'b') : O$$

$$= CA : CB, \text{ by hypothesis}$$

Therefore P' is equal to the polygon inscribed in O .

But this is impossible because the latter polygon is by hypothesis greater than the ellipse, and *a fortiori* greater than P .

Hence O is not greater than the ellipse.

II. If possible, let O be less than the ellipse.

In this case we inscribe in the ellipse a polygon P with $4n$ equal sides such that $P > O$.

Let the perpendiculars from the angular points on the axis AA' be produced to meet the auxiliary circle and let the corresponding polygon (P') in the circle be formed.

Inscribe in O a polygon similar to P .

Then $P' : P = CA : CB$

$$= (\text{circle } AbA'b') : O, \text{ by hypothesis,}$$

$$= P' : (\text{polygon inscribed in } O)$$

Therefore the polygon inscribed in O is equal to the polygon P , which is impossible because $P > O$.

Hence O being neither greater nor less than the ellipse is equal to it, and the required result follows.

PROPOSITION 5

If AA', BB' be the major and minor axis of an ellipse respectively, and if d be the diameter of any circle then

$$(\text{area of ellipse}) : (\text{area of circle}) = AA' : BB' : d^2$$

For

$$(\text{area of ellipse}) : (\text{area of auxiliary circle}) = BB' : AA' \quad [\text{Prop. 4}]$$

$$= AA' : BB' : AA'^2$$

And

$$(\text{area of aux. circle}) : (\text{area of circle with diam. } d) = AA'^2 : d^2$$

Therefore the required result follows *ex aequali*.

PROPOSITION 6

The areas of ellipses are as the rectangles under their axes.

Thus follows at once from Prop. 5.

COR. The areas of similar ellipses are as the squares of corresponding axes.

PROPOSITION 7

Given an ellipse with centre C , and a line CO drawn perpendicular to its plane it is possible to find a circular cone with vertex O and such that the given ellipse is a section of it [or, in other words to find the circular sections of the cone with vertex O passing through the circumference of the ellipse]

Conceive an ellipse with BB' as its minor axis and lying in a plane perpendicular to that of the paper. Let CO be drawn perpendicular to the plane of the ellipse and let O be the vertex of the required cone. Produce OB OC OB' and in the same plane with them draw BED meeting OC OB produced in E , D respectively and in such a direction that

$$BE \cdot ED \cdot EO^2 = CA^2 \cdot CO,$$

where CA is half the major axis of the ellipse

And this is possible, since

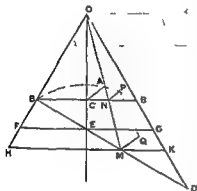
$$BE \cdot ED \cdot EO > BC \cdot CB \cdot CO^2$$

[Both the construction and this proposition are assumed as known]

Now conceive a circle with BD as diameter lying in a plane at right angles to that of the paper, and describe a cone with this circle for its base and with vertex O

We have therefore to prove that the given ellipse is a section of the cone or if P be any point on the ellipse, that P lies on the surface of the cone

Draw PN perpendicular to BB' . Join ON and produce it to meet BD in V and let MQ be drawn in the plane of the circle on BD as diameter perpendicular to BD and meeting the circle in Q . Also let FG , HK be drawn through E , M respectively parallel to BL'



We have then

$$\begin{aligned} QM^2 \cdot HM \cdot MK &= BV \cdot MD \cdot HM \cdot MK \\ &= BE \cdot ED \cdot FE \cdot DG \\ &= (BE \cdot ED \cdot EO) \cdot (EO^2 \cdot FE \cdot EG) \\ &= (CA^2 \cdot CO) \cdot (CO^2 \cdot BC \cdot CB) \\ &= CA^2 \cdot CB^2 \\ &= PN^2 \cdot BN \cdot NB \end{aligned}$$

$$\text{Therefore } QM^2 \cdot PN^2 = HM \cdot MK \cdot BN \cdot NB \\ = OV^2 \cdot ON^2$$

whence since PN QM are parallel OPQ is a straight line

But Q is on the circumference of the circle on BD as diameter, therefore OQ is a generator of the cone and hence P lies on the cone

Thus the cone passes through all points on the ellipse

PROPOSITION 8

Given an ellipse, a plane through one of its axes AA' and perpendicular to the plane of the ellipse and a line CO drawn from C the centre in the given plane through A' but not perpendicular to AA' it is possible to find a cone with vertex

O such that the given ellipse is a section of it [or, in other words to find the circular sections of the cone with vertex *O* whose surface passes through the circumference of the ellipse]

By hypothesis *OA OA* are unequal Produce *OA* to *D* so that *OA = OD*
Join *AD* and draw *FG* through *C* parallel to it

The given ellipse is to be supposed to lie in a plane perpendicular to the plane of the paper Let *BB* be the other axis of the ellipse

Conceive a plane through *AD* perpendicular to the plane of the paper, and in it describe either (a), if $CB^2 = FC \cdot CG$ a circle with diameter *AD*, or (b), if not an ellipse on *AD* as axis such that if *d* be the other axis

$$d^2 \cdot AD = CB^2 \cdot FC \cdot CG$$

Take a cone with vertex *O* whose surface passes through the circle or ellipse just drawn This is possible even when the curve is an ellipse because the line from *O* to the middle point of *AD* is perpendicular to the plane of the ellipse and the construction is effected by means of Prop 7

Let *P* be any point on the given ellipse and we have only to prove that *P* lies on the surface of the cone so described

Draw *PN* perpendicular to *AA* Join *ON* and produce it to meet *AD* in *M*
Through *M* draw *HA* parallel to *AA*

Lastly draw *MQ* perpendicular to the plane of the paper (and therefore perpendicular to both *HA* and *AD*) meeting the ellipse or circle about *AD* (and therefore the surface of the cone) in *Q*

Then

$$\begin{aligned} QM^2 \cdot HM \cdot MK &= (QM^2 \cdot DM \cdot MA) \cdot (DM \cdot MA \cdot HM \cdot MK) \\ &= (d^2 \cdot AD^2) \cdot (FC \cdot CG \cdot AC \cdot CA) \\ &= (CB^2 \cdot FC \cdot CG) \cdot (FC \cdot CG \cdot AC \cdot CA) \\ &= CB^2 \cdot C1^2 \\ &= PV^2 \cdot AN \cdot NA \end{aligned}$$

Therefore alternately

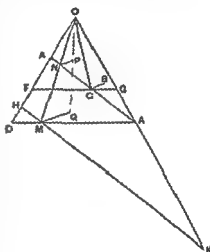
$$\begin{aligned} QM^2 \cdot PV^2 &= HM \cdot MK \cdot AN \cdot NA \\ &= OM^2 \cdot ON^2 \end{aligned}$$

Thus since *PV QM* are parallel *OPQ* is a straight line and *Q* being on the surface of the cone it follows that *P* is also on the surface of the cone

Similarly all points on the ellipse are also on the cone and the ellipse is therefore a section of the cone

PROPOSITION 9

Given an ellipse a plane through one of its axes and perpendicular to that of the ellipse and a straight line CO drawn from the centre C of the ellipse in the given plane through the axis but not perpendicular to that axis it is possible to find a



cylinder with axis OC such that the ellipse is a section of it [or in other words, to find the circular sections of the cylinder with axis OC whose surface passes through the circumference of the given ellipse]

Let AA' be an axis of the ellipse and suppose the plane of the ellipse to be perpendicular to that of the paper, so that OC lies in the plane of the paper

Draw AD AE parallel to CO , and let DE be the line through O perpendicular to both AD and $A'E$

We have now three different cases according as the other axis BB' of the ellipse is (1) equal to, (2) greater than or (3) less than, DE

(1) Suppose $BB' = DE$

Draw a plane through DE at right angles to OC and in this plane describe a circle on DE as diameter Through

this circle describe a cylinder with axis OC

This cylinder shall be the cylinder required or its surface shall pass through every point P of the ellipse

For if P be any point on the ellipse draw PN perpendicular to AA' , through N draw NV parallel to CO meeting DE in M and through M , in the plane of the circle on DE as diameter draw MQ perpendicular to DE meeting the circle in Q

Then since

$$DE = BB'$$

$$PN^2 = AN \cdot A'A = DO^2 = AC \cdot CA'$$

And

$$DV \cdot VE = AN \cdot A'A = DO^2 = AC^2,$$

since $DN \parallel CO \parallel A'E$ are parallel

Therefore

$$PN^2 = DV \cdot ME = QM^2$$

by the property of the circle

Hence since PN QM are equal as well as parallel PQ is parallel to MN and therefore to CO It follows that PQ is a generator of the cylinder, whose surface accordingly passes through P

(2) If $BB' > DE$ we take E on AE such that $DE = BB'$ and describe a circle on DE as diameter in a plane perpendicular to that of the paper, and the rest of the construction and proof is exactly similar to those given for case (1)

(3) Suppose $BB' < DE$

Take a point K on CO produced such that

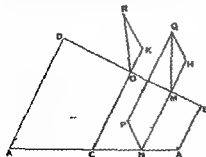
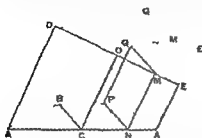
$$DO^2 - CB^2 = OK^2$$

From K draw KR perpendicular to the plane of the paper and equal to CB

$$\text{Thus } OR^2 = OK^2 + CB^2 = OD^2$$

In the plane containing DE OR describe a circle on DE as diameter Through this circle (which must pass through R) draw a cylinder with axis OC

We have then to prove that if P be any point on the given ellipse P lies on the cylinder so described



Draw PN perpendicular to AA' , and through N draw NM parallel to CO meeting DE in M . In the plane of the circle on DE as diameter draw MQ perpendicular to DE and meeting the circle in Q .

Lastly, draw QH perpendicular to NM produced. QH will then be perpendicular to the plane containing AC , DE , i.e. the plane of the paper.

Now $QH^2 = QM^2 + MN^2 = KR^2 + OR^2$ by similar triangles

And $QM \cdot AN \cdot NA = DM \cdot ME \cdot AN \cdot NA'$
 $= OD^2 \cdot CA^2$

Hence *ex aequali* since $OR = OD$

$$QH \cdot IN \cdot N I' = KR^2 \cdot CA^2$$

$$= CB^2 \cdot CA^2$$

$$= PN^2 \cdot AA' \cdot NA'$$

Thus $QH = PN$. And QH , PN are also parallel. Accordingly PQ is parallel to MN and therefore to CO so that PQ is a generator, and the cylinder passes through P .

PROPOSITION 10

It was proved by the earlier geometers that any two cones have to one another the ratio compounded of the ratios of their bases and of their heights.¹ The same method of proof will show that any segments of cones have to one another the ratio compounded of the ratios of their bases and of their heights.

The proposition that any frustum of a cylinder is triple of the conical segment which has the same base as the frustum and equal height is also proved in the same manner as the proposition that the cylinder is triple of the cone which has the same base as the cylinder and equal height.²

PROPOSITION 11

(1) If a paraboloid of revolution be cut by a plane through or parallel to the axis the section will be a parabola equal to the original parabola which by its revolution generates the paraboloid. And the axis of the section will be the intersection between the cutting plane and the plane through the axis of the paraboloid at right angles to the cutting plane.

If the paraboloid be cut by a plane at right angles to its axis the section will be a circle whose centre is on the axis.

(2) If a hyperboloid of revolution be cut by a plane through the axis parallel to the axis or through the centre the section will be a hyperbola (a) if the section be through the axis equal (b) if parallel to the axis similar (c) if through the centre not similar to the original hyperbola which by its revolution generates the hyperboloid. And the axis of the section will be the intersection of the cutting plane and the plane through the axis of the hyperboloid at right angles to the cutting plane.

Any section of the hyperboloid by a plane at right angles to the axis will be a circle whose centre is on the axis.

(3) If any of the spheroidal figures be cut by a plane through the axis or parallel to the axis the section will be an ellipse (a) if the section be through the axis, equal (b) if parallel to the axis similar to the ellipse which by its revolution gen

¹This follows from Eucl. viii. 11 and 12 taken together. Cf. *On the Sphere and Cylinder* I. Lemma 1.

²This proposition was proved by Euclid as stated in the preface to *On the Sphere and Cylinder*. Cf. Eucl. viii. 10.

erates the figure. And the axis of the section will be the intersection of the cutting plane and the plane through the axis of the spheroid at right angles to the cutting plane.

If the section be by a plane at right angles to the axis of the spheroid, it will be a circle whose centre is on the axis.

(4) If any of the said figures be cut by a plane through the axis, and if a perpendicular be drawn to the plane of section from any point on the surface of the figure but not on the section that perpendicular will fall within the section.

And the proofs of all these propositions are evident.

PROPOSITION 12

If a paraboloid of revolution be cut by a plane neither parallel nor perpendicular to the axis and if the plane through the axis perpendicular to the cutting plane intersect it in a straight line of which the portion intercepted within the paraboloid is RR the section of the paraboloid will be an ellipse whose major axis is RR and whose minor axis is equal to the perpendicular distance between the lines through R R parallel to the axis of the paraboloid.

Suppose the cutting plane to be perpendicular to the plane of the paper and let the latter be the plane through the axis ANF of the paraboloid which intersects the cutting plane at right angles in RR . Let RH be parallel to the axis of the paraboloid, and $R'H$ perpendicular to RH .

Let Q be any point on the section made by the cutting plane and from Q draw QM perpendicular to RR . QM will therefore be perpendicular to the plane of the paper.

Through M draw $DMFE$ perpendicular to the axis ANF meeting the parabolic section made by the plane of the paper in D E . Then QM is perpendicular to DE and if a plane be drawn through DE QN , it will be perpendicular to the axis and will cut the paraboloid in a circular section.

Since Q is on this circle

$$QM^2 = DM \cdot ME$$

Again if PT be that tangent to the parabolic section in the plane of the paper which is parallel to RR and if the tangent at A meet PT in O then from the property of the parabola

$$DM \cdot ME \cdot RM \cdot MR = AO \cdot OP^2 \quad [\text{Prop 3 (1)}]$$

$$= AO \cdot OT^2 \text{ since } AN = 1T$$

$$\text{Therefore } QM \cdot RV \cdot VR = AO^2 \cdot OT^2$$

$$= RH^2 \cdot RR^2$$

by similar triangles

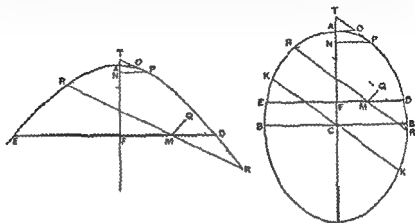
Hence Q lies on an ellipse whose major axis is RR and whose minor axis is equal to RH .

PROPOSITIONS 13 14

If a hyperboloid of revolution be cut by a plane meeting all the generators of the enveloping cone or if an oblong spheroid be cut by a plane not perpendicular to

the axis¹ and if a plane through the axis intersect the cutting plane at right angles in a straight line on which the hyperboloid or spheroid intercepts a length RR , then the section by the cutting plane will be an ellipse whose major axis is RR .

Suppose the cutting plane to be at right angles to the plane of the paper, and suppose the latter plane to be that through the axis ANF which intersects the



cutting plane at right angles in RR . The section of the hyperboloid or spheroid by the plane of the paper is thus a hyperbola or ellipse having ANF for its transverse or major axis.

Take any point on the section made by the cutting plane as Q and draw QM perpendicular to RR . QM will then be perpendicular to the plane of the paper.

Through M draw DFE at right angles to the axis ANF meeting the hyperbola or ellipse in D , E and through QM , DE let a plane be described. This plane will accordingly be perpendicular to the axis and will cut the hyperboloid or spheroid in a circular section.

Thus $QM = DM = ME$

Let PT be that tangent to the hyperbola or ellipse which is parallel to RR , and let the tangent at A meet PT in O .

Then by the property of the hyperbola or ellipse

$$DM \cdot ME \cdot RM \cdot MR = OA^2 \cdot OP$$

or $QM^2 \cdot RM \cdot MR = OA^2 \cdot OP$

Now (1) in the hyperbola $OA < OP$ because $AT < AV$ and accordingly $OT < OP$ while $OA < OT$

(2) in the ellipse if KK' be the diameter parallel to RR , and BB' the minor axis

$$BC \cdot CB \cdot KC \cdot CK = OA^2 \cdot OP^2$$

and $BC \cdot CB < KC \cdot CK'$ so that $OA < OP$

Hence in both cases the locus of Q is an ellipse whose major axis is RR .

CON. 1 If the spheroid be a flat spheroid the section will be an ellipse, and everything will proceed as before except that RR will in this case be the minor axis.

¹Archimedes begins Prop. 14 for the spheroid with the remark that when the cutting plane passes through or is parallel to the axis the case is clear Cf. Prop. 11 (3).

COR 2 In all conoids or spheroids parallel sections will be similar, since the ratio $OA^2 : OP^2$ is the same for all the parallel sections

PROPOSITION 15

(1) *If from any point on the surface of a conoid a line be drawn, in the case of the paraboloid, parallel to the axis and, in the case of the hyperboloid parallel to any line passing through the vertex of the enveloping cone the part of the straight line which is in the same direction as the convexity of the surface will fall without it, and the part which is in the other direction within it*

For, if a plane be drawn, in the case of the paraboloid, through the axis and the point and in the case of the hyperboloid through the given point and through the given straight line drawn through the vertex of the enveloping cone, the section by the plane will be (a) in the paraboloid a parabola whose axis is the axis of the paraboloid, (b) in the hyperboloid a hyperbola in which the given line through the vertex of the enveloping cone is a diameter¹

[Prop 11]

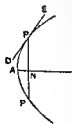
Hence the property follows from the plane properties of the conics

(2) *If a plane touch a conoid without cutting it, it will touch it at one point only, and the plane drawn through the point of contact and the axis of the conoid will be at right angles to the plane which touches it*

For if possible let the plane touch at two points. Draw through each point a parallel to the axis. The plane passing through both parallels will therefore either pass through or be parallel to the axis. Hence the section of the conoid made by this plane will be a conic [Prop 11 (1), (2)], the two points will lie on this conic and the line joining them will lie within the conic and therefore within the conoid. But this line will be in the tangent plane since the two points are in it. Therefore some portion of the tangent plane will be within the conoid which is impossible since the plane does not cut it.

Therefore the tangent plane touches in one point only.

That the plane through the point of contact and the axis is perpendicular to the tangent plane is evident in the particular case where the point of contact is the vertex of the conoid. For if two planes through the axis cut it in two comes the tangents at the vertex in both conics will be perpendicular to the axis of the conoid. And all such tangents will be in the tangent plane which must therefore be perpendicular to the axis and to any plane through the axis.



If the point of contact P is not the vertex draw the plane passing through the axis AN and the point P . It will cut the conoid in a conic whose axis is AN and the tangent plane in a line DPE touching the conic at P . Draw PNP perpendicular to the axis and draw a plane through it also perpendicular to the axis. This plane will make a circular section and meet the tangent plane in a tangent to the circle which will therefore be at right angles to PN . Hence the tangent to the circle will be at right angles to the plane containing PN, AN , and it follows that this last plane is perpendicular to the tangent plane.

¹There seems to be some error in the text here which says that the diameter (i.e. axis) of the hyperbola is the straight line drawn in the conoid from the vertex of the cone. But this straight line is not in general the axis of the section.

PROPOSITION 16

(1) *If a plane touch any of the spheroidal figures without cutting it, it will touch at one point only, and the plane through the point of contact and the axis will be at right angles to the tangent plane*

This is proved by the same method as the last proposition

(2) *If any conoid or spheroid be cut by a plane through the axis and if through any tangent to the resulting conic a plane be erected at right angles to the plane of section the plane so erected will touch the conoid or spheroid in the same point as that in which the line touches the conic*

For it cannot meet the surface at any other point. If it did the perpendicular from the second point on the cutting plane would be perpendicular also to the tangent to the conic and would therefore fall outside the surface. But it must fall within it. [Prop 11 (4)]

(3) *If two parallel planes touch any of the spheroidal figures the line joining the points of contact will pass through the centre of the spheroid*

If the planes are at right angles to the axis the proposition is obvious. If not, the plane through the axis and one point of contact is at right angles to the tangent plane at that point. It is therefore at right angles to the parallel tangent plane and therefore passes through the second point of contact. Hence both points of contact lie on one plane through the axis and the proposition is reduced to a plane one.

PROPOSITION 17

If two parallel planes touch any of the spheroidal figures and another plane be drawn parallel to the tangent planes and passing through the centre the line drawn through any point of the circumference of the resulting section parallel to the chord of contact of the tangent planes will fall outside the spheroid

This is proved at once by reduction to a plane proposition.

Archimedes adds that it is evident that if the plane parallel to the tangent planes does not pass through the centre a straight line drawn in the manner described will fall without the spheroid in the direction of the smaller segment but within it in the other direction.

PROPOSITION 18

Any spheroidal figure which is cut by a plane through the centre is divided both as regards its surface and its volume into two equal parts by that plane

To prove this Archimedes takes another equal and similar spheroid divides it similarly by a plane through the centre and then uses the method of application.

PROPOSITIONS 19 20

Given a segment cut off by a plane from a paraboloid or hyperboloid of revolution or a segment of a spheroid less than half the spheroid also cut off by a plane it is possible to inscribe in the segment one solid figure and to circumscribe about it another solid figure each made up of cylinders or frusta of cylinders of equal height, and such that the circumscribed figure exceeds the inscribed figure by a volume less than that of any given solid

Let the plane base of the segment be perpendicular to the plane of the paper

and let the plane of the paper be the plane through the axis of the conoid or spheroid which cuts the base of the segment at right angles in BC . The section in the plane of the paper is then a conic BAC [Prop 11]

Let EAF be that tangent to the conic which is parallel to BC , and let i be the point of contact. Through EAF draw a plane parallel to the plane through BC bounding the segment. The plane so drawn will then touch the conoid or spheroid at A [Prop 16]

(1) If the base of the segment is at right angles to the axis of the conoid or spheroid A will be the vertex of the conoid or spheroid and its axis AD will bisect BC at right angles

(2) If the base of the segment is not at right angles to the axis of the conoid or spheroid, we draw AD

(a) in the paraboloid, parallel to the axis

(b) in the hyperboloid, through the centre (or the vertex of the enveloping cone),

(c) in the spheroid, through the centre

and in all the cases it will follow that AD bisects BC in D

Then A will be the vertex of the segment, and AD will be its axis

Further, the base of the segment will be a circle or an ellipse with BC as diameter or as an axis respectively and with centre D . We can therefore describe through this circle or ellipse a cylinder or a 'frustum' of a cylinder whose axis is AD [Prop 9]

Dividing this cylinder or frustum continually into equal parts by planes parallel to the base we shall at length arrive at a cylinder or frustum less in volume than any given solid

Let this cylinder or frustum be that whose axis is OD , and let AD be divided into parts equal to OD at L, M

Through L, M draw lines parallel to

and through these lines draw planes parallel to the base of the segment. These will cut the conoid or spheroid in circles or similar ellipses. On each of these circles or ellipses describe two cylinders or frusta of cylinders each with axis equal to OD one of them lying in the direction of A and the other in the direction of D as shown in the figure

Then the cylinders or frusta of cylinders drawn in the direction of A make up a circumscribed figure and those in the direction of D an inscribed figure in relation to the segment

Also the cylinder or frustum PG in the circumscribed figure is equal to the cylinder or frustum PH in the inscribed figure. QI in the circumscribed figure is equal to QA in the inscribed figure, and so on

Therefore by addition

(circumscribed fig) = (inscr fig) + (cylinder or frustum whose axis is OD)

But the cylinder or frustum whose axis is OD is less than the given solid figure whence the proposition follows

'Having set out these preliminary propositions let us proceed to demonstrate the theorems propounded with reference to the figures'



PROPOSITIONS 21, 22

Any segment of a paraboloid of revolution is half as large again as the cone or segment of a cone which has the same base and the same axis

Let the base of the segment be perpendicular to the plane of the paper and let the plane of the paper be the plane through the axis of the paraboloid which cuts the base of the segment at right angles in BC and makes the parabolic section BAC

Let EF be that tangent to the parabola which is parallel to BC , and let A be the point of contact

Then (1), if the plane of the base of the segment is perpendicular to the axis of the paraboloid, that axis is the line AD bisecting BC at right angles in D

(2) If the plane of the base is not perpendicular to the axis of the paraboloid draw AD parallel to the axis of the paraboloid AD will then bisect BC , but not at right angles

Draw through EF a plane parallel to the base of the segment This will touch the paraboloid at A and A will be the vertex of the segment, AD its axis

The base of the segment will be a circle with diameter BC or an ellipse with BC as major axis

Accordingly a cylinder or a frustum of a cylinder can be found passing through the circle or ellipse and having AD for its axis [Prop 7], and likewise a cone or a segment of a cone can be drawn passing through the circle or ellipse and having A for vertex and AD for axis [Prop 8]

Suppose λ to be a cone equal to $\frac{2}{3}$ (cone or segment of cone ABC) The cone λ is therefore equal to half the cylinder or frustum of a cylinder EC [Cf Prop 10]

We shall prove that the volume of the segment of the paraboloid is equal to λ

If not the segment must be either greater or less than λ

I. If possible let the segment be greater than λ

We can then inscribe and circumscribe as in the first proposition figures made up of cylinders or frusta of cylinders with equal height and such that (circumscribed fig) - (inscribed fig) < (segment) - λ

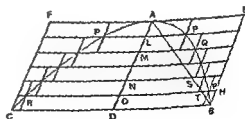
Let the greatest of the cylinders or frusta forming the circumscribed figure be that whose base is the circle or ellipse about BC and whose axis is OD and let the smallest of them be that whose base is the circle or ellipse about PP' and whose axis is AL

Let the greatest of the cylinders forming the inscribed figure be that whose base is the circle or ellipse about RH' and whose axis is OD and let the smallest be that whose base is the circle or ellipse about PP' and whose axis is LM

Produce all the plane bases of the cylinders or frusta to meet the surface of the complete cylinder or frustum EC

Now, since

$$(\text{circumscribed fig}) - (\text{inscr fig}) < (\text{segment}) - \lambda$$



it follows that (inscribed figure) $> \lambda$ (a)

Next comparing successively the cylinders or frusta with heights equal to OD and respectively forming parts of the complete cylinder or frustum EC and of the inscribed figure, we have

(first cylinder or frustum in EC) (first in inscr fig)

$$= BD \quad RO$$

$$= 4D \quad 4O$$

$$= BD \quad TO \text{ where } AB \text{ meets } OR \text{ in } T$$

And (second cylinder or frustum in EC) (second in inscr fig)

$$= HO \quad SN \text{ in like manner,}$$

and so on

Hence [Prop 1] (cylinder or frustum EC) (inscribed figure)

$$= (BD + HO + \dots) (TO + SN + \dots),$$

where BD HO , ... are all equal, and BD , TO SN , ... diminish in arithmetical progression

But [Lemma preceding Prop 1]

$$BD + HO + \dots > 2(TO + SN + \dots)$$

Therefore (cylinder or frustum EC) > 2 (inscribed fig),

or $\lambda > \text{(inscribed fig)}$,

which is impossible, by (a) above

II If possible, let the segment be less than λ

In this case we inscribe and circumscribe figures as before but such that

$$(\text{circumscriber fig}) - (\text{inscr fig}) < \lambda - (\text{segment}),$$

whence it follows that

$$(\text{circumscribed figure}) < \lambda \quad (\beta)$$

And, comparing the cylinders or frusta making up the complete cylinder or frustum CE and the circumscribed figure respectively we have

(first cylinder or frustum in CE) (first in circumscriber fig)

$$= BD \quad BD^2$$

$$= BD \quad BD$$

(second in CE) (second in circumscriber fig)

$$= HO^2 \quad RO^2$$

$$= AD \quad AO$$

$$= HO \quad TO,$$

and so on

Hence [Prop 1]

(cylinder or frustum CE) (circumscribed fig)

$$= (BD + HO + \dots) (BD + TO + \dots)$$

$$< 2 \quad 1$$

[Lemma preceding Prop 1]

and it follows that

$$\lambda < (\text{circumscribed fig})$$

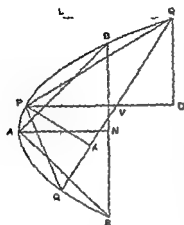
which is impossible, by (b)

Thus the segment being neither greater nor less than λ is equal to it, and therefore to $\frac{1}{3}$ (cone or segment of cone ABC)

PROPOSITION 23

If from a paraboloid of revolution two segments be cut off one by a plane perpendicular to the axis the other by a plane not perpendicular to the axis and if the axes of the segments are equal the segments will be equal in volume

Let the two planes be supposed perpendicular to the plane of the paper, and let the latter plane be the plane through the axis of the paraboloid cutting the other two planes at right angles in BB' , QQ' respectively and the paraboloid itself in the parabola $QPQ B$



Let $1A$ $P1$ be the equal axes of the segments and $1, P$ their respective vertices

Draw $Q1$ parallel to AN or PV and QL perpendicular to QL

Now, since the segments of the parabolic section cut off by BB' , QQ have equal axes, the triangles ABB , PQQ are equal [Prop 3]

Also if QD be perpendicular to PV , $QD = BV$ (as in the same Prop 3)

Construct two cones drawn with the same bases as the segments and with A, P as vertices respectively. The height of the cone PQQ is then PA where PK is perpendicular to QQ

Now the cones are in the ratio compounded of the ratios of their bases and of their heights i.e. the ratio compounded of (1) the ratio of the circle about BB' to the ellipse about QQ , and (2) the ratio of $1V$ to PA

That is to say, we have by means of Props 5, 12,

$$(\text{cone } 1BB) (\text{cone } PQQ) = (BB' : QQ) (Q1 : 1A) (1A : PA)$$

$$\text{And } BB' = 2BA = 2QD = QL \text{ while } QQ = 2Q1$$

Therefore

$$\begin{aligned} (\text{cone } 1BB) (\text{cone } PQQ) &= (QD : QV) (1N : PA) \\ &= (PA : P1) (1A : PA) \\ &= 1A : P1 \end{aligned}$$

Since $1N = P1$, the ratio of the cones is a ratio of equality, and it follows that the segments being each half as large again as the respective cones [Prop 22], are equal

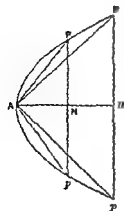
PROPOSITION 24

If from a paraboloid of revolution two segments be cut off by planes drawn in any manner, the segments will be to one another as the squares on their axes

For let the paraboloid be cut by a plane through the axis in the parabolic section $P P A p p'$, and let the axis of the parabola and paraboloid be ANN'

Measure along ANN' the lengths AN , AN' equal to the respective axes of the given segments and through N N' draw planes perpendicular to the axis making circular sections on Pp , $P p'$ as diameters respectively. With these circles as bases and with the common vertex A let two cones be described

Now the segments of the paraboloid whose bases are the circles about Pp , $P p'$ are equal to the given segments respectively, since their respective axes are



equal [Prop 23] and, since the segments APp $AP'p$ are half as large again as the cones APp $AP'p$ respectively we have only to show that the cones are in the ratio of AV to AV'

But

$$\begin{aligned} (\text{cone } APp) (\text{cone } AP'p) &= (PV^2 \cdot P'V'^2) (AV \cdot AV') \\ &= (AV \cdot AV') (AV \cdot AV') \\ &= AV^2 \cdot AV'^2 \end{aligned}$$

thus the proposition is proved.

PROPOSITION 23 20

In any hyperboloid of revolution, if A be the vertex and AD the axis of any segment cut off by a plane and if CA be the semi-diameter of the hyperboloid through A (CA being of course in the same straight line with AD) then

$$\frac{(\text{segment})}{(\text{cone with same base and axis})} = (1D + 3CA) (1D + 2CA)$$

Let the plane cutting off the segment be perpendicular to the plane of the paper and let the latter plane be the plane through the axis of the hyperboloid which intersects the cutting plane at right angles in BB' , and makes the hyperbolic segment BAB . Let C be the centre of the hyperboloid (or the vertex of the enveloping cone)

Let EF be that tangent to the hyperbolic section which is parallel to BB' . Let EF touch at A and join CA . Then CA produced will bisect BB' at D . CA will be a semi-diameter of the hyperboloid. A will be the vertex of the segment and AD its axis. Produce AC to A' and H so that $AC = CA' = A'H$.

Through EF draw a plane parallel to the base of the segment. This plane will touch the hyperboloid at A .

Then (1) if the base of the segment is at right angles to the axis of the hyperboloid A will be the vertex and AD the axis of the hyperboloid as well as of the segment, and the base of the segment will be a circle on BB' as diameter.

(2) If the base of the segment is not perpendicular to the axis of the hyperboloid the base will be an ellipse on BB' as major axis. [Prop 13]

Then we can draw a cylinder or a frustum of a cylinder FBB' passing through the circle or ellipse about BB' and having AD for its axis and so we can describe a cone or a segment of a cone through the circle or ellipse and having A for its vertex.

We have to prove that

$$(\text{segment } ABB) (\text{cone or segment of cone } ABB) = HD \cdot 1'D$$

Let 1 be a cone such that

$$1 (\text{cone or segment of cone } ABB) = HD \cdot AD \quad (\alpha)$$

and we have to prove that 1 is equal to the segment

Now

$$(\text{cylinder or frustum } EB) (\text{cone or segment of cone } ABB) = 3 \cdot 1$$

$$\text{Therefore, by means of } (\alpha) (\text{cylinder or frustum } EB) \cdot 1 = 1D \cdot \frac{HD}{3} \quad (\beta)$$

If the segment is not equal to 1 it must either be greater or less.

I. If possible let the segment be greater than 1 .

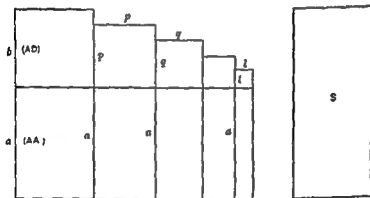
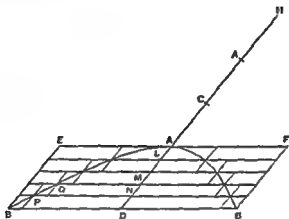
Inscribe and circumscribe to the segment figures made up of cylinders or frusta of cylinders with axes along AD and all equal to one another such that

(circumscribed fig) - (inscr fig) < (segmt) - I,

whence

(inscribed figure) > V (7)

Produce all the planes forming the bases of the cylinders or frusta of cylinders to meet the surface of the complete cylinder or frustum EB'



Then if AD be the axis of the greatest cylinder or frustum in the circumscribed figure the complete cylinder will be divided into cylinders or frusta each equal to this greatest cylinder or frustum

Let there be a number of straight lines a equal to AA' and as many in number as the parts into which AD is divided by the bases of the cylinders or frusta. To each line a apply a rectangle which shall overlap it by a square and let the greatest of the rectangles be equal to the rectangle $AD A'D$ and the least equal to the rectangle $1L A'L$ also let the sides of the overlapping squares b, p, q, l be in descending arithmetical progression. Thus b, p, q, l will be respectively equal to AD, AN, AM, AL and the rectangles $(ab + b^2), (ap + p^2), (al + l^2)$ will be respectively equal to $AD A'D, AN A'N, AL A'L$.

Suppose further that we have a series of spaces S each equal to the largest rectangle $AD A'D$ and as many in number as the diminishing rectangles.

Comparing now the successive cylinders or frusta (1) in the complete cylin

der or frustum EB' and (2) in the inscribed figure, beginning from the base of the segment, we have

$$\begin{aligned} & \text{(first cylinder or frustum in } FB) \quad \text{(first in inscr figure)} \\ & = BD \quad PA^2 \\ & = AD \quad A'D \quad AN \quad A'N, \text{ from the hyperbola} \\ & = S \quad (ap+p) \end{aligned}$$

Again

$$\begin{aligned} & \text{(second cylinder or frustum in } EB) \quad \text{(second in inscr fig)} \\ & = BD^2 \quad QM^2 \\ & = AD \quad A'D \quad 1M \quad 4'M \\ & = S \quad (aq+q^2) \end{aligned}$$

and so on

The last cylinder or frustum in the complete cylinder or frustum EB' has no cylinder or frustum corresponding to it in the inscribed figure

Combining the proportions, we have [Prop 1]

$$\begin{aligned} & \text{(cylinder or frustum } EB') : \text{(inscribed figure)} \\ & = (\text{sum of all the spaces } S) \quad (ap+p^2) + (aq+q^2) + \\ & > (a+b) \quad \left(\frac{a}{2} + \frac{b}{3}\right) \quad \text{[Prop 2]} \end{aligned}$$

$$> A'D \quad \frac{HD}{3} \text{ since } a = AA' \quad b = AD,$$

$$> (EB) \quad V \text{ by } (\beta) \text{ above}$$

Hence

(inscribed figure) $< V$

But this is impossible because by (γ) above the inscribed figure is greater than V

II Next suppose if possible, that the segment is less than V

In this case we circumscribe and inscribe figures such that

$$(\text{circumscribed fig}) - (\text{inscribed fig}) < V - (\text{segment}),$$

whence we derive

$$V > (\text{circumscribed figure}) \quad (\delta)$$

We now compare successive cylinders or frusta in the complete cylinder or frustum and in the circumscribed figure and we have

$$\text{(first cylinder or frustum in } EB') \quad \text{(first in circumscribed fig)}$$

$$\begin{aligned} & = S \quad S \\ & = S \quad (ab+b^2) \end{aligned}$$

$$\text{(second in } EB) \quad \text{(second in circumscribed fig)}$$

$$= S \quad (ap+p)$$

and so on

Hence [Prop 1]

$$\text{(cylinder or frustum } EB) : \text{(circumscribed fig)}$$

$$= (\text{sum of all spaces } S) \quad (ab+b^2) + (ap+p) +$$

$$< (a+b) \quad \left(\frac{a}{2} + \frac{b}{3}\right) \quad \text{[Prop 2]}$$

$$< A'D \quad \frac{HD}{3}$$

$$< (EB) \quad V \text{ by } (\beta) \text{ above}$$

Hence the circumscribed figure is greater than 1 which is impossible, by (8) above

Thus the segment is neither greater nor less than 1, and is therefore equal to it

Therefore by (a),

$$\begin{aligned} & (\text{segment } ABB) \quad (\text{cone or segment of cone } ABB') \\ & = (AD + 3C1) \quad (AD + 2C4) \end{aligned}$$

PROPOSITIONS 27, 28, 29, 30

(1) In any spheroid whose centre is C , if a plane meeting the axis cut off a segment not greater than half the spheroid and having A for its vertex and AD for its axis, and if $A'D$ be the axis of the remaining segment of the spheroid then

$$\begin{aligned} & (\text{first segmt}) \quad (\text{cone or segmt of cone with same base and axis}) \\ & = C1 + AD \quad A'D \\ & = 3C1 - AD \quad 2C1 - AD \end{aligned}$$

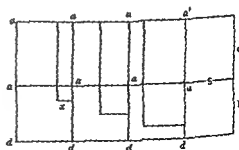
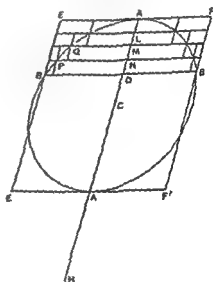
(2) As a particular case, if the plane passes through the centre so that the segment is half the spheroid, half the spheroid is double of the cone or segment of a cone which has the same vertex and axis

Let the plane cutting off the segment be at right angles to the plane of the paper and let the latter plane be the plane through the axis of the spheroid which intersects the cutting plane in BB' and makes the elliptic section ABB

Let EF $E'F'$ be the two tangents to the ellipse which are parallel to BB' let them touch it in A and through the tangents draw planes parallel to the base of the segment. These planes will touch the spheroid at I I' which will be the vertices of the two segments into which it is divided. Also AA' will pass through the centre C and bisect BB' in D

Then (1) if the base of the segments be perpendicular to the axis of the spheroid AA' will be the vertices of the spheroid as well as of the segments. AA' will be the axis of the spheroid and the base of the segments will be a circle on BB' as diameter

(2) if the base of the segments be not perpendicular to the axis of the spheroid the base of the segments will be an ellipse of which BB' is one axis and ID $I'D$ will be the axes of the segments respectively



We can now draw a cylinder or a frustum of a cylinder $EBB'F$ through the circle or ellipse about BB' and having AD for its axis, and we can also draw a cone or a segment of a cone passing through the circle or ellipse about BB' and having A for its vertex.

We have then to show that if CA' be produced to H so that $CA' = A'H$,
(segment ABB') (cone or segment of cone ABB') = $HD \cdot A'D$.

Let V be such a cone that

$$V \text{ (cone or segment of cone } ABB') = HD \cdot A'D \quad (\alpha)$$

and we have to show that the segment ABB' is equal to V

But, since

$$(\text{cylinder or frustum } EB) \text{ (cone or segment of cone } ABB') = 3 \cdot 1,$$

we have, by the rule of (α)

$$(\text{cylinder or frustum } EB) \cdot V = A'D \cdot \frac{HD}{3} \quad (\beta)$$

Now, if the segment ABB' is not equal to V , it must be either greater or less

I Suppose if possible, that the segment is greater than V

Let figures be inscribed and circumscribed to the segment consisting of cylinders or frusta of cylinders, with axes along AD and all equal to one another, such that

$$(\text{circumscribed fig}) - (\text{inscribed fig}) < (\text{segment}) - V$$

whence it follows that

$$(\text{inscribed fig}) > V \quad (\gamma)$$

Produce all the planes forming the bases of the cylinders or frusta to meet the surface of the complete cylinder or frustum EB' . Thus if ND be the axis of the greatest cylinder or frustum of a cylinder in the circumscribed figure the complete cylinder or frustum EB' will be divided into cylinders or frusta of cylinders each equal to the greatest of those in the circumscribed figure.

Take straight lines da' each equal to $A'D$ and as many in number as the parts into which AD is divided by the bases of the cylinders or frusta, and measure da along da' equal to AD . It follows that $aa' = 2CD$.

Apply to each of the lines $a'd$ rectangles with height equal to ad and draw the squares on each of the lines ad as in the figure. Let S denote the area of each complete rectangle.

From the first rectangle take away a gnomon with breadth equal to AN (i.e. with each end of a length equal to AN) take away from the second rectangle a gnomon with breadth equal to AM and so on the last rectangle having no gnomon taken from it.

Then

$$\begin{aligned} \text{the first gnomon} &= AD \cdot 1D - ND \cdot (AD - AN) \\ &= A'D \cdot AN + ND \cdot AN \\ &= AN \cdot 1N \end{aligned}$$

Similarly

$$\text{the second gnomon} = 1H \cdot 1H$$

and so on

And the last gnomon (that in the last rectangle but one) is equal to $AL \cdot 1'L$

Also after the gnomons are taken away from the successive rectangles the remainders (which we will call R_1, R_2, \dots, R_n where n is the number of rectangles and accordingly $R_n = S$) are rectangles applied to straight lines each of

length aa' and "exceeding by squares" whose sides are respectively equal to DN, DM, DA

For brevity let DV be denoted by x , and aa' or $2CD$ by c , so that

$$R_1 = cx + x^2 \quad R_2 = c \cdot 2x + (2x)^2,$$

Then, comparing successively the cylinders or frusta of cylinders (1) in the complete cylinder or frustum EB and (2) in the inscribed figure, we have

(first cylinder or frustum in EB) (first in inscribed fig)

$$= BD^2 \cdot PV^2$$

$$= AD \cdot A'D \cdot AN \cdot A'N$$

$$= S \text{ (first gnomon),}$$

(second cylinder or frustum in EB') (second in inscribed fig)

$$= S \text{ (second gnomon)}$$

and so on

The last of the cylinders or frusta in the cylinder or frustum EB has none corresponding to it in the inscribed figure, and there is no corresponding gnomon

Combining the proportions, we have [by Prop 1]

(cylinder or frustum EB) (inscribed fig)

= (sum of all spaces S) (sum of gnomons)

Now the differences between S and the successive gnomons are R_1, R_2, \dots, R_n , while

$$R_1 = cx + x^2,$$

$$R_2 = c \cdot 2x + (2x)^2,$$

$$R_n = cb + b^2 = S,$$

where $b = nx = AD$

Hence [Prop 2]

(sum of all spaces S) ($R_1 + R_2 + \dots + R_n$) $< (c+b) \left(\frac{c}{2} + \frac{b}{3}\right)$

It follows that

$$\begin{aligned} \text{(sum of all spaces } S) \text{ (sum of gnomons)} &> (c+b) \left(\frac{c}{2} + \frac{2b}{3}\right) \\ &> 1'D \cdot \frac{HD}{3} \end{aligned}$$

Thus (cylinder or frustum EB) (inscribed fig)

$$> AD \cdot \frac{HD}{3}$$

$$> (\text{cylinder or frustum } EB) \cdot V,$$

from (β) above

Therefore (inscribed fig) $< V$,

which is impossible by (γ) above

Hence the segment ABB' is not greater than V

II If possible let the segment ABB' be less than V

We then inscribe and circumscribe figures such that

$$(\text{circumscribed fig}) - (\text{inscribed fig}) < V - (\text{segment}),$$

whence

$$1 > (\text{circumscribed fig})$$

(δ)

In this case we compare the cylinders or frusta in (EB) with those in the circumscribed figure

Thus
 (first cylinder or frustum in EB') (first in circumscribed fig)
 $=S$
 (second in EB') (second in circumscribed fig)
 $=S$ (first gnomon),

and so on

Lastly (last in EB') (last in circumscribed fig)
 $=S$ (last gnomon)

Now

$$\{S + (\text{all the gnomons})\} = nS - (R_1 + R_2 + \dots + R_{n-1})$$

And $nS - R_1 - R_2 - \dots - R_{n-1} > (c+b) \left(\frac{c}{2} + \frac{b}{3}\right)$ [Prop 2]

so that

$$nS - \{S + (\text{all the gnomons})\} < (c+b) \left(\frac{c}{2} + \frac{2b}{3}\right)$$

It follows that, if we combine the above proportions as in Prop 1, we obtain
 (cylinder or frustum EB') (circumscribed fig)

$$< (c+b) \left(\frac{c}{2} + \frac{2b}{3}\right)$$

$$< A'D \frac{HD}{3}$$

$$< (EB) V, \text{ by } (\beta) \text{ above}$$

Hence the circumscribed figure is greater than V , which is impossible by (δ) above

Thus since the segment ABB is neither greater nor less than V , it is equal to it and the proposition is proved

(2) The particular case [Props 27, 28] where the segment is half the spheroid differs from the above in that the distance CD or $c/2$ vanishes, and the rectangles $cb+b$ are simply squares (b) so that the gnomons are simply the differences between b^2 and x^2 , b^2 and $(2x)^2$, and so on

Instead therefore of Prop 2 we use the Lemma to Prop 2, Cor 1 given above [On Spirals, Prop 10] and instead of the ratio $(c+b) \left(\frac{c}{2} + \frac{2b}{3}\right)$ we obtain the ratio $\frac{2}{3}$ whence (segment ABB) (cone or segment of cone ABB) = 2 : 1

PROPOSITIONS 31, 32

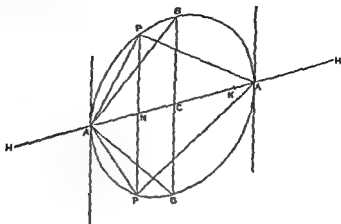
If a plane divide a spheroid into two unequal segments and if AN A^1N be the axes of the lesser and greater segments respectively while C is the centre of the spheroid then

$$(\text{greater segmt}) (\text{cone or segmt of cone with same base and axis}) \\ = CA + AN \quad AN$$

Let the plane dividing the spheroid be that through PP' perpendicular to the plane of the paper and let the latter plane be that through the axis of the spheroid which intersects the cutting plane in PP and makes the elliptic section $PAP A$

Draw the tangents to the ellipse which are parallel to PP let them touch the ellipse at A A' , and through the tangents draw planes parallel to the base of the segments These planes will touch the spheroid at A , A' , the line AA'

will pass through the centre C and bisect PP' in N , while AN , $A'N$ will be the axes of the segments



Then (1) if the cutting plane be perpendicular to the axis of the spheroid, AA' will be that axis and A, A' will be the vertices of the spheroid as well as of the segments. Also the sections of the spheroid by the cutting plane and all planes parallel to it will be circles.

(2) If the cutting plane be not perpendicular to the axis, the base of the segments will be an ellipse of which PP' is an axis and the sections of the spheroid by all planes parallel to the cutting plane will be similar ellipses.

Draw a plane through C parallel to the base of the segments and meeting the plane of the paper in BB' .

Construct three cones or segments of cones, two having A for their common vertex and the plane sections through PP' BB' for their respective bases, and a third having the plane section through PP' for its base and A' for its vertex.

Produce CA to H and CA' to H' so that

$$AH = A'H' = CA$$

We have then to prove that

$$\begin{aligned} (\text{segment } APP') &= (\text{cone or segment of cone } A'PP') \\ &= CA + AN \quad AN \\ &= NH \quad 1N \end{aligned}$$

Now halt the spheroid as double of the cone or segment of a cone ABB' [Props 27-28]. Therefore

$$(\text{the spheroid}) = 4(\text{cone or segment of cone } ABB')$$

But

$$\begin{aligned} (\text{cone or segmt of cone } ABB') &= (\text{cone or segmt of cone } APP') \\ &= (CA \quad AN) (BC^2 \quad PN^2) \\ &= (CA \quad AN) (CA \quad CA' \quad AN \quad A'N) \end{aligned} \quad (\alpha)$$

If we measure AK along AA' so that

$$\frac{AK}{AN} = \frac{AC}{AN} = \frac{CA}{AN},$$

and the compound ratio in (α) becomes

$$\frac{(AK \quad AN \quad CA \quad A'N) (CA \quad CA' \quad AN \quad A'N)}{AK \quad CA' \quad AN \quad AN}$$

Thus

$$\begin{aligned} & \text{(cone or segmt of cone } ABB') \text{ (cone or segmt of cone } APP) \\ & = AA \ CA' \ AN \ A'N \end{aligned}$$

But

$$\begin{aligned} & \text{(cone or segment of cone } APP) \text{ (segment } APP) \\ & = A'N \ NH' \\ & = AN \ A'N \ AN \ NH' \end{aligned} \quad [\text{Props 29, 30}]$$

Therefore *ex aequali*

$$\begin{aligned} & \text{(cone or segment of cone } ABB) \text{ (segment } APP') \\ & = AA \ CA' \ AN \ NH', \end{aligned}$$

so that

$$\begin{aligned} & \text{(spheroid) (segment } APP) \\ & = HH' \ AA \ AN \ NH', \end{aligned}$$

since

$$HH' = 4CA'$$

Hence

$$\begin{aligned} & \text{(segment } A'PP) \text{ (segment } APP') \\ & = (HH \ AK - AN \ NH) \ AN \ NH' \\ & = (AA \ NH + NH' \ NK) \ AN \ NH' \end{aligned}$$

Further,

$$\begin{aligned} & \text{(segment } APP) \text{ (cone or segment of cone } APP') \\ & = NH \ A'N \\ & = AN \ NH' \ AN \ A'N, \end{aligned}$$

and

$$\begin{aligned} & \text{(cone or segmt of cone } APP') \text{ (cone or segmt of cone } A'PP') \\ & = AN \ A'N \\ & = AN \ A'N \ A'N^2 \end{aligned}$$

From the last three proportions we obtain *ex aequali*

$$\begin{aligned} & \text{(segment } APP') \text{ (cone or segment of cone } A'PP') \\ & = (AA \ NH + NH' \ NK) \ A'N^2 \\ & = (AA \ NH + NH' \ NK) \ (CA^2 + NH' \ CN) \\ & = (AA \ NH + NH' \ NK) \ (AA \ AN + NH \ CN) \end{aligned} \quad (\beta)$$

But

$$\begin{aligned} AA \ NH \ AA \ AN &= NH \ AN \\ &= CA + AN \ AN \\ &= AA + CA \ CA \text{ (since } AK \ AC = AC \ AN) \\ &= HH \ CA \\ &= HH - NH \ CA - AN \\ &= NK \ CN \\ &= NH \ NK \ NH \ CN \end{aligned}$$

Hence the ratio in (β) is equal to the ratio

$$AK \ NH \ AA \ AN, \text{ or } NH \ AN$$

Therefore

$$\begin{aligned} & \text{(segment } A'PP) \text{ (cone or segment of cone } A'PP') \\ & = NH \ AN \\ & = CA + AN \ AN \end{aligned}$$

ON SPIRALS

"ARCHIMEDES to Dositheus greeting

Of most of the theorems which I sent to Conon, and of which you ask me from time to time to send you the proofs the demonstrations are already before you in the books brought to you by Heracleides and some more are also contained in that which I now send you. Do not be surprised at my taking a considerable time before publishing these proofs. This has been owing to my desire to communicate them first to persons engaged in mathematical studies and anxious to investigate them. In fact, how many theorems in geometry which have seemed at first impracticable are in time successfully worked out! Now Conon died before he had sufficient time to investigate the theorems referred to otherwise he would have discovered and made manifest all these things and would have enriched geometry by many other discoveries besides. For I know well that it was no common ability that he brought to bear on mathematics and that his industry was extraordinary. But though many years have elapsed since Conon's death I do not find that any one of the problems has been stirred by a single person. I wish now to put them in review one by one, particularly as it happens that there are two included among them which are impossible of realisation (and which may serve as a warning) how those who claim to discover everything but produce no proofs of the same may be confuted as having actually pretended to discover the impossible.

What are the problems I mean, and what are those of which you have already received the proofs, and those of which the proofs are contained in this book respectively, I think it proper to specify. The first of the problems was Given a sphere, to find a plane area equal to the surface of the sphere and this was first made manifest on the publication of the book concerning the sphere, for when it is once proved that the surface of any sphere is four times the greatest circle in the sphere, it is clear that it is possible to find a plane area equal to the surface of the sphere. The second was Given a cone or a cylinder to find a sphere equal to the cone or cylinder, the third To cut a given sphere by a plane so that the segments of it have to one another an assigned ratio the fourth To cut a given sphere by a plane so that the segments of the surface have to one another an assigned ratio the fifth To make a given segment of a sphere similar to a given segment of a sphere,¹ the sixth Given two segments of either the same or different spheres to find a segment of a sphere which shall be similar to one of the segments and have its surface equal to the surface of the other segment. The seventh was From a given sphere to cut off a segment by a plane so that the segment bears to the cone which has the same base

¹Cf. *On the Sphere and Cylinder* II. 5

as the segment and equal height an assigned ratio greater than that of three to two. Of all the propositions just enumerated Heraclides brought you the proofs. The proposition stated next after these was wrong viz that, if a sphere be cut by a plane into unequal parts the greater segment will have to the less the duplicate ratio of that which the greater surface has to the less. That this is wrong is obvious by what I sent you before, for it included this proposition. If a sphere be cut into unequal parts by a plane at right angles to any diameter in the sphere the greater segment of the surface will have to the less the same ratio as the greater segment of the diameter has to the less while the greater segment of the sphere has to the less a ratio less than the duplicate ratio of that which the greater surface has to the less but greater than the sesqui alterate¹ of that ratio. The last of the problems was also wrong viz that, if the diameter of any sphere be cut so that the square on the greater segment is triple of the square on the lesser segment, and if through the point thus arrived at a plane be drawn at right angles to the diameter and cutting the sphere, the figure in such a form as is the greater segment of the sphere is the greatest of all the segments which have an equal surface. That this is wrong is also clear from the theorems which I before sent you. For it was there proved that the hemisphere is the greatest of all the segments of a sphere bounded by an equal surface.

After these theorems the following were propounded concerning the cone.² If a section of a right-angled cone [a parabola] in which the diameter [axis] remains fixed be made to revolve so that the diameter [axis] is the axis [of revolution] let the figure described by the section of the right angled cone be called a *conoid*. And if a plane touch the conoidal figure and another plane drawn parallel to the tangent plane cut off a segment of the conoid, let the base of the segment cut off be defined as the cutting plane and the vertex as the point in which the other plane touches the conoid. Now, if the said figure be cut by a plane at right angles to the axis it is clear that the section will be a circle, but it needs to be proved that the segment cut off will be half as large again as the cone which has the same base as the segment and equal height. And if two segments be cut off from the conoid by planes drawn in any manner, it is clear that the sections will be sections of acute-angled cones [ellipses] if the cutting planes be not at right angles to the axis but it needs to be proved that the segments will bear to one another the ratio of the squares on the lines drawn from their vertices parallel to the axis to meet the cutting planes. The proofs of these propositions are not yet sent to you.

After these came the following propositions about the spiral which are as it were another sort of problem having nothing in common with the foregoing, and I have written out the proofs of them for you in this book. They are as follows. If a straight line of which one extremity remains fixed be made to revolve at a uniform rate in a plane until it returns to the position from which it started and if at the same time as the straight line revolves a point move at a uniform rate along the straight line starting from the fixed extremity the point will describe a spiral in the plane. I say then that the area bounded by the spiral and the straight line which has returned to the position from which it started is a third part of the circle described with the fixed point as centre and with radius the length traversed by the point along the straight line during

¹See On the Sphere and Cylinder II 8

²This should be presumably the *conoid* not the cone

the one revolution. And, if a straight line touch the spiral at the extreme end of the spiral and another straight line be drawn at right angles to the line which has revolved and resumed its position from the fixed extremity of it, so as to meet the tangent I say that the straight line so drawn to meet it is equal to the circumference of the circle. Again, if the revolving line and the point moving along it make several revolutions and return to the position from which the straight line started I say that the area added by the spiral in the third revolution will be double of that added in the second that in the fourth three times, that in the fifth four times and generally the areas added in the later revolutions will be multiples of that added in the second revolution according to the successive numbers while the area bounded by the spiral in the first revolution is a sixth part of that added in the second revolution. Also if on the spiral described in one revolution two points be taken and straight lines be drawn joining them to the fixed extremity of the revolving line and if two circles be drawn with the fixed point as centre and radii the lines drawn to the fixed extremity of the straight line and the shorter of the two lines be produced I say that (1) the area bounded by the circumference of the greater circle in the direction of (the part of) the spiral included between the straight lines the spiral (itself) and the produced straight line will bear to (2) the area bounded by the circumference of the lesser circle the same (part of the) spiral and the straight line joining their extremities the ratio which (3) the radius of the lesser circle together with two thirds of the excess of the radius of the greater circle over the radius of the lesser bears to (4) the radius of the lesser circle together with one third of the said excess.

The proofs then of these theorems and others relating to the spiral are given in the present book. Prefixed to them, after the manner usual in other geometrical works are the propositions necessary to the proofs of them. And here too as in the books previously published I assume the following lemma, that if there be (two) unequal lines or (two) unequal areas the excess by which the greater exceeds the less can by being [continually] added to itself be made to exceed any given magnitude among those which are comparable with [it and with] one another.

PROPOSITION 1

If a point move at a uniform rate along any line, and two lengths be taken on it, they will be proportional to the times of describing them.

Two unequal lengths are taken on a straight line and two lengths on another straight line representing the times and they are proved to be proportional by taking equimultiples of each length and the corresponding time after the manner of Eucl. V Def. 5.

PROPOSITION 2

If each of two points on different lines respectively move along them each at a uniform rate and if lengths be taken one on each line forming pairs, such that each pair are described in equal times the lengths will be proportionals.

This is proved at once by equating the ratio of the lengths taken on one line to that of the times of description which must also be equal to the ratio of the lengths taken on the other line.

PROPOSITION 3

Given any number of circles it is possible to find a straight line greater than the sum of all their circumferences

For we have only to describe polygons about each and then take a straight line equal to the sum of the perimeters of the polygons

PROPOSITION 4

Given two unequal lines, viz a straight line and the circumference of a circle, it is possible to find a straight line less than the greater of the two lines and greater than the less

For by the Lemma, the excess can, by being added a sufficient number of times to itself, be made to exceed the lesser line

Thus e.g., if $c > l$ (where c is the circumference of the circle and l the length of the straight line) we can find a number n such that

$$n(c-l) > l$$

Therefore

$$c-l > \frac{l}{n}$$

and

$$c > l + \frac{l}{n} > l$$

Hence we have only to divide l into n equal parts and add one of them to l . The resulting line will satisfy the condition

PROPOSITION 5

Given a circle with centre O and the tangent to it at a point A it is possible to draw from O a straight line OPF , meeting the circle in P and the tangent in F , such that, if c be the circumference of any given circle whatever

$$FP \cdot OP < (\text{arc } AP) \cdot c$$

Take a straight line as D greater than the circumference c [Prop 3]

Through O draw OH parallel to the given tangent and draw through A a line APH meeting the circle in P and OH in H such that the portion PH intercepted between the circle and the line OH may be equal to D . Join OP and produce it to meet the tangent in F .

$$\begin{aligned} \text{Then } FP \cdot OP &= AP \cdot PH \text{ by parallels} \\ &= AP \cdot D \\ &< (\text{arc } AP) \cdot c \end{aligned}$$

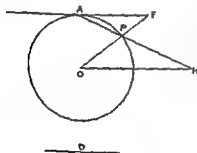
PROPOSITION 6

Given a circle with centre O , a chord AB less than the diameter and OM the perpendicular on AB from O it is possible to draw a straight line OFP meeting the chord AB in F and the circle in P such that

$$FP \cdot PB = D \cdot E$$

where $D \cdot E$ is any given ratio less than $BM \cdot MO$

Draw OH parallel to AB and BT perpendicular to BO meeting OH in T



Then the triangles BMO , OBT are similar, and therefore

$$BM \cdot MO = OB \cdot BT,$$

$$D \cdot E < OB \cdot BT$$

whence

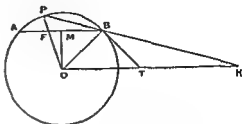
Suppose that a line PH (greater than BT) is taken such that

$$D \cdot E = OB \cdot PH$$

and let PH be so placed that it passes through B and P lies on the circumference of the circle, while H is on the line OH (PH will fall outside BT , because $PH > BT$). Join OP meeting AB in F .

We now have

$$\begin{aligned} FI \cdot PB &= OP \cdot PH \\ &= OB \cdot PH \\ &= D \cdot E \end{aligned}$$



PROPOSITION 7

Given a circle with centre O a chord AB less than the diameter, and OM the perpendicular on it from O it is possible to draw from O a straight line OPF , meeting the circle in P and AB produced in F such that

$$FP \cdot PB = D \cdot E$$

where $D \cdot E$ is any given ratio greater than $BM \cdot MO$

Draw OT parallel to AB and BT perpendicular to BO meeting OT in T .

In this case

$$D \cdot E > BM \cdot MO$$

$$> OB \cdot BT, \text{ by similar triangles}$$

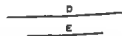
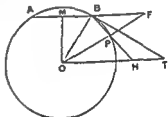
Take a line PH (less than BT) such that

$$D \cdot E = OB \cdot PH,$$

and place PH so that P is on the circle and H is on OT respectively while HP produced passes through B .

Then

$$\begin{aligned} FP \cdot PB &= OP \cdot PH \\ &= D \cdot E \end{aligned}$$



PROPOSITION 8

Given a circle with centre O , a chord AB less than the diameter the tangent at B , and the perpendicular OM from O on AB it is possible to draw from O a straight line OPG , meeting the chord AB in F the circle in P and the tangent in G , such that

$$FP \cdot PG = D \cdot E$$

where $D \cdot E$ is any given ratio less than $BM \cdot MO$

If OT be drawn parallel to AB meeting the tangent at B in T ,

$$BM \cdot MO = OB \cdot BT,$$

so that

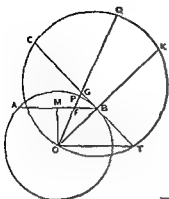
$$D \cdot E < OB \cdot BT$$

Take a point C on TB produced such that

$$D \cdot E = OB \cdot BC,$$

whence

$$BC > BT$$



Through the points O, T, C describe a circle and let OB be produced to meet this circle in A .

Then, since $BC > BT$, and OB is perpendicular to CT it is possible to draw from O a straight line OGQ meeting CT in G and the circle about OTC in Q , such that $GQ = BH$.

Let OGQ meet AB in F and the original circle in P

| | | | |
|---------|------|---------|-------|
| Now | CG | $GT=OG$ | $GQ,$ |
| and | OF | $OG=BT$ | $GT,$ |
| so that | OF | $GT=OG$ | BT |

It follows that

$$\begin{array}{llll}
 CG & GT & OF & GT=OG & GQ & OG & BT, \\
 & & CG & OF=GQ & BT & & \\
 & & & =BA & BT & \text{by construction,} \\
 & & & =BC & OB & \\
 & & & =BC & OP & \\
 OP & OF=BC & CG & & & \\
 PF & OP=BG & BC & & & \\
 PF & BG=OP & BC & & & \\
 & & =OB & BC & & \\
 & & =D & E & &
 \end{array}$$

or

Hence
and therefore
or

PROPOSITION 9

Given a circle with centre O a chord AB less than the diameter the tangent at B , and the perpendicular OM from O on AB it is possible to draw from O a straight line $OPGF$, meeting the circle in P , the tangent in G , and AB produced in F , such that

ГР ВГ-Д Е.

where D/E is any given ratio greater than BM/MO

Let OT be drawn parallel to AB
meeting the tangent at B in T

Then

$D \ E > BM \ MO$

$\angle OB$ BT by similar triangles

Produce TB to C so that

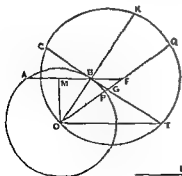
$$DE = OB \quad BC$$

whence $BC < BT$

Describe a circle through the points O T C and produce OB to meet this circle in K .

Then since $TB > BC$, and $OB \perp$ perpendicular to CT it is possible to draw from O a line OGQ meeting CT in G and the circle about OTC

in Q such that $GQ = BA$. Let OQ meet the original circle in P and AB produced in F .



1 1 1

We now prove, exactly as in the last proposition, that

$$\begin{aligned} CG \quad OF &= BA \quad BT \\ &= BC \quad OP \end{aligned}$$

Thus as before,

$$OP \quad OF = BC \quad CG,$$

and
whence

$$OP \quad PF = BC \quad BG,$$

$$PF \quad BC = OP \quad BC$$

$$= OB \quad BC$$

$$= D \quad F$$

PROPOSITION 10

If $A_1, A_2, A_3 \dots A$ be n lines forming an ascending arithmetical progression in which the common difference is equal to A_1 , the least term, then

$$(n+1)A^2 + A_1(A_1 + A_2 + \dots + A) = 3(1^2 + 2^2 + \dots + A^2)$$

[Archimedes' proof of this proposition is given above, pp 456-7, and it is there pointed out that the result is equivalent to

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}]$$

COR 1 It follows from this proposition that

$$n+1 < 3(1^2 + 2^2 + \dots + A^2),$$

and also that

$$n A^2 > 3(A_1^2 + A_2^2 + \dots + A_{n-1}^2)$$

[For the proof of the latter inequality see p 457 above]

COR 2 All the results will equally hold if similar figures are substituted for squares

PROPOSITION 11

If $A_1, A_2, A_3 \dots A$ be n lines forming an ascending arithmetical progression (in which the common difference is equal to the least term A_1), then

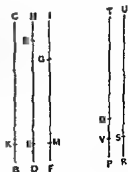
$$(n-1)A^2 - (A - A_1)^2 < A_1^2 \{ 1 + A_1 + \frac{1}{2}(A - A_1)^2 \}$$

but

$$(n-1)A_1^2 - (A - A_1)^2 > 1 - \{ 1 + A_1 + \frac{1}{2}(1 - A_1)^2 \}$$

[Archimedes sets out the terms side by side in the manner shown in the figure where $BC = A$, $DE = A - A_1$, $RS = A_1$ and produces DE , FG , RS until they are respectively equal to BC or A so that EH , GI , SU in the figure are respectively equal to A_1 , A_2 , $A_3 \dots A_{n-1}$. He further measures lengths BA , DL , FM , PI along BC , DE , FG , PQ respectively each equal to RS .

The figure makes the relations between the terms easier to see with the eye but the use of so large a number of letters makes the proof somewhat difficult to follow, and it may be more clearly represented as follows]



It is evident that $(A_n - A_1) = A_{n-1}$

The following proposition is therefore obviously true viz

$$(n-1)A^2 - (n-1)(A - A_1 + \frac{1}{2}A_{n-1}) = A_n^2 \{ 1 + A_1 + \frac{1}{2}(A_n - A_1)^2 \}$$

In order therefore to prove the desired result we have only to show that

$$(n-1) \frac{1}{2} A_1 + \frac{1}{2}(n-1)A_{n-1} < (A_{n-2}^2 + A_{n-3}^2 + \dots + A_1^2)$$

but

I To prove the first inequality we have

$$(n-1)A_1 + \frac{1}{2}(n-1)A_{n-1} = (n-1) \frac{1}{2} A_1^2 + (n-1)A_1 \frac{1}{2} A_{n-1} + \frac{1}{2}(n-1)A_{n-1}^2 \quad (1)$$

And

$$\begin{aligned} A_n^2 + A_{n-1}^2 + \dots + A_1^2 &= (A_{n-2} + A_1)^2 + (A_{n-3} + A_2)^2 + \dots + (A_1 + A_2)^2 \\ &= (A_{n-2}^2 + A_{n-3}^2 + \dots + A_1^2) \\ &\quad + (n-1)A_1^2 \\ &\quad + 2A_1(A_{n-2} + A_{n-3} + \dots + A_1) \\ &= (A_{n-2}^2 + A_{n-3}^2 + \dots + A_1^2) \\ &\quad + (n-1)A_1^2 \\ &\quad + A_1\{A_{n-2} + A_{n-3} + \dots + A_1 + A_1 + A_2 + \dots + A_{n-2} + A_{n-1}\} \\ &= (A_{n-2}^2 + A_{n-3}^2 + \dots + A_1^2) \\ &\quad + (n-1)A_1^2 \\ &\quad + nA_1 A_{n-1} \end{aligned} \quad (2)$$

Comparing the right hand sides of (1) and (2), we see that $(n-1) \frac{1}{2} A_1$ is common to both sides, and

$$(n-1)A_1 A_{n-1} < nA_1 A_{n-1}$$

while by Prop 10 Cor 1

$$\frac{1}{2}(n-1)A_{n-1}^2 < A_{n-2}^2 + A_{n-3}^2 + \dots + A_1^2$$

It follows therefore that

$$(n-1) \frac{1}{2} A_1 + \frac{1}{2}(n-1)A_{n-1}^2 < (A_{n-2}^2 + A_{n-3}^2 + \dots + A_1^2)$$

and hence the first part of the proposition is proved

II We have now in order to prove the second result, to show that

$$(n-1)A_n \frac{1}{2} A_1 + \frac{1}{2}(n-1)A_{n-1}^2 > (A_{n-2}^2 + A_{n-3}^2 + \dots + A_1^2)$$

The right hand side is equal to

$$\begin{aligned} (A_{n-2} + A_1)^2 + (A_{n-3} + A_2)^2 + \dots + (A_1 + A_2)^2 + A_1^2 \\ = A_{n-2}^2 + A_{n-3}^2 + \dots + A_1^2 \\ + (n-1)A_1^2 \\ + 2A_1(A_{n-2} + A_{n-3} + \dots + A_1) \\ = (A_{n-2}^2 + A_{n-3}^2 + \dots + A_1^2) \\ + (n-1)A_1^2 \\ + A_1\{A_{n-2} + A_{n-3} + \dots + A_1 + A_1 + A_2 + \dots + A_{n-2} + A_{n-1}\} \\ = (A_{n-2}^2 + A_{n-3}^2 + \dots + A_1^2) \\ + (n-1)A_1^2 \\ + (n-2)A_1 A_{n-1} \end{aligned} \quad (3)$$

Comparing this expression with the right hand side of (1) above, we see that $(n-1) \frac{1}{2} A_1$ is common to both sides and

$$(n-1) \frac{1}{2} A_1 A_{n-1} > (n-2)A_1 A_{n-1}$$

while by Prop 10 Cor 1,

$$\frac{1}{2}(n-1)A_{n-1}^2 > (A_{n-2}^2 + A_{n-3}^2 + \dots + A_1^2)$$

Hence $(n-1)A_n \frac{1}{2} A_1 + \frac{1}{2}(n-1)A_{n-1}^2 > (A_{n-2}^2 + A_{n-3}^2 + \dots + A_1^2)$,

and the second required result follows

CON The results in the above proposition are equally true if similar figures be substituted for squares on the several lines

DEFINITIONS

1 If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started, and if, at the same time as the line revolves a point move at a uniform rate along the straight line beginning from the extremity which remains fixed the point will describe a *spiral* (ὀρίξ) in the plane

2 Let the extremity of the straight line which remains fixed while the straight line revolves be called the *origin* of the spiral

3 And let the position of the line from which the straight line began to revolve be called the *initial line* in the revolution

4 Let the length which the point that moves along the straight line describes in one revolution be called the *first distance* that which the same point describes in the second revolution the *second distance*, and similarly let the distances described in further revolutions be called after the number of the particular revolution

5 Let the area bounded by the spiral described in the first revolution and the *first distance* be called the *first area* that bounded by the spiral described in the second revolution and the *second distance* the *second area*, and similarly for the rest in order

6 If from the origin of the spiral any straight line be drawn let that side of it which is in the same direction as that of the revolution be called *forward* (πρὸς τοὺς μενῶν), and that which is in the other direction *backward* (ἐπὶ τοὺς μενῶν)

7 Let the circle drawn with the *origin* as centre and the *first distance* as radius be called the *first circle* that drawn with the same centre and twice the radius the *second circle*, and similarly for the succeeding circles

PROPOSITION 12

If any number of straight lines drawn from the origin to meet the spiral make equal angles with one another the lines will be in arithmetical progression

[The proof is obvious]

PROPOSITION 13

If a straight line touch the spiral it will touch it in one point only

Let O be the origin of the spiral and BC a tangent to it

If possible let BC touch the spiral in two points P Q Join OP OQ and bisect the angle POQ by the straight line OR meeting the spiral in R

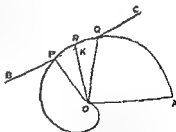
Then [Prop 12] OR is an arithmetic mean between OP and OQ or

$$OP + OQ = 2OR$$

But in any triangle POQ if the bisector of the angle POQ meets PQ in K

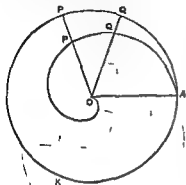
$$OP + OQ > 2OK$$

Therefore $OK < OR$ and it follows that some point on BC between P and Q lies within the spiral Hence BC cuts the spiral, which is contrary to the hypothesis



PROPOSITION 14

If O be the origin, and P Q two points on the first turn of the spiral and if OP OQ produced meet the 'first circle' $AKPQ$ in P' , Q' respectively, OA being the initial line, then



$$OP \cdot OQ = (\text{arc } AKP) \cdot (\text{arc } AKQ)$$

For while the revolving line OA moves about O the point A on it moves uniformly along the circumference of the circle $AKP'Q'$ and at the same time the point describing the spiral moves uniformly along OA

Thus while A describes the arc AKP' , the moving point on OA describes the length OP , and while A describes the arc AKQ' the moving point on OA describes the distance OQ

Hence

$$OP \cdot OQ = (\text{arc } AKP) \cdot (\text{arc } AKQ)$$

[Prop 2]

PROPOSITION 15

If P Q be points on the second turn of the spiral, and OP , OQ meet the 'first circle' $AKPQ$ in P' , Q' , as in the last proposition and if c be the circumference of the 'first circle' then

$$OP \cdot OQ = c + (\text{arc } AKP') \cdot c + (\text{arc } AKQ')$$

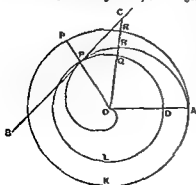
For, while the moving point on OA describes the distance OP the point A describes the whole of the circumference of the first circle together with the arc AKP' and while the moving point on OA describes the distance OQ the point A describes the whole circumference of the first circle together with the arc AKQ'

Con. Similarly, if P Q are on the n th turn of the spiral

$$OP \cdot OQ = (n-1)c + (\text{arc } AKP') \cdot (n-1)c + (\text{arc } AKQ')$$

PROPOSITIONS 16 17

If BC be the tangent at P , any point on the spiral PC being the forward part of BC , and if OP be joined, the angle OPC is obtuse while the angle OPB is acute



I Suppose P to be on the first turn of the spiral

Let OA be the initial line AKP the first circle. Draw the circle DLP with centre O and radius OP meeting OA in D . This circle must then in the forward direction from P fall within the spiral and in the backward direction outside it since the radii vectores of the spiral are on the forward side greater and on the 'backward' side less than OP . Hence the angle OPC cannot be acute since it can not be less than the angle between OP and

the tangent to the circle at P which is a right angle

It only remains therefore to prove that OPC is not a right angle

If possible let it be a right angle BC will then touch the circle at P

Therefore [Prop 5] it is possible to draw a line OQC meeting the circle through P in Q and BC in C such that

$$CQ \cdot OQ < (\text{arc } PQ) \quad (\text{arc } DLP) \quad (1)$$

Suppose that OC meets the spiral in R and the first circle in R and produce OP to meet the "first circle" in P'

From (1) it follows *componendo* that

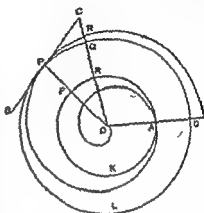
$$\begin{aligned} CO \cdot OQ &< (\text{arc } DLQ) \quad (\text{arc } DLP) \\ &< (\text{arc } AAR') \quad (\text{arc } AAP') \\ &< OR \cdot OP \quad [\text{Prop 14}] \end{aligned}$$

But this is impossible because $OQ = OP$, and $OR < OC$

Hence the angle OPC is not a right angle. It was also proved not to be acute

Therefore the angle OPC is obtuse and the angle OPB consequently acute

II If P is on the second or the n th turn the proof is the same except that in the proportion (1) above we have to substitute for the arc DLP an arc equal to $(p + \text{arc } DLP)$ or $(n-1)p + \text{arc } DLP$ where p is the perimeter of the circle DLP through P . Similarly in the later steps p or $(n-1)p$ will be added to each of the arcs DLQ and DLP and c or $(n-1)c$ to each of the arcs AAR' , AAP' where c is the circumference of the first circle $IA P$



PROPOSITIONS 18-19

I If $O 1$ be the initial line A the end of the first turn of the spiral, and if the tangent to the spiral at A be drawn the straight line OB drawn from O perpendicular to $O 1$ will meet the said tangent in some point B and OB will be equal to the circumference of the first circle

II If A be the end of the second turn the perpendicular OB will meet the tangent at A in some point B and OB will be equal to 2 (circumference of second circle)

III Generally if A be the end of the n th turn and OB meet the tangent at A in B then $OB = nc$ where c is the circumference of the n th circle

I Let $IA C$ be the first circle. Then meet the backward angle between OA and the tangent at A as acute [Prop 16, the tangent will meet the first circle in a second point C . And the angles $C 1 O$, BOA are together less than two right angles therefore OB will meet $1C$ produced in some point B

Then if c be the circumference of the first circle we have to prove that

$$OB = c$$

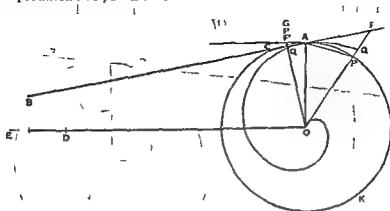
If not OB must be either greater or less than c

(1) If possible suppose $OB > c$

Measure along OB a length OD less than OB but greater than c

We have then a circle $IA C$ a chord AC in it less than the diameter and a ratio $AO : OD$ which is greater than the ratio $1 O : OB$ or (what is by similar

triangles equal to it) the ratio of $\frac{1}{2}AC$ to the perpendicular from O on AC . Therefore [Prop 7] we can draw a straight line OPF , meeting the circle in P and CA produced in F , such that



$$FP \cdot PA = AO \cdot OD$$

Thus alternately, since $AO = PO$,

$$FP \cdot PO = PA \cdot OD < (\text{arc } PA) \cdot c$$

since $(\text{arc } PA) > PA$ and $OD > c$

Componendo

$$FO \cdot PO < (c + \text{arc } PA) \cdot c < OQ \cdot OA$$

where OF meets the spiral in Q

[Prop 15]

Therefore since $OA = OP$, $FO < OQ$ which is impossible

Hence $OB > c$

(2) If possible suppose $OB < c$

Measure OE along OB so that OE is greater than OB but less than c

In this case since the ratio $AO : OE$ is less than the ratio $AO : OB$ (or the ratio of $\frac{1}{2}AC$ to the perpendicular from O on AC), we can [Prop 8] draw a line $OP'G$ meeting AC in F' , the circle in P' and the tangent at A to the circle in G , such that

$$FP' \cdot AG = AO \cdot OE$$

Let $OP'G$ cut the spiral in Q

Then we have alternately

$$FP' \cdot PO = AG \cdot OE > (\text{arc } AP') \cdot c$$

because $AG > (\text{arc } AP')$ and $OE < c$

Therefore

$$FO \cdot PO < (\text{arc } AP') \cdot c < OQ \cdot OA$$

[Prop 14]

But this is impossible since $OA = OP$ and $OQ < OF$

Hence $OB < c$

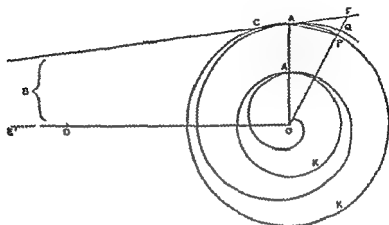
Since therefore OB is neither greater nor less than c

$$OB = c$$

II Let AKC be the second circle AC being the tangent to the spiral

at A' (which will cut the second circle, since the backward angle $O'AC$ is acute) Thus as before, the perpendicular OB to OA' will meet $A'C$ produced in some point B

If then \square is the circumference of the 'second circle, we have to prove that $OB = 2c$



For if not OB must be either greater or less than $2c$

(1) If possible suppose $OB > 2c$

Measure OD along OB so that OD is less than OB but greater than $2c$

Then as in the case of the first circle above we can draw a straight line OPF meeting the second circle in P and C & produced in F such that

$$FP \cdot PA = AO \cdot OD$$

Let OF meet the spiral in Q

We now have since

$$\begin{aligned} AO &= PO \\ FP \cdot PO &= PA \cdot OD \\ &< (\text{arc } AP) \cdot 2c, \end{aligned}$$

because $(\text{arc } AP) > AP$ and $OD > 2c$

$$\begin{aligned} \text{Therefore } FO \cdot PO &< (2c + \text{arc } AP) \cdot 2c \\ &< OQ \cdot OI \end{aligned}$$

[Prop 15, Cor]

Hence $FO < OQ$ which is impossible

Thus

$$OB > 2c$$

Similarly as in the case of the first circle we can prove that

$$OB < 2c$$

Therefore

$$OB = 2c$$

III Proceeding in like manner to the third and succeeding circles we shall prove that

$$OB = nc$$

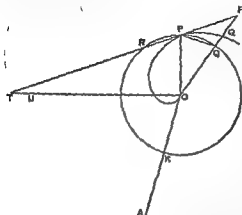
PROPOSITION 20

I If P be any point on the first turn of the spiral and OT be drawn perpendicular to OP OT will meet the tangent at P to the spiral in some point T and, if the circle drawn with centre O and radius OP meet the initial line in K then OT is equal to the arc of this circle between K and P measured in the forward direction of the spiral

II Generally, if P be a point on the n th turn, and the notation be as before, while p represents the circumference of the circle with radius OP ,

$$OT = (n-1)p + \text{arc } KP \text{ (measured 'forward')}$$

I Let P be a point on the first turn of the spiral OA the initial line, PR the tangent at P taken in the 'backward' direction



Then [Prop 16] the angle OPR is acute. Therefore PR meets the circle through P in some point R , and also OT will meet PR produced in some point T .

If now OT is not equal to the arc KRP , it must be either greater or less.

(1) If possible let OT be greater than the arc KRP .

Measure OU along OT less than OT but greater than the arc KRP .

Then, since the ratio $PO : OU$ is greater than the ratio $PO : OT$, or (what is by similar triangles,

equal to it) the ratio of $\frac{1}{2}PR$ to the perpendicular from O on PR , we can draw a line OQF meeting the circle in Q and RP produced in F such that

$$FQ : PQ = PO : OU \quad [\text{Prop 7}]$$

Let OF meet the spiral in Q .

We have then

$$FQ : QO = PQ : OU \\ < (\text{arc } PQ) : (\text{arc } KRP), \text{ by hypothesis}$$

Componendo

$$FO : QO < (\text{arc } KRQ) : (\text{arc } KRP) \\ < OQ : OP \quad [\text{Prop 14}] \\ QO = OP$$

But

Therefore $FO < OQ$ which is impossible.

Hence

$$OT = (\text{arc } KRP)$$

(2) The proof that $OT < (\text{arc } KRP)$ follows the method of Prop 18 I (2), exactly as the above follows that of Prop 18 I (1).

Since then OT is neither greater nor less than the arc KRP it is equal to it.

II If P be on the second turn the same method shows that

$$OT = p + (\text{arc } KRP)$$

and, similarly we have for a point P on the n th turn

$$OT = (n-1)p + (\text{arc } KRP)$$

PROPOSITIONS 21, 22, 23

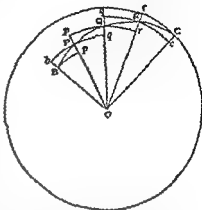
Given an area bounded by any arc of a spiral and the lines joining the extremities of the arc to the origin it is possible to circumscribe about the area one figure, and to inscribe in it another figure each consisting of similar sectors of circles and such that the circumscribed figure exceeds the inscribed by less than any assigned area.

For let BC be any arc of the spiral O the origin. Draw the circle with centre

O and radius OC , where C is the "forward" end of the arc

Then, by bisecting the angle BOC bisecting the resulting angles and so on continually we shall ultimately arrive at an angle COr cutting off a sector of the circle less than any assigned area. Let COr be this sector

Let the other lines dividing the angle BOC into equal parts meet the spiral in P, Q and let Or meet it in R . With O as centre and radii OB, OP, OQ, OR respectively describe arcs of circles Bp, bBq, pQr, qRc each meeting the adjacent radii as shown in the figure. In each case the arc in the 'forward' direction from each point will fall within and the arc in the backward direction outside the spiral.



We have now a circumscribed figure and an inscribed figure each consisting of similar sectors of circles. To compare their areas we take the successive sectors of each beginning from OC and compare them.

The sector OCr in the circumscribed figure stands alone
And

$$(\text{sector } ORq) = (\text{sector } ORc),$$

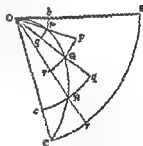
$$(\text{sector } OQp) = (\text{sector } OQr)$$

$$(\text{sector } OPb) = (\text{sector } OPq)$$

while the sector OBp in the inscribed figure stands alone

Hence if the equal sectors be taken away the difference between the circumscribed and inscribed figures is equal to the difference between the sectors OCr and OBp and this difference is less than the sector OCr which is itself less than any assigned area.

The proof is exactly the same whatever be the number of angles into which the angle BOC is divided the only difference being that when the arc begins from the origin the smallest sectors OPb, OPq in each figure are equal and there is therefore no inscribed sector standing by itself so that the difference between the circumscribed and inscribed figures is equal to the sector OCr itself.



Thus the proposition is universally true.

COR. Since the area bounded by the spiral is intermediate in magnitude between the circumscribed and inscribed figures it follows that

- (1) a figure can be circumscribed to the area such that it exceeds the area by less than any assigned space
- (2) a figure can be inscribed such that the area exceeds it by less than any assigned space

PROPOSITION 24

The area bounded by the first turn of the spiral and the initial line is equal to one third of the first circle [$= \frac{1}{3}\pi(2\pi a)^2$ where the spiral is $r = a\theta$]

[The same proof shows equally that, if OP be any radius vector in the first turn of the spiral, the area of the portion of the spiral bounded thereby is equal to one third of that sector of the circle drawn with radius OP which is bounded by the initial line and OP , measured in the "forward" direction from the initial line]

Let O be the origin, OA the initial line, A the extremity of the first turn

Draw the "first circle," i.e. the circle with O as centre and OA as radius

Then if C_1 be the area of the first circle R_1 that of the first turn of the spiral bounded by OA , we have to prove that

$$R_1 = \frac{1}{3}C_1$$

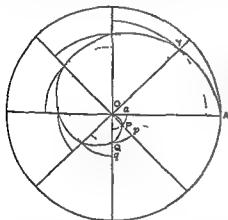
For, if not, R_1 must be either greater or less than C_1

I If possible suppose $R_1 < \frac{1}{3}C_1$

We can then circumscribe a figure about R_1 made up of similar sectors of circles such that, if F be the area of this figure

$$F - R_1 < \frac{1}{3}C_1 - R_1$$

whence $F < \frac{1}{3}C_1$



Let OP OQ , be the radii of the circular sectors beginning from the smallest. The radius of the largest is of course OA

The radii then form an ascending arithmetical progression in which the common difference is equal to the least term OP . If n be the number of the sectors, we have [by Prop 10 Cor 1]

$n \cdot OA^2 < 3(OP^2 + OQ^2 + \dots + OA^2)$, and since the similar sectors are proportional to the squares on their radii it follows that

$$C_1 < 3F,$$

or

$$F > \frac{1}{3}C_1$$

But this is impossible since F was less than $\frac{1}{3}C_1$

Therefore $R_1 < \frac{1}{3}C_1$

II If possible, suppose $R_1 > \frac{1}{3}C_1$

We can then inscribe a figure made up of similar sectors of circles such that, if f be its area

$$R_1 - f < R_1 - \frac{1}{3}C_1$$

whence $f > \frac{1}{3}C_1$

If there are $(n-1)$ sectors their radii as OP OQ , form an ascending arithmetical progression in which the least term is equal to the common difference and the greatest term as OA , is equal to $(n-1)OP$

Thus [Prop 10 Cor 1]

$$n \cdot OA^2 > 3(OP + OQ + \dots + OA^2)$$

whence

$$C_1 > 3f$$

or

$$f < \frac{1}{3}C_1$$

which is impossible since

$$f > \frac{1}{3}C_1$$

Therefore

$$R_1 = \frac{1}{3}C_1$$

Since then R_1 is neither greater nor less than $\frac{1}{3}C_1$,

$$R_1 = \frac{1}{3}C_1$$

PROPOSITIONS 25, 26, 27

[Prop 25] If A_1 be the end of the second turn of the spiral the area bounded by the second turn and OA_1 is to the area of the "second circle" in the ratio of 7 to 12 being the ratio of $\{r_1 r_2 + \frac{1}{3}(r_2 - r_1)^2\}$ to r_2^2 , where r_1, r_2 are the radii of the "first" and "second" circles respectively

[Prop 26] If BC be any arc measured in the forward direction on any turn of a spiral not being greater than the complete turn, and if a circle be drawn with O as centre and OC as radius meeting OB in B' then

$$\begin{aligned} & (\text{area of spiral between } OB \text{ } OC) \quad (\text{sector } OB'C) \\ & = \{OC \cdot OB + \frac{1}{3}(OC - OB)^2\} \cdot OC^2 \end{aligned}$$

[Prop 27] If R_1 be the area of the first turn of the spiral bounded by the initial line R_2 the area of the ring added by the second complete turn R_3 that of the ring added by the third turn, and so on then

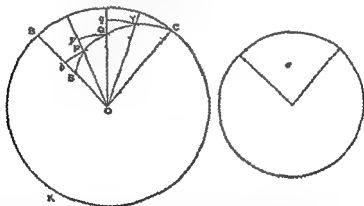
$$R_3 = 2R_1 \quad R_4 = 3R_1 \quad R_5 = 4R_1 \quad R = (n-1)R_1$$

Also

$$R_2 = 6R_1$$

[Archimedes' proof of Prop 25 is *mutatis mutandis* the same as his proof of the more general Prop 26. The latter will accordingly be given here and applied to Prop 25 as a particular case.]

Let BC be an arc measured in the forward direction on any turn of the spiral. CKB the circle drawn with O as centre and OC as radius



Take a circle such that the square of its radius = equal to

$$OC \cdot OB + \frac{1}{3}(OC - OB)^2,$$

and let σ be a sector in it whose central angle is equal to the angle BOC

Thus σ (sector $OB'C$) = $\{OC \cdot OB + \frac{1}{3}(OC - OB)^2\} \cdot OC^2$

and we have therefore to prove that

$$(\text{area of spiral } OBC) = \sigma$$

For if not the area of the spiral OBC (which we will call S) must be either greater or less than σ

I Suppose if possible $S < \sigma$

Circumscribe to the area S a figure made up of similar sectors of circles such that if F be the area of the figure

$$F - S < \sigma - S$$

$$F < \sigma$$

whence

Let the radii of the successive sectors starting from OB be OP, OQ, OC
Produce OP, OQ to meet the circle CKB' ,

If then the lines OB, OP, OQ, OC be n in number, the number of sectors in the circumscribed figure will be $(n-1)$, and the sector $OB'C$ will also be divided into $(n-1)$ equal sectors. Also OB, OP, OQ, OC will form an ascending arithmetical progression of n terms.

Therefore [see Prop 11 and Cor.]

$$(n-1)OC^2 (OP^2 + OQ^2 + \dots + OC^2) < OC \{OC \cdot OB + \frac{1}{2}(OC - OB)^2\} \\ < (\text{sector } OB'C) \cdot \sigma \text{ by hypothesis}$$

Hence, since similar sectors are as the squares of their radii

$$(\text{sector } OB'C) \cdot F < (\text{sector } OB'C) \cdot \sigma$$

so that

$$F > \sigma$$

But this is impossible because

$$F < \sigma$$

Therefore

$$S < \sigma$$

II Suppose if possible $S > \sigma$

Inscribe in the area S a figure made up of similar sectors of circles such that if f be its area,

$$S - f < S - \sigma,$$

whence

$$f > \sigma$$

Suppose OB, OP, OY to be the radii of the successive sectors making up the figure f being $(n-1)$ in number.

We shall have in this case [see Prop 11 and Cor.]

$$(n-1)OC^2 (OB^2 + OP^2 + \dots + OY^2) > OC \{OC \cdot OB + \frac{1}{2}(OC - OB)^2\},$$

whence

$$(\text{sector } OB'C) \cdot f > (\text{sector } OB'C) \cdot \sigma$$

so that

$$f < \sigma$$

But this is impossible because

$$f > \sigma$$

Therefore

$$S > \sigma$$

Since then S is neither greater nor less than σ , it follows that

$$S = \sigma$$

In the particular case where B coincides with A_1 the end of the first turn of the spiral, and C with A_2 the end of the second turn the sector $OB'C$ becomes the complete 'second circle' that, namely with OA_2 (or r_2) as radius

Thus (area of spiral bounded by OA_2) ('second circle')

$$= \{r_2 r_1 + \frac{1}{2}(r_2 - r_1)^2\} \cdot r \\ = (2 + \frac{1}{2}) \cdot 4 \quad (\text{since } r_2 = 2r_1) \\ = 7 \cdot 12$$

Again the area of the spiral bounded by OA_2 is equal to $R_1 + R_2$ (R_1 = the area bounded by the first turn and OA_1 together with the ring added by the second turn). Also the 'second circle' is four times the 'first circle' and therefore equal to $12 R_1$.

Hence

$$(R_1 + R_2) \cdot 12 R_1 = 7 \cdot 12$$

or

$$R_1 + R_2 = 7 R_1$$

Thus

$$R_2 = 6 R_1$$

(1)

Next, for the third turn we have

$$(R_1 + R_2 + R_3) \cdot (\text{third circle}) = \{r_3 r_2 + \frac{1}{2}(r_3 - r_2)^2\} \cdot r_2^2 \\ = (3 \cdot 2 + \frac{1}{2}) \cdot 3^2 \\ = 19 \cdot 27,$$

and

$$(\text{'third circle'}) = 9(\text{'first circle'}) \\ = 27 R_1$$

therefore

$$R_1 + R_2 + R_3 = 19R_1,$$

and, by (1) above it follows that

$$\begin{aligned} R_2 &= 12R_1 \\ &= 2R_3 \end{aligned} \quad (2)$$

and so on

Generally, we have

$$\begin{aligned} (R_1 + R_2 + \dots + R_n) \text{ (nth circle)} &= \{r_n r_{n-1} + \frac{1}{2}(r_n - r_{n-1})^2\} r_n^2, \\ (R_1 + R_2 + \dots + R_{n-1}) \text{ (n-1th circle)} &= \{r_{n-1} r_{n-2} + \frac{1}{2}(r_{n-1} - r_{n-2})^2\} r_{n-1}^2, \\ \text{and} \quad (nth \text{ circle}) \quad (n-1th \text{ circle}) &= r^2 r_{n-1}^2 \end{aligned}$$

Therefore

$$\begin{aligned} (R_1 + R_2 + \dots + R_n) \quad (R_1 + R_2 + \dots + R_{n-1}) \\ = \{n(n-1) + \frac{1}{2}\} \quad \{(n-1)(n-2) + \frac{1}{2}\} \\ = \{3n(n-1) + 1\} \quad \{3(n-1)(n-2) + 1\} \end{aligned}$$

Dividing,

$$R \quad (R_1 + R_2 + \dots + R_{n-1}) = 6(n-1) \quad \{3(n-1)(n-2) + 1\} \quad (\alpha)$$

Similarly

$$R_{n-1} \quad (R_1 + R_2 + \dots + R_{n-2}) = 6(n-2) \quad \{3(n-2)(n-3) + 1\},$$

from which we derive

$$\begin{aligned} R_{n-1} \quad (R_1 + R_2 + \dots + R_{n-1}) \\ = 6(n-2) \quad \{6(n-2) + 3(n-2)(n-3) + 1\} \\ = 6(n-2) \quad \{3(n-1)(n-2) + 1\} \end{aligned} \quad (\beta)$$

Combining (α) and (β) we obtain

$$R \quad R_{n-1} = (n-1) \quad (n-2)$$

Thus

$$R_1 \quad R_2 \quad R_3 \quad R \quad \text{are in the ratio of the successive numbers } 1 \quad 2, 3 \quad (n-1)$$

PROPOSITION 28

If O be the origin and BC any arc measured in the 'forward' direction on any turn of the spiral let two circles be drawn (1) with centre O and radius OB meeting OC in C and (2) with centre O and radius OC , meeting OB produced in B . Then if E denote the area bounded by the larger circular arc BC , the line BB and the spiral BC while F denotes the area bounded by the smaller arc BC , the line CC and the spiral BC

$$E \quad F = \{OB + \frac{1}{2}(OC - OB)\} \quad \{OB + \frac{1}{2}(OC - OB)\}$$

Let σ denote the area of the lesser sector OBC

then the larger sector $OB'C$ is equal to $\sigma + F + E$

Thus [Prop 26]

$$\begin{aligned} (\sigma + F) \quad (\sigma + F + E) = \\ \{OC \quad OB + \frac{1}{2}(OC - OB)^2\} \quad OC^2 \end{aligned} \quad (1)$$

whence

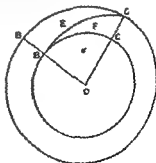
$$\begin{aligned} E \quad (\sigma + F) &= \{OC(OC - OB) - \frac{1}{2}(OC - OB)^2\} \\ &\quad \{OC \quad OB + \frac{1}{2}(OC - OB)^2\} \\ &= \{OB(OC - OB) + \frac{1}{2}(OC - OB)^2\} \\ &\quad \{OC \quad OB + \frac{1}{2}(OC - OB)^2\} \end{aligned} \quad (2)$$

Again

$$(\sigma + F + E) \quad \sigma = OC^2 \quad OB^2$$

Therefore by the first proportion above *ex aequali*

$$(\sigma + F) \quad \sigma = \{OC \quad OB + \frac{1}{2}(OC - OB)^2\} \quad OB^2,$$



whence

$$\begin{aligned} \{\sigma + F\} \quad F = & \{OC \quad OB + \frac{1}{2}(OC - OB)^2\} \\ & \{OB(OC - OB) + \frac{1}{2}(OC - OB)^2\} \end{aligned}$$

Combining this with (2) above we obtain

$$\begin{aligned} E \quad F = & \{OB(OC - OB) + \frac{1}{2}(OC - OB)^2\} \quad \{OB(OC - OB) + \frac{1}{2}(OC - OB)^2\} \\ = & \{OB + \frac{1}{2}(OC - OB)\} \quad \{OB + \frac{1}{2}(OC - OB)\} \end{aligned}$$

ON THE EQUILIBRIUM OF PLANES OR THE CENTRES OF GRAVITY OF PLANES

BOOK ONE

"I POSTULATE the following

1 Equal weights at equal distances are in equilibrium, and equal weights at unequal distances are not in equilibrium but incline towards the weight which is at the greater distance '

2 If, when weights at certain distances are in equilibrium, something be added to one of the weights they are not in equilibrium but incline towards that weight to which the addition was made "

3 ' Similarly if anything be taken away from one of the weights they are not in equilibrium but incline towards the weight from which nothing was taken '

4 "When equal and similar plane figures coincide if applied to one another, their centres of gravity similarly coincide '

5 ' In figures which are unequal but similar, the centres of gravity will be similarly situated By points similarly situated in relation to similar figures I mean points such that, if straight lines be drawn from them to the equal angles they make equal angles with the corresponding sides

6 "If magnitudes at certain distances be in equilibrium, (other) magnitudes equal to them will also be in equilibrium at the same distances

7 "In any figure whose perimeter is concave in (one and) the same direction the centre of gravity must be within the figure

PROPOSITION 1

Weights which balance at equal distances are equal

For, if they are unequal take away from the greater the difference between the two The remainders will then not balance [Post 3], which is absurd

Therefore the weights cannot be unequal

PROPOSITION 2

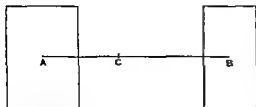
Unequal weights at equal distances will not balance but will incline towards the greater weight

For take away from the greater the difference between the two The equal remainders will therefore balance [Post 1] Hence if we add the difference again the weights will not balance but incline towards the greater [Post 2]

PROPOSITION 3

Unequal weights will balance at unequal distances, the greater weight being at the less distance

Let A B be two unequal weights (of which A is the greater) balancing about C at distances AC BC respectively



Then shall AC be less than BC . For, if not, take away from A the weight $(A-B)$. The remainders will then incline towards B [Post 3]. But this is impossible for (1) if $AC=CB$ the equal remainders will balance or (2) if $AC>CB$, they will incline towards A at the greater distance [Post 1].

Hence $AC < CB$

Conversely, if the weights balance, and $AC < CB$, then $A > B$

PROPOSITION 4

If two equal weights have not the same centre of gravity, the centre of gravity of both taken together is at the middle point of the line joining their centres of gravity [Proved from Prop 3 by *reductio ad absurdum*]

PROPOSITION 5

If three equal magnitudes have their centres of gravity on a straight line at equal distances, the centre of gravity of the system will coincide with that of the middle magnitude

[This follows immediately from Prop 4]

COR 1 The same is true of any odd number of magnitudes if those which are at equal distances from the middle one are equal while the distances between their centres of gravity are equal

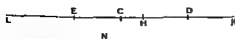
COR 2 If there be an even number of magnitudes with their centres of gravity situated at equal distances on one straight line and if the two middle ones be equal while those which are equidistant from them (on each side) are equal respectively, the centre of gravity of the system is the middle point of the line joining the centres of gravity of the two middle ones

PROPOSITIONS 6, 7

Two magnitudes whether commensurable [Prop 6] or incommensurable [Prop 7] balance at distances reciprocally proportional to the magnitudes

I Suppose the magnitudes A, B to be commensurable, and the points A, B to be their centres of gravity. Let DE be a straight line so divided at C that

$$A \cdot B = DC \cdot CE$$



We have then to prove that if A be placed at E and B at D C is the centre of gravity of the two taken together

Since A, B are commensurable so are DC, CE . Let N be a common measure of DC, CE . Make DH, DA each equal to CE and EL (on CE produced) equal to CD . Then $EH = CD$, since $DH = CE$. Therefore LH

is bisected at E as HK is bisected at D

Thus LH, HK must each contain N an even number of times

Take a magnitude O such that O is contained as many times in A as Λ is contained in LH , whence

$$\begin{array}{l} \text{But} \quad A \quad O = LH \quad N \\ \quad \quad B \quad A = CE \quad DC \\ \quad \quad \quad = HK \quad LH \end{array}$$

Hence *ex aequali* $B \quad O = HK \quad N$, or O is contained in B as many times as Λ is contained in HK .

Thus O is a common measure of $A \quad B$.

Divide LH , HK into parts each equal to Λ and $A \quad B$ into parts each equal to O . The parts of A will therefore be equal in number to those of LH , and the parts of B equal in number to those of HK . Place one of the parts of A at the middle point of each of the parts N of LH and one of the parts of B at the middle point of each of the parts N of HK .

Then the centre of gravity of the parts of A placed at equal distances on LH will be at E the middle point of LH [Prop 5 Cor 2] and the centre of gravity of the parts of B placed at equal distances along HK will be at D the middle point of HK .

Thus we may suppose A itself applied at E and B itself applied at D .

But the system formed by the parts O of A and B together is a system of equal magnitudes even in number and placed at equal distances along LH . And since $LE = CD$ and $EC = DK$, $LC = CK$ so that C is the middle point of LK . Therefore C is the centre of gravity of the system ranged along LK .

Therefore A acting at E and B acting at D balance about the point C .

II Suppose the magnitudes to be incommensurable, and let them be $(A+a)$ and B respectively. Let DE be a line divided at C so that

$$(A+a) \quad B = DC \quad CE$$

Then if $(A+a)$ placed at E and B placed at D do not balance about C , $(A+a)$ is either too great to balance B or not great enough.

Suppose if possible that $(A+a)$ is too great to balance B . Take from $(A+a)$ a magnitude a smaller than the deduction which would make the remainder balance B but such that the remainder A and the magnitude B are commensurable.



Then since $A \quad B$ are commensurable and

$$A \quad B < DC \quad CE$$

A and B will not balance [Prop 6] but D will be depressed.

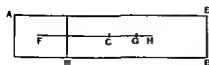
But this is impossible, since the deduction a is an insufficient deduction from $(A+a)$ to produce equilibrium so that E was still depressed.

Therefore $(A+a)$ is not too great to balance B and similarly it may be proved that B is not too great to balance $(A+a)$.

Hence $(A+a) \quad B$ taken together have their centre of gravity at C .

PROPOSITION 8

If AB be a magnitude whose centre of gravity is C and AD a part of it whose centre of gravity is F then the centre of gravity of the remaining part will be a point G on FC produced such that



$$GC \cdot CF = (AD) \cdot (DE)$$

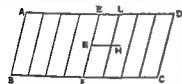
For, if the centre of gravity of the remainder (DE) be not G , let it be a point H . Then an absurdity follows at once from Props 6, 7

PROPOSITION 9

The centre of gravity of any parallelogram lies on the straight line joining the middle points of opposite sides

Let $ABCD$ be a parallelogram, and let EF join the middle points of the opposite sides AD, BC

If the centre of gravity does not lie on EF suppose it to be H , and draw HA parallel to AD or BC meeting EF in K



Then it is possible by bisecting ED then bisecting the halves and so on continually to arrive at a length EL less than KH . Divide both AE and ED into parts each equal to EL and through the points of division draw parallels to AB or CD

We have then a number of equal and similar parallelograms, and if any one be applied to any other their centres of gravity coincide [Post 4]. Thus we have an even number of equal magnitudes whose centres of gravity lie at equal distances along a straight line. Hence the centre of gravity of the whole parallelogram will lie on the line joining the centres of gravity of the two middle parallelograms [Prop 5, Cor 2]

But this is impossible for H is outside the middle parallelograms

Therefore the centre of gravity cannot but lie on EF

PROPOSITION 10

The centre of gravity of a parallelogram is the point of intersection of its diagonals

For by the last proposition the centre of gravity lies on each of the lines which bisect opposite sides. Therefore it is at the point of their intersection, and this is also the point of intersection of the diagonals

Alternative proof

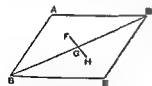
Let $ABCD$ be the given parallelogram and BD a diagonal. Then the triangles ABD, CDB are equal and similar, so that [Post 4] if one be applied to the other their centres of gravity will fall one upon the other

Suppose F to be the centre of gravity of the triangle ABD . Let G be the middle point of BD . Join FG and produce it to H , so that $FG = GH$

If we then apply the triangle ABD to the triangle CDB so that AD falls on CB and AB on CD the point F will fall on H

But [by Post 4] F will fall on the centre of gravity of CDB . Therefore H is the centre of gravity of CDB

Hence since F, H are the centres of gravity of the two equal triangles the centre of gravity of the whole parallelogram is at the middle point of FH i.e. at the middle point of BD which is the intersection of the two diagonals



PROPOSITION 11

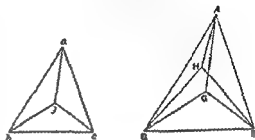
If abc ABC be two similar triangles and g G two points in them similarly situated with respect to them respectively, then if g be the centre of gravity of the triangle abc G must be the centre of gravity of the triangle ABC

Suppose

$$ab : bc : ca = AB : BC : CA$$

The proposition is proved by an obvious *reductio ad absurdum*. For if G be not the centre of gravity of the triangle ABC suppose H to be its centre of gravity.

Post. 5 requires that g H shall be similarly situated with respect to the triangles respectively and this leads at once to the absurdity that the angles HAB , GAB are equal



PROPOSITION 12

Given two similar triangles abc ABC and d D the middle points of bc BC respectively then, if the centre of gravity of abc lie on ad , that of ABC will lie on AD

Let g be the point on ad which is the centre of gravity of abc

Take G on AD such that

$$ad : ag = AD : AG$$

and join gb , gc GB GC

Then since the triangles are similar and bd BD are the halves of bc BC respectively

$$ab : bd = AB : BD$$

and the angles abd ABD are equal

Therefore the triangles abd ABD are similar and

$$\angle bad = \angle BAD$$

Also

$$ba : ad = BA : AD$$

while from above

$$ad : ag = AD : AG$$

Therefore $ba : ag = BA : AG$ while the angles bag BAG are equal

Hence the triangles bag BAG are similar and

$$\angle abg = \angle ABG$$

And since the angles abd ABD are equal it follows that

$$\angle gbd = \angle GBD$$

In exactly the same manner we prove that

$$\angle gac = \angle GAC$$

$$\angle acg = \angle ACG$$

$$\angle gcd = \angle GCD$$

Therefore g G are similarly situated with respect to the triangles respectively whence [Prop. 11] G is the centre of gravity of ABC

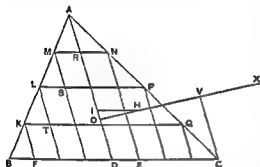


PROPOSITION II

In any triangle the centre of gravity lies on the straight line joining any angle to the middle point of the opposite side

Let ABC be a triangle and D the middle point of BC Join AD Then shall the centre of gravity lie on AD

For if possible let this not be the case and let H be the centre of gravity Draw HI parallel to CB meeting AD in I



Then if we bisect DC then bisect the halves, and so on, we shall at length arrive at a length as DE less than HI Divide both BD and DC into lengths each equal to DE and through the points of division draw lines each parallel to DA meeting BA and AC in points as K, L, M and N, P, Q respectively

Join MN, LP, KQ which lines will then be each parallel to BC

We have now a series of parallelograms as FQ, TP, SN and AD bisects opposite sides in each Thus the centre of gravity of each parallelogram lies on AD [Prop 9] and therefore the centre of gravity of the figure made up of them all lies on AD

Let the centre of gravity of all the parallelograms taken together be O Join OH and produce it, also draw CV parallel to DA meeting OH produced in V

Now if n be the number of parts into which AC is divided

$$\begin{aligned}\triangle ADC \text{ (sum of triangles on } AN, NP, \dots) &= AC^2 (AN^2 + NP^2 + \dots) \\ &= n^2 \cdot n \\ &= n \cdot 1 \\ &= AC \cdot AN\end{aligned}$$

Similarly

$$\triangle ABD \text{ (sum of triangles on } AM, ML, \dots) = AB \cdot AM$$

And

$$AC \cdot AN = AB \cdot AM$$

It follows that

$$\begin{aligned}\triangle ABC \text{ (sum of all the small } \triangle s) &= CA \cdot AN \\ &> VO \cdot OH \text{ by parallels}\end{aligned}$$

Suppose OV produced to λ so that

$$\triangle ABC \text{ (sum of small } \triangle s) = \lambda O \cdot OH$$

whence *dividendo*

$$\text{(sum of parallelograms) (sum of small } \triangle s) = \lambda H \cdot HO$$

Since then the centre of gravity of the triangle ABC is at H and the centre of gravity of the part of it made up of the parallelograms is at O it follows from Prop 8 that the centre of gravity of the remaining portion consisting of all the small triangles taken together is at λ

But this is impossible since all the triangles are on one side of the line through λ parallel to AD

Therefore the centre of gravity of the triangle cannot but lie on AD

Alternative proof

Suppose if possible that H not lying on AD , is the centre of gravity of the triangle ABC . Join AH , BH , CH . Let E , F be the middle points of CA , AB respectively and join DE , EF , FD . Let EF meet AD in M .

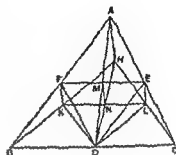
Draw FA , EL parallel to AH meeting BH , CH in K , L respectively. Join KD , HD , LD , KL . Let KL meet DH in N and join MN .

Since DE is parallel to AB the triangles ABC , EDC are similar.

And, since $CE = EA$ and EL is parallel to AH it follows that $CL = LH$. And $CD = DB$. Therefore BH is parallel to DL .

Thus in the similar and similarly situated triangles ABC , EDC the straight lines AH , BH are respectively parallel to EL , DL and it follows that H , L are similarly situated with respect to the triangles respectively.

But H is by hypothesis the centre of gravity of ABC . Therefore L is the centre of gravity of EDC . [Prop 11]



Similarly the point K is the centre of gravity of the triangle FBD . And the triangles FBD , EDC are equal so that the centre of gravity of both together is at the middle point of KL i.e. at the point N .

The remainder of the triangle ABC after the triangles FBD , EDC are deducted is the parallelogram $AFDE$ and the centre of gravity of this parallelogram is at M the intersection of its diagonals.

It follows that the centre of gravity of the whole triangle ABC must be on MN that is MN must pass through H which is impossible (since MN is parallel to AH).

Therefore the centre of gravity of the triangle ABC cannot but be on AD .

PROPOSITION 14

It follows at once from the last proposition that the centre of gravity of any triangle is at the intersection of the lines drawn from any two angles to the middle points of the opposite sides respectively.

PROPOSITION 15

If AD , BC be the two parallel sides of a trapezium $ABCD$, AD being the smaller, and if AD , BC be bisected at E , F respectively then the centre of gravity of the trapezium is at a point G on EF such that

$$GE : GF = (2BC + AD) : (2AD + BC)$$

Produce BA , CD to meet at O . Then FE produced will also pass through O , since $AE = ED$ and $BF = FC$.

Now the centre of gravity of the triangle OAD will be on OE and that of the triangle OBC will be on OF . [Prop 13]

It follows that the centre of gravity of the remainder the trapezium $ABCD$ will also be on OF . [Prop 8]

Join BD and divide it at L , M into three equal parts. Through L , M draw PQ , RS parallel to BC meeting BA in P , R , FE in W , V and CD in Q , S respectively.

Join DF , BE meeting PQ in H and RS in K respectively

Now, since

$$BL = \frac{1}{2}BD,$$

$$FH = \frac{1}{3}FD$$

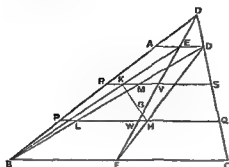
Therefore H is the centre of gravity of the triangle DBC

Similarly since $EK = \frac{1}{3}BE$, it follows that K is the centre of gravity of the triangle ADB

Therefore the centre of gravity of the triangles DBC , ADB together, i.e. of the trapezium, lies on the line HK

But it also lies on OF

Therefore if OF HK meet in G G is the centre of gravity of the trapezium



Hence [Props 6, 7]

$$\triangle DBC \quad \triangle ABD = \frac{KG}{GH}$$

$$= \frac{VG}{GW}$$

$$\triangle DBC \quad \triangle ABD = \frac{BC}{AD}$$

$$\frac{BC}{AD} = \frac{VG}{GW}$$

But

Therefore

It follows that

$$(2BC + AD) (2AD + BC) = (2VG + GW) (2GW + VG) \\ = EG \quad GF$$

Q.E.D.

ON THE EQUILIBRIUM OF PLANES

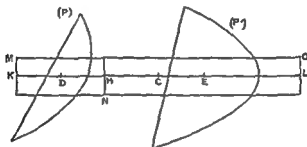
BOOK TWO

PROPOSITION I

If P, P' be two parabolic segments and D, E their centres of gravity respectively the centre of gravity of the two segments taken together will be at a point C on DE determined by the relation

$$P \cdot P' = CE \cdot CD$$

In the same straight line with DE measure EH, EL each equal to DC , and DK equal to DH whence it follows at once that $DK = CE$, and also that $AC = CL$



Apply a rectangle MN equal in area to the parabolic segment P to a base equal to AK , and place the rectangle so that AK bisects it, and is parallel to its base

Then D is the centre of gravity of MN since $AD = DH$

Produce the sides of the rectangle which are parallel to AK , and complete the rectangle NO whose base is equal to HL . Then E is the centre of gravity of the rectangle NO

Now

$$\begin{aligned} (MN) \cdot (NO) &= AK \cdot HL \\ &= DH \cdot EH \\ &= CE \cdot CD \\ &= P \cdot P' \end{aligned}$$

But

$$\begin{aligned} (MN) &= P \\ (NO) &= P' \end{aligned}$$

Therefore

Also since C is the middle point of AL C is the centre of gravity of the whole parallelogram made up of the two parallelograms $(MN) \cdot (NO)$, which are equal to, and have the same centres of gravity as P, P' respectively

Hence C is the centre of gravity of P, P' taken together

DEFINITION AND LEMMAS PRELIMINARY TO PROPOSITION III

If in a segment bounded by a straight line and a section of a right angled cone [a parabola] a triangle be inscribed having the same base as the segment and equal height, if again triangles be inscribed in the remaining segments having the same bases as the segments and equal height and if in the remaining segments triangles be inscribed in the same manner let the resulting figure be said to be inscribed in the recognised manner in the segment

"And it is plain

(1) that the lines joining the two angles of the figure so inscribed which are nearest to the vertex of the segment and the next pairs of angles in order will be parallel to the base of the segment"

(2) "that the said lines will be bisected by the diameter of the segment, and

(3) ' that they will cut the diameter in the proportions of the successive odd numbers, the number one having reference to [the length adjacent to] the vertex of the segment

"And these properties will have to be proved in their proper places"

PROPOSITION 2

If a figure be "inscribed in the recognised manner" in a parabolic segment, the centre of gravity of the figure so inscribed will lie on the diameter of the segment

For, in the figure of the foregoing lemmas, the centre of gravity of the trapezium $BRRb$ must lie on AO , that of the trapezium $RQqr$ on WX , and so on while the centre of gravity of the triangle PAP lies on AV

Hence the centre of gravity of the whole figure lies on AO

PROPOSITION 3

If BAB , bab' be two similar parabolic segments whose diameters are AO ao respectively and if a figure be inscribed in each segment in the recognised manner ' the number of sides in each figure being equal, the centres of gravity of the inscribed figures will divide AO ao in the same ratio "

Suppose $BRQPAP$ $Q R' B$ $brqpap$ $q r' b'$ to be the two figures inscribed "in the recognised manner" Join PP' , QQ' RR' meeting AO in L M , N , and pp' qq' rr' meeting ao in l m n

Then [Lemma (3)]

$$AL \quad LM \quad MN \quad NO = 1 \quad 3 \quad 5 \quad 7 \\ = al \quad lm \quad mn \quad no$$

so that AO ao are divided in the same proportion

Also, by reversing the proof of Lemma (3) we see that

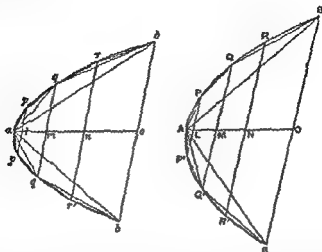
$$PP' \quad pp' = QQ' \quad qq' = RR' \quad rr' = BB' \quad bb'$$

Since then $RR' \quad BB' = rr' \quad bb'$, and these ratios respectively determine the proportion in which AO , ao are divided by the centres of gravity of the trapezia $BRRB$ $brrb$ [I 15] it follows that the centres of gravity of the trapezia divide NO no in the same ratio

Similarly the centres of gravity of the trapezia $RQQR'$ $rqqr'$ divide MN mn in the same ratio respectively, and so on

¹Archimedes enunciates this proposition as true of similar segments but it is equally true of segments which are not similar as the course of the proof will show

Lastly, the centres of gravity of the triangles PAP , pap' divide AL , al respectively in the same ratio



Moreover the corresponding trapezia and triangles are each to each in the same proportion (since their sides and heights are respectively proportional), while AO ao are divided in the same proportion

Therefore the centres of gravity of the complete inscribed figures divide AO ao in the same proportion

PROPOSITION 4

The centre of gravity of any parabolic segment cut off by a straight line lies on the diameter of the segment

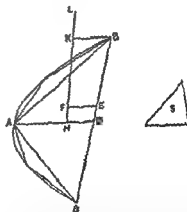
Let BAB be a parabolic segment A its vertex and AO its diameter

Then if the centre of gravity of the segment does not lie on AO suppose it to be if possible the point F Draw FE parallel to AO meeting BB in E

Inscribe in the segment the triangle ABB having the same vertex and height as the segment and take an area S such that

$$\triangle ABB \quad S \approx BE \quad EO$$

We can then inscribe in the segment ' in the recognised manner ' a figure such that the segments of the parabola left over are together less than S '



¹For Prop. 20 of the *Quadrature of the Parabola* proves that if in any segment the triangle with the same base and height be inscribed the triangle is greater than half the segment whence it appears that each time that we increase the number of the sides of the figure inscribed in the recognised manner we take away more than half of the remaining segments

Let the inscribed figure be drawn accordingly, its centre of gravity then lies on AO [Prop 2] Let it be the point H

Join HF and produce it to meet in K the line through B parallel to AO

Then we have

$$\begin{aligned} (\text{inscribed figure}) (\text{remainder of segmt}) &> \triangle ABB \quad S \\ &> BE \quad EO \\ &> HF \quad FH \end{aligned}$$

Suppose L taken on HK produced so that the former ratio is equal to the ratio $LF \quad FH$

Then since H is the centre of gravity of the inscribed figure, and F that of the segment, L must be the centre of gravity of all the segments taken together which form the remainder of the original segment [I 8]

But this is impossible since all these segments lie on one side of the line drawn through L parallel to AO (Ch Post 7)

Hence the centre of gravity of the segment cannot but lie on AO

PROPOSITION 5

If in a parabolic segment a figure be inscribed¹ in the recognised manner, the centre of gravity of the segment is nearer to the vertex of the segment than the centre of gravity of the inscribed figure is

Let BAB be the given segment and AO its diameter First let ABB be the triangle inscribed² in the recognised manner

Divide AO in F so that $AF = 2FO$ F is then the centre of gravity of the triangle ABB

Bisect AB in D and AB' in D' respectively and join DD' meeting AO in E Draw DQ $D'Q'$ parallel to OA to meet the curve QD $Q'D'$ will then be the diameters of the segments whose bases are AB AB' , and the centres of gravity of those segments will be respectively on QD $Q'D'$ [Prop 4] Let them be H , H' and join HH' meeting AO in A

Now QD $Q'D'$ are equal³ and therefore the segments of which they are the diameters are equal {On Conoids and Spheroids Prop 3}

Also since QD $Q'D'$ are parallel and $DE = ED'$ A is the middle point of HH'

Hence the centre of gravity of the equal segments AQB AQB' taken together is K where K lies between E and A And the centre of gravity of the triangle ABB is F

It follows that the centre of gravity of the whole segment BAB lies between A and F and is therefore nearer to the vertex A than F is

Secondly take the five sided figure $BQAQB$ inscribed in the recognised manner QD $Q'D'$ being as before the diameters of the segments AQB AQB'

Then by the first part of this proposition the centre of gravity of the segment AQB (lying of course on QD) is nearer to Q than the centre of gravity of

¹This may either be inferred from Lemma (1) above (since QQ' DD' are both parallel to BB') or from Prop 19 of the *Quadrature of the Parabola* which applies equally to Q or Q'

the triangle AQB is I . Let the centre of gravity of the segment be H and that of the triangle I .

Similarly let H' be the centre of gravity of the segment AQB , and I' that of the triangle AQB' .

It follows that the centre of gravity of the two segments AQB, AQB' taken together is K , the middle point of HH' and that of the two triangles AQB, AQB' is L , the middle point of II' .

If now the centre of gravity of the triangle ABB be F , the centre of gravity of the whole segment BAB (i.e. that of the triangle ABB and the two segments AQB, AQB' taken together) is a point G on AF determined by the proportion

$$(\text{sum of segments } AQB, AQB') : \triangle ABB = FG : GA \quad [167]$$

And the centre of gravity of the inscribed figure $BQAQB$ is a point F on LI determined by the proportion

$$(\triangle AQB + \triangle AQB') : \triangle ABB = GF : FL \quad [167]$$

[Hence

$$FG : GA > GF : FL$$

or

$$GK : FG < GL : FL$$

and *componendo* $FA : FG < FL : FF$ while $FA > FL$]

Therefore $FG > FF$ or G lies nearer than F to the vertex A .

Using this last result and proceeding in the same way we can prove the proposition for any figure inscribed in the recognised manner.

PROPOSITION 6

Given a segment of a parabola cut off by a straight line it is possible to inscribe in it in the recognised manner a figure such that the distance between the centres of gravity of the segment and of the inscribed figure is less than any assigned length.

Let BAB be the segment AO its diameter C its centre of gravity and ABB the triangle inscribed in the recognised manner.

Let D be the assigned length and S an area such that

$$AG : D = \triangle ABB : S$$

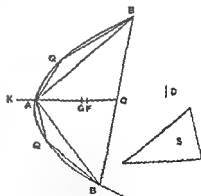
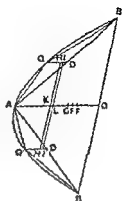
In the segment inscribe in the recognised manner a figure such that the sum of the segments left over is less than S . Let F be the centre of gravity of the inscribed figure.

We shall prove that $FG < D$.

For if not FG must be either equal to or greater than D .

And clearly

$$\begin{aligned} & (\text{inscribed fig}) : (\text{sum of remaining segmts}) \\ & > \triangle ABB : S \\ & > AG : D \\ & > AG : FG \text{ by hypothesis (since } FG < D) \end{aligned}$$



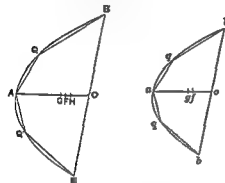
Let the first ratio be equal to the ratio $AG : FG$ (where K lies on GA produced), and it follows that K is the centre of gravity of the small segments taken together [I 8]

But this is impossible, since the segments are all on the same side of a line drawn through K parallel to BB'

Hence FG cannot but be less than D

PROPOSITION 7

If there be two similar parabolic segments their centres of gravity divide their diameters in the same ratio



Let $BAB' bab'$ be the two similar segments AO, ao their diameters and G, g their centres of gravity respectively

Then if G, g do not divide AO, ao respectively in the same ratio, suppose H to be such a point on AO that

$$AH : HO = ag : go$$

and inscribe in the segment BAB' "in the recognised manner" a figure such that if F be its centre of gravity,

$$GF < GH \quad [\text{Prop 6}]$$

Inscribe in the segment bab' "in the recognised manner" a similar

figure then if f be the centre of gravity of this figure

$$ag < af$$

[Prop 5]

And by Prop 3 $af : fo = AF : FO$

But $AF : FO < AH : HO$

$$< ag : go \text{ by hypothesis}$$

Therefore $af : fo < ag : go$ which is impossible

It follows that G, g cannot but divide AO, ao in the same ratio

PROPOSITION 8

If AO be the diameter of a parabolic segment, and G its centre of gravity then

$$AG = \frac{3}{2}GO$$

Let the segment be BAB' . Inscribe the triangle ABB' in the recognised manner and let F be its centre of gravity

Bisect AB, AB' in D, D' , and draw $DQ, D'Q$ parallel to OA to meet the curve so that $QD, Q'D'$ are the diameters of the segments $AQB, A'Q'B'$ respectively

Let H, H' be the centres of gravity of the segments $AQB, A'Q'B'$ respectively. Join QQ', HH' meeting AO in V, K respectively

K is then the centre of gravity of the two segments $AQB, A'Q'B'$ taken together

Now

$$AG : GO = QH : HD$$

[Prop 7]

whence

$$AO : OG = QD : HD$$

But $AO = 4QD$ [as is easily proved by means of Lemma (3) p. 511]

Therefore

$$OG = 4HD$$

and by subtraction

$$AG = 4QH$$

Also by Lemma (2) QQ is parallel to BB' and therefore to DD' . It follows from Prop 7 that HH' is also parallel to QQ or DD' ,

and hence

$$QH = VK$$

Therefore

$$AG = 4VK$$

and

$$AV + KG = 3VK$$

Measuring VK along VK so that $VL = \frac{1}{3}AV$, we have

$$KG = 3LK \quad (1)$$

Again

$$AO = 4AV \quad [\text{Lemma (3)}]$$

$$= 34L, \text{ since } AV = 3VL$$

whence

$$AL = \frac{1}{3}AO = OF \quad (2)$$

Now, by I 6, 7

$\triangle ABB'$ (sum of segmts $AQB, AQ'B$) = $KG + GF$,

and $\triangle ABB = 3(\text{sum of segments } AQB, AQ'B')$

[since the segment ABB' is equal to $\frac{1}{3}\triangle ABB'$ (*Quadrature of the Parabola* Props 17, 24)]

Hence

$$KG = 3GF$$

But

$$KG = 3LK \text{ from (1) above}$$

Therefore

$$LF = LK + KG + GF \\ = 5GF$$

And, from (2),

$$LF = (AO - AL - OF) = \frac{1}{3}AO = OF$$

Therefore

$$OF = 5GF,$$

and

$$OG = 6GF$$

But

$$AO = 3OF = 15GF$$

Therefore, by subtraction

$$AG = 9GF \\ = \frac{3}{2}GO$$

PROPOSITION 9 (LEMMA)

If a, b, c, d be four lines in continued proportion and in descending order of magnitude and if

$$d(a-d) = x \cdot \frac{1}{2}(a-c),$$

and $(2a+4b+6c+3d)(5a+10b+10c+5d) = y(a-c)$,

it is required to prove that

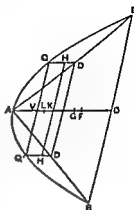
$$x+y = \frac{1}{2}a$$

[The following is the proof given by Archimedes with the only difference that it is set out in algebraical instead of geometrical notation. This is done in the particular case simply in order to make the proof easier to follow. Archimedes exhibits his lines in the figure reproduced in the margin but now that it is possible to use algebraical notation there is no advantage in using the figure and the more cumbersome notation which only obscures the course of the proof. The relation between Archimedes' figure and the letters used below is as follows]

$$AB = a \quad \Gamma B = b \quad \Delta B = c \quad EB = d, \quad 2H = x, \quad H\Theta = y \quad \Delta O = z]$$

We have

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} \quad (1)$$



whence

$$\frac{a-b}{b} = \frac{b-c}{c} = \frac{c-d}{d}$$

and therefore

$$\frac{a-b}{b-c} = \frac{b-c}{c-d} = \frac{a}{b} = \frac{b}{c} = \frac{c}{d} \quad (2)$$

Now

$$\frac{2(a+b)}{2c} = \frac{a+b}{c} = \frac{a+b}{b} \cdot \frac{b}{c} = \frac{a-c}{b-c} \cdot \frac{b-c}{c-d} = \frac{a-c}{c-d}$$

And in like manner,

$$\frac{b+c}{d} = \frac{b+c}{c} \cdot \frac{c}{d} = \frac{a-c}{c-d}$$

It follows from the last two relations that

$$\frac{a-c}{c-d} = \frac{2a+3b+c}{2c+d} \quad (3)$$

Suppose z to be so taken that

$$\frac{2a+4b+4c+2d}{2c+d} = \frac{a-c}{z} \quad (4)$$

so that $z < (c-d)$

Therefore

$$\frac{a-c+z}{a-c} = \frac{2a+4b+6c+3d}{2(a+d)+4(b+c)}$$

And by hypothesis

$$\frac{a-c}{y} = \frac{5(a+d)+10(b+c)}{2a+4b+6c+3d},$$

so that

$$\frac{a-c+z}{y} = \frac{5(a+d)+10(b+c)}{2(a+d)+4(b+c)} = \frac{5}{2} \quad (5)$$

Again dividing (3) by (4) crosswise we obtain

$$\frac{z}{c-d} = \frac{2a+3b+c}{2(a+d)+4(b+c)},$$

whence

$$\frac{c-d-z}{c-d} = \frac{b+3c+2d}{2(a+d)+4(b+c)} \quad (6)$$

But by (2)

$$\frac{c-d}{d} = \frac{a-b}{b} = \frac{3(b-c)}{3c} = \frac{2(c-d)}{2d},$$

so that

$$\frac{c-d}{d} = \frac{(a-b)+3(b-c)+2(c-d)}{b+3c+2d} \quad (7)$$

Combining (6) and (7) we have

$$\frac{c-d-z}{d} = \frac{(a-b)+3(b-c)+2(c-d)}{2(a+d)+4(b+c)}$$

whence

$$\frac{c-z}{d} = \frac{3a+6b+3c}{2(a+d)+4(b+c)} \quad (8)$$

And since [by (1)]

$$\frac{c-d}{c+d} = \frac{b-c}{b+c} = \frac{a-b}{a+b}$$

we have

$$\frac{c-d}{a-c} = \frac{c+d}{b+c+a+b}$$

whence

$$\frac{a-d}{a-c} = \frac{a+2b+2c+d}{a+2b+c} = \frac{2(a+d)+4(b+c)}{2(a+c)+4b} \quad (9)$$

Thus

$$\frac{a-d}{\frac{2}{3}(a-c)} = \frac{2(a+d)+4(b+c)}{\frac{2}{3}[2(a+c)+4b]},$$

and therefore, by hypothesis

$$\frac{d}{x} = \frac{2(a+d)+4(b+c)}{\frac{2}{3}[2(a+c)+4b]}$$

But by (8),

$$\frac{c-z}{d} = \frac{3a+6b+3c}{2(a+d)+4(b+c)}$$

and it follows *ex aequali*, that

$$\frac{c-z}{x} = \frac{3(a+c)+6b}{\frac{2}{3}[2(a+c)+4b]} = \frac{5}{3} \cdot \frac{3}{2} = \frac{5}{2}$$

And, by (5)

$$\frac{a-c+z}{y} = \frac{5}{2}$$

Therefore

$$\frac{5}{2} = \frac{a}{x+y}$$

or

$$x+y = \frac{2}{5}a$$

PROPOSITION 10

If $PPBB$ be the portion of a parabola intercepted between two parallel chords PP BB' bisected respectively in N O by the diameter ANO (N being nearer than O to A , the vertex of the segments), and if NO be divided into five equal parts of which LM is the middle one (L being nearer than M to N), then, if G be a point on LM such that

$$LG \cdot GM = BO^2 (2PN + BO) \quad PN^2 (2BO + PN),$$

G will be the centre of gravity of the area $PPBB$

Take a line ao equal to AO and an on it equal to AN . Let p q be points on the line ao such that

$$ao \cdot aq = aq \cdot an \quad (1)$$

$$ao \cdot an = aq \cdot ap \quad (2)$$

[whence $ao \cdot aq = aq \cdot an = an \cdot ap$ or $ao \cdot aq \cdot an \cdot ap$ are lines in continued proportion and in descending order of magnitude]

Measure along GA a length GF such that

$$op \cdot ap = OL \cdot GF \quad (3)$$

Then since PN BO are ordinates to ANO

$$BO \cdot PN^2 = AO \cdot AN$$

$$= ao \cdot an$$

$$= ao^2 \cdot aq^2 \text{ by (1),}$$

so that

$$BO \cdot PN = ao \cdot aq \quad (4)$$

and

$$BO^2 \cdot PN^2 = ao^2 \cdot aq^2$$

$$= (ao \cdot aq) (aq \cdot an) (an \cdot ap) \quad (5)$$

$$= ao \cdot ap$$

Thus

$$(\text{segment } BAB) \cdot (\text{segment } PAP')$$

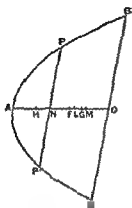
$$= \triangle BAB \cdot \triangle PAP$$

$$= BO^2 \cdot PN^2$$

$$= ao \cdot ap$$

whence

$$(\text{area } PPBB) \cdot (\text{segment } PAP) = op \cdot ap$$



$$=OL \quad GF, \text{ by (3)}$$

$$= \frac{1}{2}OL \quad GF \quad (6)$$

Now

$$BO \quad (2PV + BO) \quad BO^2$$

$$= (2PN + BO) \quad BO$$

$$= (2aq + ao) \quad ao \text{ by (4)}$$

$$BO^2 \quad PN^2$$

$$= ao \quad ap, \text{ by (5),}$$

and

$$PV^2 \quad PV \quad (2BO + PN)$$

$$= PV \quad (2BO + PN)$$

$$= aq \quad (2ao + aq), \text{ by (4)}$$

$$= ap \quad (2an + ap) \text{ by (2)}$$

Hence ex aequali

$$BO^2 \quad (2PV + BO) \quad PN^2$$

$$(2BO + PN) = (2aq + ao)$$

$$(2an + ap),$$

so that by hypothesis

$$LG \quad GM = (2aq + ao) \quad (2an + ap)$$

Componendo and multiplying the antecedents by 5

$$OL \quad GM = \{5(ao + ap) + 10(aq + an)\} \quad (2an + ap)$$

But

$$OL \quad OM = 5 \quad 2 = \{5(ao + ap) + 10(aq + an)\} \quad \{2(ao + ap) + 4(aq + an)\}$$

It follows that

$$OL \quad OG = \{5(ao + ap) + 10(aq + an)\} \quad (2ao + 4aq + 6an + 3ap)$$

Therefore

$$(2ao + 4aq + 6an + 3ap) \quad \{5(ao + ap) + 10(aq + an)\} = OL \quad ON$$

$$= OG \quad on$$

And

$$ap \quad (ao - ap) = ap \quad op$$

$$= GF \quad OL \text{ by hypothesis}$$

$$= GF \quad \frac{1}{2}on$$

while $ao \quad aq \quad an \quad ap$ are in continued proportion

Therefore by Prop 9

$$GF + OG = OF = \frac{1}{2}ao = \frac{1}{2}OA$$

Thus F is the centre of gravity of the segment BAB

[Prop 8]

Let H be the centre of gravity of the segment PAP so that $AH = \frac{1}{2}AP$

$$\text{And, since} \quad AF = \frac{1}{2}AO$$

we have by subtraction

$$HF = \frac{1}{2}OV$$

But by (6) above

$$(\text{area } PPBB) \quad (\text{segment } PAP) = \frac{1}{2}ON \quad GF$$

$$= HF \quad FG$$

Thus since $F \quad H$ are the centres of gravity of the segments $BAB \quad PAP$ respectively it follows [by 1.6.7] that G is the centre of gravity of the area $PLBB$

THE SAND-RECKONER

THERE are some King Gelon who think that the number of the sand is infinite in multitude and I mean by the sand not only that which exists about Syracuse and the rest of Sicily but also that which is found in every region whether inhabited or uninhabited. Again there are some who without regarding it as infinite yet think that no number has been named which is great enough to exceed its multitude. And it is clear that they who hold this view if they imagined a mass made up of sand in other respects as large as the mass of the earth including in it all the seas and the hollows of the earth filled up to a height equal to that of the highest of the mountains would be many times further still from recognising that any number could be expressed which exceeded the multitude of the sand so taken. But I will try to show you by means of geometrical proofs which you will be able to follow that of the numbers named by me and given in the work which I sent to Zeuxippus some exceed not only the number of the mass of sand equal in magnitude to the earth filled up in the way described but also that of a mass equal in magnitude to the universe. Now you are aware that universe is the name given by most astronomers to the sphere whose centre is the centre of the earth and whose radius is equal to the straight line between the centre of the sun and the centre of the earth. This is the common account (*τα γυαφόμενα*) as you have heard from astronomers. But Aristarchus of Samos brought out a book consisting of some hypotheses in which the premisses lead to the result that the universe is many times greater than that now so called. His hypotheses are that the fixed stars and the sun remain unmoved that the earth revolves about the sun in the circumference of a circle the sun lying in the middle of the orbit, and that the sphere of the fixed stars situated about the same centre as the sun is so great that the circle in which he supposes the earth to revolve bears such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface. Now it is easy to see that this is impossible for since the centre of the sphere has no magnitude we cannot conceive it to bear any ratio whatever to the surface of the sphere. We must however take Aristarchus to mean this since we conceive the earth to be as it were the centre of the universe the ratio which the earth bears to what we describe as the universe is the same as the ratio which the sphere containing the circle in which he supposes the earth to revolve bears to the sphere of the fixed stars. For he adapts the proofs of his results to a hypothesis of this kind and in particular he appears to suppose the magnitude of the sphere in which he represents the earth as moving to be equal to what we call the universe.

I say then that even if a sphere were made up of the sand as great as Aristarchus supposes the sphere of the fixed stars to be I shall still prove that,

of the numbers named in the *Principles*,¹ some exceed in multitude the number of the sand which is equal in magnitude to the sphere referred to, provided that the following assumptions be made "

1 'The perimeter of the earth is about 3 000 000 stadia and not greater

'It is true that some have tried as you are of course aware to prove that the said perimeter is about 300 000 stadia. But I go further and, putting the magnitude of the earth at ten times the size that my predecessors thought it, I suppose its perimeter to be about 3,000 000 stadia and not greater'

2 'The diameter of the earth is greater than the diameter of the moon, and the diameter of the sun is greater than the diameter of the earth

"In this assumption I follow most of the earlier astronomers "

3 "The diameter of the sun is about 30 times the diameter of the moon and not greater

'It is true that, of the earlier astronomers, Eudoxus declared it to be about nine times as great and Pheidias my father twelve times while Aristarchus tried to prove that the diameter of the sun is greater than 18 times but less than 20 times the diameter of the moon. But I go even further than Aristarchus, in order that the truth of my proposition may be established beyond dispute, and I suppose the diameter of the sun to be about 30 times that of the moon and not greater'

4 'The diameter of the sun is greater than the side of the chiliagon inscribed in the greatest circle in the (sphere of the) universe

"I make this assumption because Aristarchus discovered that the sun appeared to be about $\frac{1}{180}$ th part of the circle of the zodiac and I myself tried by a method which I will now describe to find experimentally (*ὁργανικώς*) the angle subtended by the sun and having its vertex at the eye

[Up to this point the treatise has been literally translated because of the historical interest attaching to the *ipsissima verba* of Archimedes on such a subject. The rest of the work can now be more freely reproduced and, before proceeding to the mathematical contents of it it is only necessary to remark that Archimedes next describes how he arrived at a higher and a lower limit for the angle subtended by the sun. Thus he did by taking a long rod or ruler fastening on the end of it a small cylinder or disc pointing the rod in the direction of the sun just after its rising (so that it was possible to look directly at it), then putting the cylinder at such a distance that it just concealed and just failed to conceal, the sun and lastly measuring the angles subtended by the cylinder. He explains also the correction which he thought it necessary to make because the eye does not see from one point but from a certain area.]

The result of the experiment was to show that the angle subtended by the diameter of the sun was less than $\frac{1}{180}$ th part and greater than $\frac{1}{200}$ th part of a right angle

To prove that (on this assumption) the diameter of the sun is greater than the side of a chiliagon, or figure with 1000 equal sides inscribed in a great circle of the universe

Suppose the plane of the paper to be the plane passing through the centre of the sun the centre of the earth and the eye at the time when the sun has

¹A lost work of Archimedes

just risen above the horizon. Let the plane cut the earth in the circle EHL and the sun in the circle FAG , the centres of the earth and sun being C , O respectively, and E being the position of the eye.

Further let the plane cut the sphere of the 'universe' (i.e. the sphere whose centre is C and radius CO) in the great circle AOB .

Draw from E two tangents to the circle FAG touching it at P , Q and from C draw two other tangents to the same circle touching it in F , G respectively.

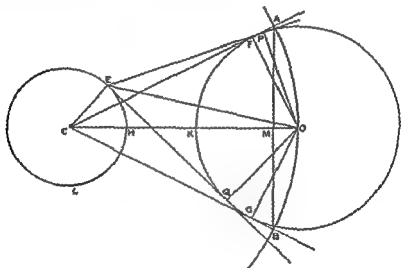
Let CO meet the sections of the earth and sun in H , K respectively and let CF , CG produced meet the great circle AOB in A , B .

Join EO , OF , OG , OP , OQ , AB and let AB meet CO in M .

Now $CO > EO$ since the sun is just above the horizon.

Therefore $\angle PEQ > \angle FCG$

And $\angle PEQ > \frac{1}{100}R$
but $< \frac{1}{100}R$ where R represents a right angle



Thus

$\angle FCG < \frac{1}{100}R$ a fortiori

and the chord AB subtends an arc of the great circle which is less than $\frac{1}{100}$ th of the circumference of that circle i.e.

$AB < (\text{side of 656-sided polygon inscribed in the circle})$

Now the perimeter of any polygon inscribed in the great circle is less than $\frac{1}{100}CO$ [Cf. Measurement of a circle Prop 3]

Therefore $AB < CO < 11 \frac{1}{8}$

and a fortiori $AB < \frac{1}{100}CO$ (a)

Again since $CH = CO$ and AM is perpendicular to CO while OF is perpendicular to CF

$AM = OF$

Therefore $AB = 2AM = (\text{diameter of sun})$

Thus $(\text{diameter of sun}) < \frac{1}{100}CO$ by (a),

and a fortiori $(\text{diameter of earth}) < \frac{1}{100}CO$ [Assumption 2]

Hence $CH + OK < \frac{1}{100}CO$

so that $HK > \frac{1}{100}CO$

or $CO \quad HA < 100 \quad 99$
 And $CO > CF$
 while $HA < EQ$
 Therefore $CF \quad EQ < 100 \quad 99$ (β)
 Now in the right-angled triangles $CFO \quad EQO$ of the sides about the right angles $OF = OQ$, but $EQ < CF$ (since $EO < CO$)
 Therefore $\angle OEQ \quad \angle OCF > CO \quad EO$
 but $< CF \quad EQ$
 Doubling the angles

$$\begin{aligned} \angle PEQ \quad \angle ACB &< CF \quad EQ \\ &< 100 \quad 99, \text{ by } (\beta) \text{ above} \\ \text{But } \angle PEQ &> \frac{1}{100} R, \text{ by hypothesis} \\ \text{Therefore } \angle ACB &> \frac{1}{100} R \\ &> \frac{1}{100} R \end{aligned}$$

It follows that the arc AB is greater than $\frac{1}{100}$ th of the circumference of the great circle AOB

Hence, *a fortiori*

$AB >$ (side of chiliagon inscribed in great circle)
 and AB is equal to the diameter of the sun as proved above
 The following results can now be proved

(diameter of 'universe') $< 10 \, 000$ (diameter of earth)
 and (diameter of "universe") $< 10 \, 000 \, 000 \, 000$ stadia
 (1) Suppose, for brevity that d represents the diameter of the 'universe'
 d that of the sun d that of the earth and d_m that of the moon
 By hypothesis $d > 30d_m$ [Assumption 3]
 and $d > d_m$, [Assumption 2]
 therefore $d < 30d$

Now by the last proposition,
 $d >$ (side of chiliagon inscribed in great circle),
 so that (perimeter of chiliagon) $< 1000d$
 $< 30 \, 000d$

But the perimeter of any regular polygon with more sides than \blacksquare inscribed in a circle is greater than that of the inscribed regular hexagon and therefore greater than three times the diameter Hence

$$(\text{perimeter of chiliagon}) > 3d$$

It follows that $d < 10,000d$
 (2) (Perimeter of earth) $> 3 \, 000 \, 000$ stadia [Assumption 1]
 and (perimeter of earth) $> 3d$

Therefore $d < 1 \, 000 \, 000$ stadia
 whence $d < 10 \, 000 \, 000 \, 000$ stadia

Assumption 5

Suppose a quantity of sand taken not greater than a poppy seed and suppose that it contains not more than 10 000 grains

The proposition here assumed is of course equivalent to the trigonometrical formula which states that, if $\alpha \quad \beta$ are the circular measures of two angles each less than a right angle of which α is the greater then

$$\frac{\tan \alpha}{\tan \beta} > \frac{\alpha}{\beta} > \frac{\sin \alpha}{\sin \beta}$$

Next suppose the diameter of the poppy seed to be not less than $\frac{1}{10}$ th of a finger breadth

ORDERS AND PERIODS OF NUMBERS

I We have traditional names for numbers up to a myriad (10 000), we can therefore express numbers up to a myriad myriads (100 000 000). Let these numbers be called numbers of the *first order*

Suppose the 100 000 000 to be the unit of the *second order* and let the *second order* consist of the numbers from that unit up to (100 000 000)²

Let this again be the unit of the *third order* of numbers ending with (100 000 000)² and so on until we reach the 100 000 000th order of numbers ending with (100 000 000)^{100 000 000} which we will call *P*

II Suppose the numbers from 1 to *P* just described to form the *first period*

Let *P* be the unit of the *first order of the second period* and let this consist of the numbers from *P* up to 100 000 000*P*

Let the last number be the unit of the *second order of the second period* and let this end with (100 000 000)²*P*

We can go on in this way till we reach the 100 000 000th order of the *second period* ending with (100 000 000)^{100 000 000}*P* or *P*²

III Taking *P*² as the unit of the *first order of the third period* we proceed in the same way till we reach the 100 000 000th order of the *third period* ending with *P*³

IV Taking *P*³ as the unit of the *first order of the fourth period*, we continue the same process until we arrive at the 100 000 000th order of the 100 000 000th period ending with *P*^{100 000 000}. This last number is expressed by Archimedes as

a myriad myriad units of the myriad myriad th order of the myriad myriad th period (αἱ μυριάκις μυριάδες περιόδων μυριάκις μυριάδων ἀριθμῶν μυριάς μυριάδες), which is easily seen to be 100 000 000 times the product of (100 000 000)^{100 000 000} and *P*^{100 000 000} i.e. *P*^{100 000 000}

OCTADS

Consider the series of terms in continued proportion of which the first is 1 and the second 10 [i.e. the geometrical progression 1 10¹ 10² 10³ ...] The *first octad* of these terms [i.e. 1 10¹ 10² ... 10⁷] fall accordingly under the *first order of the first period* above described the *second octad* [i.e. 10⁸ 10⁹ ... 10¹⁵] under the *second order of the first period* the first term of the octad being the unit of the corresponding order in each case. Similarly for the *third octad*, and so on. We can in the same way place any number of octads

THEOREM

If there be any number of terms of a series in continued proportion say *A*₁ *A*₂ *A*₃ ... *A*_{*m*} *A* *A*_{*m*+1} ... of which *A*₁=1 *A*₂=10 [so that the series forms the geometrical progression 1 10¹ 10² ... 10^{*m*-1} 10⁻¹ 10^{*m*+1} ...] and if any two terms as *A*_{*m*} *A* be taken and multiplied the product *A*_{*m*} *A* will be a term in the same series and will be as many terms distant from *A* as *A*_{*m*} is distant from *A*₁ also it will be distant from *A*₁ by a number of terms less by one than the sum of the numbers of terms by which *A*_{*m*} and *A* respectively are distant from *A*₁

Take the term which is distant from *A*_{*m*} by the same number of terms as *A*_{*m*}

is distant from A_1 . This number of terms is m (the first and last being both counted). Thus the term to be taken is m terms distant from A_1 , and is therefore the term A_{m+1} .

We have therefore to prove that

$$A_m : A_1 = A_{m+1} : A$$

Now terms equally distant from other terms in the continued proportion are proportional

$$\text{Thus } \frac{A_m}{A_1} = \frac{A_{m+1}}{A}$$

$$\text{But } A_m = A_{m+1} : A_1 \text{ since } A_1 = 1$$

$$\text{Therefore } A_{m+1} = A_m : A \quad (1)$$

The second result is now obvious since A_m is m terms distant from A_1 , A is n terms distant from A_1 and A_{m+1} is $(m+n-1)$ terms distant from A_1 .

APPLICATION TO THE NUMBER OF THE SAND

By Assumption 5 [p. 523],

$$(\text{diam. of poppy seed}) < \frac{1}{10} (\text{finger breadth}),$$

and since spheres are to one another in the triplicate ratio of their diameters it follows that

$$(\text{sphere of diam. 1 finger breadth}) > 64\,000 \text{ poppy seeds}$$

$$> 64\,000 \times 10\,000$$

$$> 640\,000\,000$$

$$> 6 \text{ units of second order} \left. \begin{array}{l} \text{grains} \\ \text{order} + 40,000,000 \\ \text{units of first order} \end{array} \right\} \text{ of sand}$$

$$(\text{a fortiori}) < 10 \text{ units of second order of numbers}$$

We now gradually increase the diameter of the supposed sphere multiplying it by 100 each time. Thus remembering that the sphere is thereby multiplied by 100^3 or 1 000 000 the number of grains of sand which could be contained in a sphere with each successive diameter may be arrived at as follows

| Diameter of sphere | Corresponding number of grains of sand |
|---|--|
| (1) 100 finger breadths | $< 1\,000\,000 \times 10 \text{ units of second order}$ $< (7\text{th term of series}) \times (10\text{th term of series})$ $< 16\text{th term of series} \quad [i.e. 10^{16}]$ $< [10^7 \text{ or } 10\,000\,000 \text{ units of the second order}]$ |
| (2) 10 000 finger breadths | $< 1\,000\,000 \times (\text{last number})$ $< (7\text{th term of series}) \times (16\text{th term})$ $< 22\text{nd term of series} \quad [i.e. 10^{22}]$ $< [10^8 \text{ or } 100\,000 \text{ units of third order}]$ |
| (3) 1 stadium ($< 10\,000$ finger breadths) | $< 100\,000 \text{ units of third order}$ |
| (4) 100 stadia | $< 1\,000\,000 \times (\text{last number})$ $< (7\text{th term of series}) \times (22\text{nd term})$ $< 28\text{th term of series} \quad [10^{28}]$ $< [10^9 \text{ or } 1\,000 \text{ units of fourth order}]$ |
| (5) 10 000 stadia | $< 1\,000\,000 \times (\text{last number})$ $< (7\text{th term of series}) \times (28\text{th term})$ $< 34\text{th term of series} \quad [10^{34}]$ $< 10 \text{ units of fifth order}$ |

| | | |
|---------------------------|--|---------------------|
| (6) 1 000 000 stadia | <(7th term of series)×(34th term) | [10 ³⁹] |
| | <40th term | |
| | <[10 ⁷ or] 10 000 000 units of <i>fifth order</i> | |
| (7) 100 000 000 stadia | <(7th term of series)×(40th term) | [10 ⁴¹] |
| | <46th term | |
| | <[10 ⁸ or] 100 000 units of <i>sixth order</i> | |
| (8) 10 000 000 000 stadia | <(7th term of series)×(46th term) | [10 ⁴³] |
| | <52nd term of series | |
| | <[10 ⁹ or] 1 000 units of <i>seventh order</i> | |

But, by the proposition above [p 523]

(diameter of 'universe') < 10,000 000 000 stadia

Hence the number of grains of sand which could be contained in a sphere of the size of our "universe" is less than 1,000 units of the seventh order of numbers [or 10⁶¹]

From this we can prove further that a sphere of the size attributed by Aristarchus to the sphere of the fixed stars would contain a number of grains of sand less than 10 000 000 units of the eighth order of numbers [or 10⁶⁴⁺⁷ = 10⁶³]

For by hypothesis

(earth) ('universe') = ("universe") (sphere of fixed stars)

And [p 523]

(diameter of "universe") < 10 000 (diam of earth),

whence

(diam of sphere of fixed stars) < 10 000 (diam of 'universe')

Therefore

(sphere of fixed stars) < (10 000)³ ("universe")

It follows that the number of grains of sand which would be contained in a sphere equal to the sphere of the fixed stars

< (10,000)³ × 1 000 units of *seventh order*

< (13th term of series) × (52nd term of series)

< 64th term of series

< [10⁷ or] 10,000,000 units of *eighth order* of numbers

[i.e. 10⁶³]

CONCLUSION

I conceive that these things, King Gelon will appear incredible to the great majority of people who have not studied mathematics but that to those who are conversant therewith and have given thought to the question of the distances and sizes of the earth, the sun and moon and the whole universe the proof will carry conviction And it was for this reason that I thought the subject would be not inappropriate for your consideration

QUADRATURE OF THE PARABOLA

ARCHIMEDES to DOSITHEUS greeting

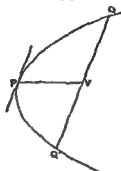
' When I heard that Conon, who was my friend in his lifetime was dead but that you were acquainted with Conon and withal versed in geometry while I grieved for the loss not only of a friend but of an admirable mathematician I set myself the task of communicating to you, as I had intended to send to Conon a certain geometrical theorem which had not been investigated before but has now been investigated by me and which I first discovered by means of mechanics and then exhibited by means of geometry Now some of the earlier geometers tried to prove it possible to find a rectilineal area equal to a given circle and a given segment of a circle, and after that they endeavoured to square the area bounded by the section of the whole cone and a straight line assuming lemmas not easily conceded so that it was recognised by most people that the problem was not solved But I am not aware that any one of my predecessors has attempted to square the segment bounded by a straight line and a section of a right-angled cone (a parabola), of which problem I have now discovered the solution For it is here shown that every segment bounded by a straight line and a section of a right angled cone (a parabola) is four thirds of the triangle which has the same base and equal height with the segment and for the demonstration of this property the following lemma is assumed that the excess by which the greater of (two) unequal areas exceeds the less can by being added to itself be made to exceed any given finite area The earlier geometers have also used this lemma for it is by the use of this same lemma that they have shown that circles are to one another in the duplicate ratio of their diameters and that spheres are to one another in the triplicate ratio of their diameters and further that every pyramid is one third part of the prism which has the same base with the pyramid and equal height also, that every cone is one third part of the cylinder having the same base as the cone and equal height they proved by assuming a certain lemma similar to that aforesaid And in the result each of the aforesaid theorems has been accepted no less than those proved without the lemma As therefore my work now published has satisfied the same test as the propositions referred to I have written out the proof and send it to you, first as investigated by means of mechanics and afterwards too as demonstrated by geometry Prefixed are also the elementary propositions in conics which are of service in the proof Farewell

PROPOSITION 1

If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as PV , and if QQ be a chord parallel to the tangent to the parabola at P and meeting PV in V then

$$QV = VQ$$

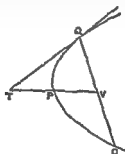
Conversely, if $QV = VQ'$, the chord QQ' will be parallel to the tangent at P



PROPOSITION 2

If in a parabola QQ be a chord parallel to the tangent at P , and if a straight line be drawn through P which is either itself the axis or parallel to the axis, and which meets QQ in V and the tangent at Q to the parabola in T , then

$$PV = PT$$



PROPOSITION 3

If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis as PV and if from two other points Q, Q' on the parabola straight lines be drawn parallel to the tangent at P and meeting PV in V, V' respectively, then

$$PV \cdot PV' = QV^2 \cdot Q'V'^2$$

'And these propositions are proved in the elements of conics''

PROPOSITION 4

If Qq be the base of any segment of a parabola and P the vertex of the segment, and if the diameter through any other point R meet Qq in O and QP (produced if necessary) in F then

$$QV \cdot VO = OF \cdot FR$$

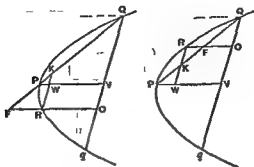
Draw the ordinate RW to PV meeting QP in K

Then $PV \cdot PW = QV^2 \cdot RW^2$,

whence, by parallels $PQ \cdot PK = PQ^2 \cdot PF^2$

¹ i.e. in the treatises on conics by Euclid and Apollonius

In other words, PQ, PF, PK are in continued proportion therefore



$$\begin{aligned} PQ \cdot PF &= PF \cdot PK \\ &= PQ \pm PF \quad PF \pm PK \\ &= QF \cdot AF \end{aligned}$$

Hence by parallels, $QV \cdot VO = OF \cdot FR$

PROPOSITION 5

If Qq be the base of any segment of a parabola P the vertex of the segment, and PV its diameter and if the diameter of the parabola through any other point R meet Qq in O and the tangent at Q in E , then

$$QO \cdot Oq = ER \cdot RO$$

Let the diameter through R meet QP in F

Then, by Prop 4,

$$QV \cdot VO = OF \cdot FR$$

Since $QV = Vq$ it follows that

$$QV \cdot qO = OF \cdot OR \quad (1)$$

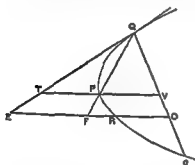
Also if VP meet the tangent in T ,

$$PT = PV, \text{ and therefore } EF = OF$$

Accordingly doubling the antecedents in (1), we have

$$Qq \cdot qO = OE \cdot OR$$

whence $QO \cdot Oq = ER \cdot RO$



PROPOSITIONS II 7¹

Suppose a lever AOB placed horizontally and supported at its middle point O . Let a triangle BCD in which the angle C is right or obtuse be suspended from B and O , so that C is attached to O and CD is in the same vertical line with O . Then if P be such an area as when suspended from A will keep the system in equilibrium

$$P = \frac{1}{2} \Delta BCD$$

¹In Prop II Archimedes takes the separate case in which the angle BCD of the triangle is a right angle so that C coincides with O in the figure and F with E . He then proves in Prop 7 the same property for the triangle in which BCD is an obtuse angle by treating the triangle as the difference between two right-angled triangles BOD BOC and using the result of Prop 6. I have combined the two propositions in one proof for the sake of brevity. The same remark applies to the propositions following Prop 6 7

Take a point E on OB such that $BE = 2OE$, and draw EFH parallel to CD meeting BC , BD in F , H respectively
 Let G be the middle point of FH

Then G is the centre of gravity of the triangle BCD

Hence, if the angular points B , C be set free and the triangle be suspended by attaching F to E , the triangle will hang in the same position as before, because EFG is a vertical straight line "For this is proved"¹

Therefore as before, there will be equilibrium

$$\begin{aligned} \text{Thus } P \triangle BCD &= OE \cdot AO \\ &= 1 \cdot 3 \\ P &= \frac{1}{3} \triangle BCD \end{aligned}$$

or

PROPOSITIONS 8 9

Suppose a lever AOB placed horizontally and supported at its middle point O . Let a triangle BCD right angled or obtuse-angled at C , be suspended from the points B , E on OB the angular point C being so attached to E that the side CD is in the same vertical line with E . Let Q be an area such that

$$AO \cdot OE = \triangle BCD \cdot Q$$

Then if an area P suspended from A keep the system in equilibrium,
 $P < \triangle BCD$ but $> Q$

Take G the centre of gravity of the triangle BCD and draw GH parallel to DC i.e. vertically meeting BO in H

We may now suppose the triangle BCD suspended from H and since there is equilibrium

$$\triangle BCD \cdot P = AO \cdot OH,$$

$$P < \triangle BCD$$

$$\triangle BCD \cdot Q = AO \cdot OE$$

$$\triangle BCD \cdot Q > \triangle BCD \cdot P,$$

$$P > Q$$

(1)

whence

Also

Therefore by (1),

and

PROPOSITIONS 10 11

Suppose a lever AOB placed horizontally and supported at O its middle point. Let $CDEF$ be a trapezium which can be so placed that its parallel sides CD , FE are vertical while C is vertically below O , and the other sides CF , DE meet in B . Let EF meet BO in H and let the trapezium be suspended by attaching F to H and C to O . Further suppose Q to be an area such that

$$AO \cdot OH = (\text{trapezium } CDEF) \cdot Q$$

Then if P be the area which when suspended from A , keeps the system in equilibrium,

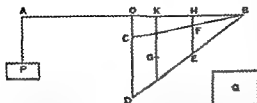
$$P < Q$$

¹Doubtless in the lost book π σι γγω

The same is true in the particular case where the angles at C, F are right and consequently C, F coincide with O, H respectively

Divide OH in K so that

$$(2CD + FE) (2FE + CD) = HK \cdot KO$$



Draw KG parallel to OD and let G be the middle point of the portion of KG intercepted with in the trapezium. Then G is the centre of gravity of the trapezium [On the equilibrium of planes I 15]

Thus we may suppose the trapezium suspended from A and

the equilibrium will remain undisturbed

Therefore $AO \cdot OK = (\text{trapezium } CDEF) \cdot P$

and, by hypothesis $AO \cdot OH = (\text{trapezium } CDEF) \cdot Q$

Since $OK < OH$, it follows that

$$P < Q$$

PROPOSITIONS 12 13

If the trapezium $CDEF$ be placed as in the last propositions except that CD is vertically below a point L on OB instead of being below O and the trapezium is suspended from L, H suppose that Q, R are areas such that

$$AO \cdot OH = (\text{trapezium } CDEF) \cdot Q$$

$$AO \cdot OL = (\text{trapezium } CDEF) \cdot R$$

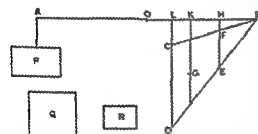
and

If then an area P suspended from A keep the system in equilibrium

$$P > R \text{ but } < Q$$

Take the centre of gravity G of the trapezium as in the last propositions and let the line through G parallel to DC meet OB in K

Then we may suppose the trapezium suspended from A and there will still be equilibrium



Therefore

$$(\text{trapezium } CDEF) \cdot P = AO \cdot OK$$

$$\text{Hence } (\text{trapezium } CDEF) \cdot P > (\text{trapezium } CDEF) \cdot Q$$

$$\text{but } < (\text{trapezium } CDEF) \cdot R$$

$$\text{It follows that } P < Q \text{ but } > R$$

PROPOSITIONS 14 15

Let Qq be the base of any segment of a parabola. Then if two lines be drawn from Q, q each parallel to the axis of the parabola and on the same side of Qq as the segment is either (1) the angles so formed at Q, q are both right angles or (2) one is acute and the other obtuse. In the latter case let the angle at q be the obtuse angle

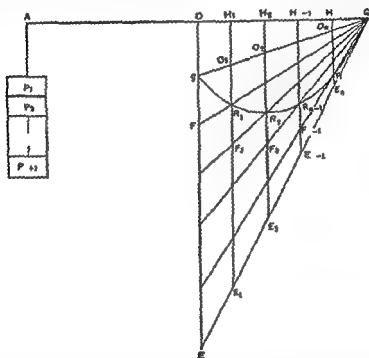
Divide Qq into any number of equal parts at the points O_1, O_2, \dots, O . Draw

through q, O_1, O_2, \dots, O diameters of the parabola meeting the tangent at Q in E, E_1, E_2, \dots, E and the parabola itself in q, R_1, R_2, \dots, R . Join QR_1, QR_2, \dots, QR meeting $qE, O_1E_1, O_2E_2, \dots, O_{n-1}E_{n-1}$ in $F, F_1, F_2, \dots, F_{n-1}$.

Let the diameters $Eg, E_1O_1, E_2O_2, \dots, E_nO_n$ meet a straight line QOA drawn through Q perpendicular to the diameters in the points O, H_1, H_2, \dots, H_n respectively. (In the particular case where Qq is itself perpendicular to the diameters q will coincide with O, O_1 with H_1 and so on.)

It is required to prove that

- (1) $\triangle E q Q < 3(\text{sum of trapezia } FO_1, F_1O_2, \dots, F_{n-1}O \text{ and } \triangle E O Q),$
- (2) $\triangle E q Q > 3(\text{sum of trapezia } R_1O_2, R_2O_3, \dots, R_{n-1}O \text{ and } \triangle R O Q)$



Suppose AO made equal to OQ and conceive QOA as a lever placed horizontally and supported at O . Suppose the triangle $E q Q$ suspended from OQ in the position drawn and suppose that the trapezium EO_1 in the position drawn is balanced by an area P_1 suspended from A the trapezium E_1O_2 in the position drawn is balanced by the area P_2 suspended from A and so on the triangle $E O Q$ being in like manner balanced by P_{n+1} .

Then $P_1 + P_2 + \dots + P_{n+1}$ will balance the whole triangle $E q Q$ as drawn and therefore $P_1 + P_2 + \dots + P_{n+1} = \frac{1}{3} \triangle E q Q$ [Props 6 7]

Again $AO \cdot OH_1 = QO \cdot OH_1$

$$= Qq \cdot qO_1$$

$$= E_1O_1 \cdot O_1R_1 \text{ [by means of Prop 5]}$$

$$= (\text{trapezium } EO_1) \cdot (\text{trapezium } FO_1),$$

whence [Props 10 11]

$$(FO_1) > P_1$$

$$\begin{aligned} \text{Next} \quad & AO \quad OH_1 = E_1O_1 \quad O_1R_1 \\ & = (E_1O_2) \quad (R_1O_2), \quad (\alpha) \\ \text{while} \quad & AO \quad OH_2 = E_2O_2 \quad O_2R_2 \\ & = (E_1O_2) \quad (F_1O_2) \quad (\beta) \end{aligned}$$

and, since (α) and (β) are simultaneously true we have by Props 12, 13,
 $(F_1O_2) > P_2 > (R_1O_2)$

Similarly it may be proved that
 $(F_2O_2) > P_3 > (R_2O_2),$

and so on

Lastly [Props 8, 9] $\triangle E O_n Q > P_{n+1} > \triangle R_n O_n Q$

By addition, we obtain

$$(1) \quad (FO_1) + (F_1O_2) + \dots + (F_{n-1}O) + \triangle E O Q > P_1 + P_2 + \dots + P_{n+1}$$

$$\text{or} \quad \triangle E q Q < 3(FO_1 + F_1O_2 + \dots + F_{n-1}O + \triangle E O Q)$$

$$(2) \quad (R_1O_2) + (R_2O_3) + \dots + (R_{n-1}O) + \triangle R_n O Q < P_2 + P_3 + \dots + P_{n+1}$$

$$\text{or} \quad \triangle E q Q > 3(R_1O_2 + R_2O_3 + \dots + R_{n-1}O + \triangle R O Q)$$

PROPOSITION 16

Suppose Qq to be the base of a parabolic segment, q being not more distant than Q from the vertex of the parabola Draw through q the straight line qE parallel to the axis of the parabola to meet the tangent at Q in E It is required to prove that

$$(\text{area of segment}) = \frac{1}{3} \triangle E q Q$$

For, if not, the area of the segment must be either greater or less than $\frac{1}{3} \triangle E q Q$

I Suppose the area of the segment greater than $\frac{1}{3} \triangle E q Q$ Then the excess can if continually added to itself be made to exceed $\triangle E q Q$ And it is possible to find a submultiple of the triangle $E q Q$ less than the said excess of the segment over $\frac{1}{3} \triangle E q Q$

Let the triangle $F q Q$ be such a submultiple of the triangle $E q Q$ Divide $E q$ into equal parts each equal to qF , and let all the points of division including F be joined to Q meeting the parabola in $R_1 R_2 \dots R$ respectively Through $R_1 R_2 \dots R$ draw diameters of the parabola meeting qQ in $O_1 O_2 \dots O$ respectively

Let $O_1 R_1$ meet QR_2 in F_1

Let $O_2 R_2$ meet QR_1 in D_1 and QR_3 in F_2

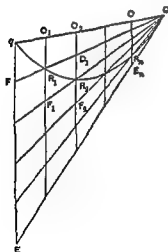
Let $O_3 R_3$ meet QR_2 in D_2 and QR_4 in F_3 , and so on

We have, by hypothesis,

$$\triangle F q Q < (\text{area of segment}) - \frac{1}{3} \triangle E q Q$$

$$\text{or} \quad (\text{area of segment}) - \triangle F q Q > \frac{1}{3} \triangle E q Q \quad (\alpha)$$

Now, since all the parts of qE as qF and the rest are equal $O_1 R_1 = R_1 F_1$, $O_2 D_1 = D_1 R_2 = R_2 F_2$ and so on therefore



$$\begin{aligned}\Delta FqQ &= (FO_1 + R_1O_2 + D_1O_3 + \dots) \\ &= (FO_1 + F_1D_1 + F_2D_2 + \dots + F_{n-1}D_{n-1} + \Delta E R Q) \quad (\beta)\end{aligned}$$

But (area of segment) $< (FO_1 + F_1O_2 + \dots + F_{n-1}O + \Delta E O Q)$

Subtracting we have

$$(\text{area of segment}) - \Delta FqQ < (R_1O_2 + R_2O_3 + \dots + R_{n-1}O + \Delta R O Q),$$

whence *a fortiori*, by (α)

$$\frac{1}{3}\Delta EqQ < (R_1O_2 + R_2O_3 + \dots + R_{n-1}O + \Delta R O Q)$$

But this is impossible since [Props 14, 15]

$$\frac{1}{3}\Delta EqQ > (R_1O + R_2O_2 + \dots + R_{n-1}O + \Delta R O Q)$$

Therefore

$$(\text{area of segment}) > \frac{1}{3}\Delta EqQ$$

II If possible suppose the area of the segment less than $\frac{1}{3}\Delta EqQ$

Take a submultiple of the triangle EqQ as the triangle FqQ less than the excess of $\frac{1}{3}\Delta EqQ$ over the area of the segment, and make the same construction as before

Since $\Delta FqQ < \frac{1}{3}\Delta EqQ - (\text{area of segment})$,
it follows that

$$\begin{aligned}\Delta FqQ + (\text{area of segment}) &< \frac{1}{3}\Delta EqQ \\ &< (FO_1 + F_1O_2 + \dots + F_{n-1}O + \Delta E O Q) \quad [\text{Props 14, 15}]\end{aligned}$$

Subtracting from each side the area of the segment, we have

$$\begin{aligned}\Delta FqQ &< (\text{sum of spaces } qFR_1, R_1\Gamma, \Gamma R_2, \dots, E R Q) \\ &< (FO_1 + F_1D_1 + \dots + F_{n-1}D_{n-1} + \Delta E R Q), \text{ a fortiori}\end{aligned}$$

which is impossible because, by (β) above

$$\Delta FqQ = FO_1 + F_1D_1 + \dots + F_{n-1}D_{n-1} + \Delta E R Q$$

Hence

$$(\text{area of segment}) < \frac{1}{3}\Delta EqQ$$

Since then the area of the segment is neither less nor greater than $\frac{1}{3}\Delta EqQ$ it is equal to it

PROPOSITION 17

It is now manifest that the area of any segment of a parabola is four thirds of the triangle which has the same base as the segment and equal height

Let Qq be the base of the segment P its vertex
Then PQq is the inscribed triangle with the same base as the segment and equal height

Since P is the vertex of the segment the diameter through P bisects Qq . Let V be the point of bisection

Let VP and qE drawn parallel to it, meet the tangent at Q in T E respectively

Then by parallels

$$qE = 2VT,$$

and

$$PV = PT$$

so that

$$VT = 2PV$$

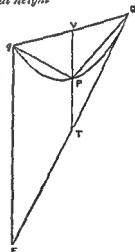
[Prop 2]

Hence

$$\Delta EqQ = 4\Delta PQq$$

But, by Prop 16 the area of the segment is equal to $\frac{1}{3}\Delta EqQ$

Therefore (area of segment) $= \frac{1}{3}\Delta PQq$



DEF In segments bounded by a straight line and any curve I call the

straight line the *base* and the *height* the greatest perpendicular drawn from the curve to the base of the segment, and the *vertex* the point from which the greatest perpendicular is drawn'

PROPOSITION 18

If Qq be the base of a segment of a parabola and V the middle point of Qq , and if the diameter through V meet the curve in P , then P is the vertex of the segment

For Qq is parallel to the tangent at P [Prop 1] Therefore, of all the perpendiculars which can be drawn from points on the segment to the base Qq , that from P is the greatest Hence, by the definition P is the vertex of the segment

PROPOSITION 19

If Qq be a chord of a parabola bisected in V by the diameter PV and if RM be a diameter bisecting QV in M , and RW be the ordinate from R to PV then

$$PV = \frac{4}{3} RM$$

For, by the property of the parabola,

$$\begin{aligned} PV \cdot PV &= QV^2 \\ PV \cdot PV &= 4RW^2 \end{aligned}$$

so that
whence

$$\begin{aligned} PV &= 4RW \\ PV &= \frac{4}{3} RM \end{aligned}$$

PROPOSITION 20

If Qq be the base, and P the vertex, of a parabolic segment, then the triangle PQq is greater than half the segment PQq

For the chord Qq is parallel to the tangent at P , and the triangle PQq is half the parallelogram formed by Qq the tangent at P and the diameters through Q and q

Therefore the triangle PQq is greater than half the segment

COR It follows that it is possible to inscribe in the segment a polygon such that the segments left over are together less than any assigned area

PROPOSITION 21

If Qq be the base and P the vertex of any parabolic segment and if R be the vertex of the segment cut off by PQ , then

$$\Delta PQq = 8 \Delta PRQ$$

The diameter through R will bisect the chord PQ , and therefore also QV where PV is the diameter bisecting Qq Let the diameter through R bisect PQ in Y and QV in M Join PM

By Prop 19

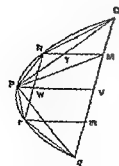
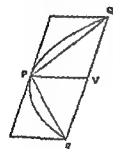
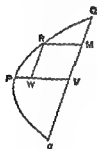
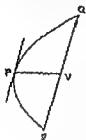
$$PY = \frac{4}{3} RM$$

Also

$$PV = 2YM$$

Therefore

$$YM = 2RY$$



and
Hence
and

$$\begin{aligned}\Delta PQV &= 2\Delta PRQ \\ \Delta PQV &= 4\Delta PRQ, \\ \Delta PQq &= 8\Delta PRQ\end{aligned}$$

Also, if RW , the ordinate from R to PV , be produced to meet the curve again in r ,

$$RW = rW,$$

and the same proof shows that

$$\Delta PQq = 8\Delta Prq$$

PROPOSITION 22

If there be a series of areas A, B, C, D, \dots each of which is four times the next in order, and if the largest, A be equal to the triangle PQq inscribed in a parabolic segment PQq and having the same base with it and equal height, then

$$(A + B + C + D + \dots) < (\text{area of segment } PQq)$$

For, since $\Delta PQq = 8\Delta PRQ = 8\Delta Prq$ where R, r are the vertices of the segments cut off by PQ, Pq as in the last proposition,

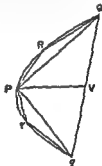
$$\Delta PQq = 4(\Delta PQR + \Delta Prq)$$

Therefore since $\Delta PQq = A$,

$$\Delta PQR + \Delta Prq = B$$

In like manner we prove that the triangles similarly inscribed in the remaining segments are together equal to the area C , and so on

Therefore $A + B + C + D + \dots$ is equal to the area of a certain inscribed polygon, and is therefore less than the area of the segment



PROPOSITION 23

Given a series of areas A, B, C, D, \dots, Z of which A is the greatest and each is equal to four times the next in order, then

$$A + B + C + \dots + Z + \frac{1}{3}Z = \frac{1}{3}A$$

Take areas b, c, d, \dots such that

$$b = \frac{1}{3}B,$$

$$c = \frac{1}{3}C,$$

$$d = \frac{1}{3}D \text{ and so on}$$

Then since $b = \frac{1}{3}B$

$$\text{and } B = \frac{1}{3}A$$

$$B + b = \frac{1}{3}A$$

$$\text{Similarly } C + c = \frac{1}{3}B$$

Therefore

$$B + C + D + \dots + Z + b + c + d + \dots + z = \frac{1}{3}(A + B + C + \dots + Y)$$

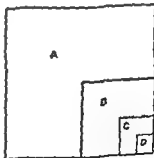
But

$$b + c + d + \dots + z = \frac{1}{3}(B + C + D + \dots + Y)$$

Therefore, by subtraction

$$\begin{aligned}B + C + D + \dots + Z + z &= \frac{1}{3}A \\ A + B + C + \dots + Z + \frac{1}{3}Z &= \frac{1}{3}A\end{aligned}$$

or



PROPOSITION 24

Every segment bounded by a parabola and a chord PQ is equal to four thirds of the triangle which has the same base as the segment and equal height

Suppose

$$K = \frac{4}{3} \Delta PQq$$

where P is the vertex of the segment, and we have then to prove that the area of the segment is equal to K

For if the segment be not equal to K , it must either be greater or less

I Suppose the area of the segment greater than K

If then we inscribe in the segments cut off by PQ Pq triangles which have the same base and equal height, i.e. triangles with the same vertices R r as those of the segments and if in the remaining segments we inscribe triangles in the same manner, and so on, we shall finally have segments remaining whose sum is less than the area by which the segment PQq exceeds K

Therefore the polygon so formed must be greater than the area K , which is impossible since [Prop 23]

$$A + B + C + \dots + Z < \frac{4}{3} A,$$

where

$$A = \Delta PQq$$

Thus the area of the segment cannot be greater than K

II Suppose if possible that the area of the segment is less than K

If then $\Delta PQq = A$, $B = \frac{1}{4}A$, $C = \frac{1}{4}B$, and so on until we arrive at an area X such that X is less than the difference between K and the segment we have

$$A + B + C + \dots + Y + \frac{1}{4}X = \frac{4}{3}A \quad [\text{Prop 23}]$$

$$= K$$

Now since K exceeds $A + B + C + \dots + X$ by an area less than X and the area of the segment by an area greater than X it follows that

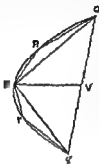
$$A + B + C + \dots + X > (\text{the segment})$$

which is impossible by Prop 22 above

Hence the segment is not less than K

Thus, since the segment is neither greater nor less than K ,

$$(\text{area of segment } PQq) = K = \frac{4}{3} \Delta PQq$$



ON FLOATING BODIES

BOOK ONE

POSTULATE 1

"Let it be supposed that a fluid is of such a character that, its parts lying evenly and being continuous, that part which is thrust the less is driven along by that which is thrust the more, and that each of its parts is thrust by the fluid which is above it in a perpendicular direction if the fluid be sunk in any thing and compressed by anything else."

PROPOSITION 1

If a surface be cut by a plane always passing through a certain point, and if the section be always a circumference [of a circle] whose centre is the aforesaid point, the surface is that of a sphere

For if not there will be some two lines drawn from the point to the surface which are not equal

Suppose O to be the fixed point, and let A, B to be two points on the surface such that $OA \neq OB$. Let the surface be cut by a plane passing through O, A, B . Then the section is by hypothesis a circle whose centre is O .

Thus $OA = OB$, which is contrary to the assumption. Therefore the surface cannot but be a sphere.

PROPOSITION 2

The surface of any fluid at rest is the surface of a sphere whose centre is the same as that of the earth

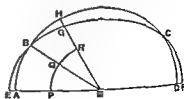
Suppose the surface of the fluid cut by a plane through O , the centre of the earth, in the curve $ABCD$.

$ABCD$ shall be the circumference of a circle.

For if not, some of the lines drawn from O to the curve will be unequal. Take one of them OB such that OB is greater than some of the lines from O to the curve and less than others. Draw a circle with OB as radius. Let it be EBF which will therefore fall partly within and partly without the surface of the fluid.

Draw OGH making with OB an angle equal to the angle EOB and meeting the surface in H and the circle in G . Draw also in the plane an arc of a circle PQR with centre O and within the fluid.

Then the parts of the fluid along PQR are uniform and continuous and the part PQ is compressed by the part between it and AB , while the part QR is compressed by the part between QR and BH .



Therefore the parts along PQ , QR will be unequally compressed, and the part which is compressed the less will be set in motion by that which is compressed the more

Therefore there will not be rest, which is contrary to the hypothesis

Hence the section of the surface will be the circumference of a circle whose centre is O and so will all other sections by planes through O

Therefore the surface is that of a sphere with centre O

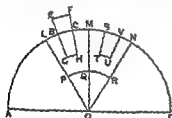
PROPOSITION 3

Of solids those which are for size, are of equal weight with a fluid will if let down into the fluid be immersed so that they do not project above the surface but do not sink lower

If possible, let a certain solid $ETHG$ of equal weight volume for volume with the fluid remain immersed in it so that part of it $EBCF$, projects above the surface

Draw through O the centre of the earth, and through the solid a plane cutting the surface of the fluid in the circle $ABCD$

Conceive a pyramid with vertex O and base a parallelogram at the surface of the fluid such that it includes the immersed portion of the solid. Let this pyramid be cut by the plane of $ABCD$ in OL , OM . Also let a sphere within the fluid and below GH be described with centre O and let the plane of $ABCD$ cut this sphere in PQR



Conceive also another pyramid in the fluid with vertex O , continuous with the former pyramid and equal and similar to it. Let the pyramid so described be cut in OM , ON by the plane of $ABCD$

Lastly let $STUV$ be a part of the fluid within the second pyramid equal and similar to the part $BGHC$ of the solid, and let SV be at the surface of the fluid

Then the pressures on PQ , QR are unequal that on PQ being the greater. Hence the part at QR will be set in motion by that at PQ , and the fluid will not be at rest, which is contrary to the hypothesis

Therefore the solid will not stand out above the surface

Nor will it sink further, because all the parts of the fluid will be under the same pressure

PROPOSITION 4

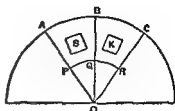
A solid lighter than a fluid will if immersed in it not be completely submerged but part of it will project above the surface

In this case after the manner of the previous proposition we assume the solid if possible to be completely submerged and the fluid to be at rest in that position and we conceive (1) a pyramid with its vertex at O the centre of the earth, including the solid (2) another pyramid continuous with the former and equal and similar to it with the same vertex O (3) a portion of the fluid within this latter pyramid equal to the immersed solid in the other pyramid (4) a sphere with centre O whose surface is below the immersed solid and the part of the fluid in the second pyramid corresponding thereto. We suppose a plane to be drawn through the centre O cutting the surface of the fluid in the circle

ABC , the solid in S , the first pyramid in OA , OB , the second pyramid in OB , OC , the portion of the fluid in the second pyramid in K , and the inner sphere in PQR

Then the pressures on the parts of the fluid at PQ , QR are unequal, since S is lighter than K . Hence there will not be rest, which is contrary to the hypothesis

Therefore the solid S cannot, in a condition of rest, be completely submerged

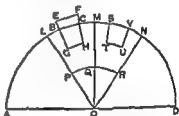


PROPOSITION 5

Any solid lighter than a fluid will, if placed in the fluid be so far immersed that the weight of the solid will be equal to the weight of the fluid displaced

For let the solid be $EGHF$, and let $BGHC$ be the portion of it immersed when the fluid is at rest. As in Prop 3, conceive a pyramid with vertex O including the solid, and another pyramid with the same vertex continuous with the former and equal and similar to it. Suppose a portion of the fluid $STUV$ at the base of the second pyramid to be equal and similar to the immersed portion of the solid and let the construction be the same as in Prop 3

Then since the pressure on the parts of the fluid at PQ , QR must be equal in order that the fluid may be at rest, it follows that the weight of the portion $STUV$ of the fluid must be equal to the weight of the solid $EGHF$. And the former is equal to the weight of the fluid displaced by the immersed portion of the solid $BGHC$



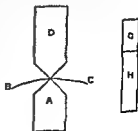
PROPOSITION 6

If a solid lighter than a fluid be forcibly immersed in it the solid will be driven upwards by a force equal to the difference between its weight and the weight of the fluid displaced

For let A be completely immersed in the fluid, and let G represent the weight of A and $(G+H)$ the weight of an equal volume of the fluid. Take a solid D , whose weight is H and add it to A . Then the weight of $(A+D)$ is less than that of an equal volume of the fluid, and if $(A+D)$ is immersed in the fluid, it will project so that its weight will be equal to the weight of the fluid displaced. But its weight is $(G+H)$

Therefore the weight of the fluid displaced is $(G+H)$ and hence the volume of the fluid displaced is the volume of the solid A . There will accordingly be rest with A immersed and D projecting

Thus the weight of D balances the upward force exerted by the fluid on A and therefore the latter force is equal to H which is the difference between the weight of A and the weight of the fluid which A displaces

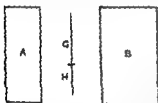


PROPOSITION 7

A solid heavier than a fluid will, if placed in it, descend to the bottom of the fluid, and the solid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced

(1) The first part of the proposition is obvious, since the part of the fluid under the solid will be under greater pressure, and therefore the other parts will give way until the solid reaches the bottom

(2) Let A be a solid heavier than the same volume of the fluid and let $(G+H)$ represent its weight, while G represents the weight of the same volume of the fluid



Take a solid B lighter than the same volume of the fluid and such that the weight of B is G , while the weight of the same volume of the fluid is $(G+H)$

Let A and B be now combined into one solid and immersed. Then since $(A+B)$ will be of the same weight as the same volume of fluid both weights being equal to $(G+H)+G$, it follows that $(A+B)$ will remain stationary in the fluid

Therefore the force which causes A by itself to sink must be equal to the upward force exerted by the fluid on B by itself. This latter is equal to the difference between $(G+H)$ and G [Prop 6]. Hence A is depressed by a force equal to H , i.e. its weight in the fluid is H or the difference between $(G+H)$ and G

POSTULATE 2

"Let it be granted that bodies which are forced upwards in a fluid are forced upwards along the perpendicular [to the surface] which passes through their centre of gravity

PROPOSITION 8

If a solid in the form of a segment of a sphere and of a substance lighter than a fluid, be immersed in it so that its base does not touch the surface the solid will rest in such a position that its axis is perpendicular to the surface and if the solid be forced into such a position that its base touches the fluid on one side and be then set free it will not remain in that position but will return to the symmetrical position

PROPOSITION 9

If a solid in the form of a segment of a sphere and of a substance lighter than a fluid be immersed in it so that its base is completely below the surface, the solid will rest in such a position that its axis is perpendicular to the surface

[The proof of this proposition has only survived in a mutilated form. It deals moreover with only one case out of three which are distinguished at the beginning viz that in which the segment is greater than a hemisphere]

Suppose first that the segment is greater than a hemisphere. Let it be cut by a plane through its axis and the centre of the earth and if possible let it be at rest in the position shown in the figure where AB is the intersection of

the plane with the base of the segment, DE its axis, C the centre of the sphere of which the segment is a part, O the centre of the earth

The centre of gravity of the portion of the segment outside the fluid as F , lies on OC produced its axis passing through C

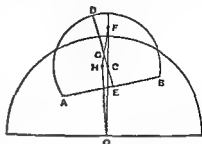
Let G be the centre of gravity of the segment Join FG , and produce it to H so that

$FG : GH = (\text{volume of immersed portion})$
(rest of solid)

Join OH

Then the weight of the portion of the solid outside the fluid acts along FO and the pressure of the fluid on the immersed portion along OH , while the weight of the immersed portion acts along HO and is by hypothesis less than the pressure of the fluid acting along OH

Hence there will not be equilibrium but the part of the segment towards A will ascend and the part towards B descend, until DE assumes a position perpendicular to the surface of the fluid



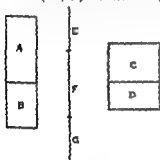
ON FLOATING BODIES

BOOK TWO

PROPOSITION 1

If a solid lighter than a fluid be at rest in it the weight of the solid will be to that of the same volume of the fluid as the immersed portion of the solid is to the whole

Let $(A+B)$ be the solid, B the portion immersed in the fluid



Let $(C+D)$ be an equal volume of the fluid, C being equal in volume to A and B to D

Further suppose the line E to represent the weight of the solid $(A+B)$ $(F+G)$ to represent the weight of $(C+D)$, and G that of D

Then

$$\frac{\text{weight of } (A+B)}{E} = \frac{\text{weight of } (C+D)}{(F+G)} \quad (1)$$

And the weight of $(A+B)$ is equal to the weight of a volume B of the fluid [I 5], i.e. to the weight of D

That is to say $E=G$

Hence by (1)

$$\begin{aligned} \frac{\text{weight of } (A+B)}{\text{weight of } (C+D)} &= \frac{G}{F+G} \\ &= \frac{D}{C+D} \\ &= \frac{B}{A+B} \end{aligned}$$

PROPOSITION 2

If a right segment of a paraboloid of revolution whose axis is not greater than $\frac{1}{2}p$ (where p is the principal parameter of the generating parabola) and whose specific gravity is less than that of a fluid be placed in the fluid with its axis inclined to the vertical at any angle but so that the base of the segment does not touch the surface of the fluid the segment of the paraboloid will not remain in that position but will return to the position in which its axis is vertical

Let the axis of the segment of the paraboloid be AN and through AN draw a plane perpendicular to the surface of the fluid. Let the plane intersect the paraboloid in the parabola BAB the base of the segment of the paraboloid in BB and the plane of the surface of the fluid in the chord QQ of the parabola

Then since the axis AN is placed in a position not perpendicular to QQ , BB will not be parallel to QQ

Draw the tangent PT to the parabola which is parallel to QQ and let P be the point of contact

¹The rest of the proof is given in brackets as supplied by Commandinus

[From P draw PV parallel to AN meeting QQ' in V . Then PV will be a diameter of the parabola, and also the axis of the portion of the paraboloid immersed in the fluid]

Let C be the centre of gravity of the paraboloid BAB' , and F that of the portion immersed in the fluid. Join FC and produce it to H so that H is the centre of gravity of the remaining portion of the paraboloid above the surface.

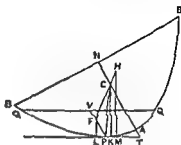
Then, since $AN = \frac{3}{2}AC$,
and $AN > \frac{1}{2}p$,

it follows that $AC > \frac{p}{2}$

Therefore if CP be joined the angle CPT is acute. Hence if CA be drawn perpendicular to PT , A will fall between P and T . And if FL HM be drawn parallel to CA to meet PT , they will each be perpendicular to the surface of the fluid.

Now the force acting on the immersed portion of the segment of the paraboloid will act upwards along LF , while the weight of the portion outside the fluid will act downwards along HM .

Therefore there will not be equilibrium but the segment will turn so that B will rise and B' will fall, until AN takes the vertical position.]



PROPOSITION 3

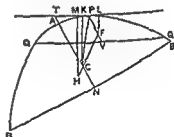
If a right segment of a paraboloid of revolution whose axis is not greater than $\frac{1}{2}p$ (where p is the parameter) and whose specific gravity is less than that of a fluid, be placed in the fluid with its axis inclined at any angle to the vertical, but so that its base is entirely submerged, the solid will not remain in that position but will return to the position in which the axis is vertical.

Let the axis of the paraboloid be AN , and through AN draw a plane perpendicular to the surface of the fluid intersecting the paraboloid in the parabola BAB' the base of the segment in BNB' and the plane of the surface of the fluid in the chord QQ' of the parabola.

Then since AN as placed is not perpendicular to the surface of the fluid QQ and BB' will not be parallel.

Draw PT parallel to QQ and touching the parabola at P . Let PT meet NA produced in T . Draw the diameter PV bisecting QQ in V . PV is then the axis of the portion of the paraboloid above the surface of the fluid.

Let C be the centre of gravity of the whole segment of the paraboloid F that of the portion above the surface. Join FC and produce it to H so that H is the centre of gravity of the immersed portion.



Then since $AC > \frac{p}{2}$ the angle CPT is an acute angle, as in the last proposition.

Hence, if CK be drawn perpendicular to PT , K will fall between P and T . Also, if HM , FL be drawn parallel to CK , they will be perpendicular to the surface of the fluid.

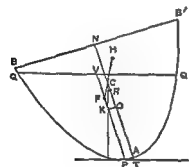
And the force acting on the submerged portion will act upwards along HM , while the weight of the rest will act downwards along LF produced.

Thus the paraboloid will turn until it takes the position in which AN is vertical.

PROPOSITION 4

Given a right segment of a paraboloid of revolution whose axis AN is greater than $\frac{1}{2}p$ (where p is the parameter), and whose specific gravity is less than that of a fluid but bears to it a ratio not less than $(AN - \frac{1}{2}p)^2 : AN^2$ if the segment of the paraboloid be placed in the fluid with its axis at any inclination to the vertical but so that its base does not touch the surface of the fluid, it will not remain in that position but will return to the position in which its axis is vertical.

Let the axis of the segment of the paraboloid be AN , and let a plane be drawn through AN perpendicular to the surface of the fluid and intersecting the segment in the parabola BAB' , the base of the segment in BB' and the surface of the fluid in the chord QQ' of the parabola.



Then AN as placed will not be perpendicular to QQ' .

Draw PT parallel to QQ' and touching the parabola at P . Draw the diameter PV bisecting QQ' in V . Thus PV will be the axis of the submerged portion of the solid.

Let C be the centre of gravity of the whole solid, F that of the immersed portion. Join FC and produce it to H so that H is the centre of gravity of the remaining portion.

$$AN = \frac{3}{2}AC$$

$$AN > \frac{1}{2}p$$

$$AC > \frac{p}{2}$$

Now, since
and

it follows that

Measure CO along CA equal to $\frac{p}{2}$ and OR along OC equal to $\frac{1}{2}AO$.

Then since
and
we have by subtraction

$$AN = \frac{3}{2}AC,$$

$$AR = \frac{3}{2}AO$$

$$NR = \frac{3}{2}OC$$

$$AN - AR = \frac{3}{2}OC$$

$$= \frac{1}{2}p$$

$$AR = (AN - \frac{1}{2}p)$$

$$(AN - \frac{1}{2}p)^2 : AN^2 = AR^2 : AN^2$$

and therefore the ratio of the specific gravity of the solid to that of the fluid is, by the enunciation, not less than the ratio $AR^2 : AN^2$.

But, by Prop. 1 the former ratio is equal to the ratio of the immersed portion to the whole solid, i.e. to the ratio $PV^2 : AN^2$ [On Conoids and Spheroids Prop. 24].

Hence

$$PV^2 : AN^2 < AR^2 : AN^2$$

or

It follows that

$$\begin{aligned} PV &< AR \\ PF (= \frac{2}{3}PV) &< \frac{2}{3}AR \\ &< AO \end{aligned}$$

If, therefore OA be drawn from O perpendicular to OP , it will meet PF between P and F

Also if CA be joined the triangle ACO is equal and similar to the triangle formed by the normal the subnormal and the ordinate at P (since $CO = \frac{1}{2}p$ or the subnormal and AO is equal to the ordinate)

Therefore CA is parallel to the normal at P , and therefore perpendicular to the tangent at P and to the surface of the fluid

Hence if parallels to CA be drawn through F H they will be perpendicular to the surface of the fluid and the force acting on the submerged portion of the solid will act upwards along the former while the weight of the other portion will act downwards along the latter

Therefore the solid will not remain in its position but will turn until AO assumes a vertical position

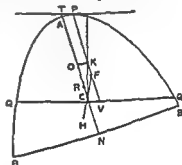
PROPOSITION 5

Given a right segment of a paraboloid of revolution such that its axis AN is greater than $\frac{2}{3}p$ (where p is the parameter), and its specific gravity is less than that of a fluid but in a ratio to it not greater than the ratio $\{1N^2 - (AN - \frac{2}{3}p)^2\} : AN^2$, if the segment be placed in the fluid with its axis inclined at any angle to the vertical, but so that its base is completely submerged, it will not remain in that position but will return to the position in which AN is vertical

Let a plane be drawn through AN as placed perpendicular to the surface of the fluid and cutting the segment of the paraboloid in the parabola BAB' , the base of the segment in BB' and the plane of the surface of the fluid in the chord QQ of the parabola

Draw the tangent PT parallel to QQ and the diameter PI bisecting QQ will accordingly be the axis of the portion of the paraboloid above the surface of the fluid

Let F be the centre of gravity of the portion above the surface C that of the whole solid and produce FC to H the centre of gravity of the immersed portion



As in the last proposition $AC > \frac{p}{2}$ and we measure CO along CA equal to $\frac{p}{2}$ and OR along OC equal to $\frac{1}{2}AO$

Then $1N = \frac{2}{3}AC$ and $1R = \frac{2}{3}AO$
and we derive as before $AR = (AN - \frac{2}{3}p)$

Now by hypothesis

$$\begin{aligned} &(\text{spec gravity of solid}) < (\text{spec gravity of fluid}) \\ &> \{1N^2 - (AN - \frac{2}{3}p)^2\} : AN^2 \\ &> (AN^2 - 1R^2) : AN^2 \end{aligned}$$

Therefore

$$\begin{aligned} &(\text{portion submerged}) < (\text{whole solid}) \\ &> (AN^2 - AR^2) : AN^2 \end{aligned}$$

and (whole solid) (portion above surface)

Thus
whence
and

$$\begin{aligned} & > AN^2 \quad AR^2 \\ AN^2 \quad PV & > 4N^2 \quad AR^2, \\ PV & < AR \\ PF & < \frac{3}{2}AR \\ & < AO \end{aligned}$$

Therefore if a perpendicular to AC be drawn from O it will meet PF in some point K between P and F

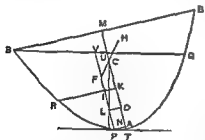
And, since $CO = \frac{1}{2}p$ CK will be perpendicular to PT as in the last proposition

Now the force acting on the submerged portion of the solid will act upwards through H and the weight of the other portion downwards through F in directions parallel in both cases to CK : whence the proposition follows

PROPOSITION 6

If a right segment of a paraboloid lighter than a fluid be such that its axis AM is greater than $\frac{1}{2}p$ but $AM < \frac{15}{4}$ and if the segment be placed in the fluid with its axis so inclined to the vertical that its base touches the fluid it will never remain in such a position that the base touches the surface in one point only

Suppose the segment of the paraboloid to be placed in the position described and let the plane through the axis AM perpendicular to the surface of the fluid intersect the segment of the paraboloid in the parabolic segment BAB and the plane of the surface of the fluid in BQ



Take C on AM such that $AC = 2CM$ (or so that C is the centre of gravity of the segment of the paraboloid) and measure CK along CA such that

$$AM \cdot CK = \frac{15}{4}$$

Thus $AM \cdot CK > AM \cdot \frac{1}{2}p$ by hypothesis therefore $CK < \frac{1}{2}p$

Measure CO along CA equal to $\frac{1}{2}p$. Also draw KR perpendicular to AC meeting the parabola in R

Draw the tangent PT parallel to BQ and through P draw the diameter PV bisecting BQ in V and meeting KR in I

Then

$$PV \cdot PI_{\text{or}} > KM \cdot AK,$$

for this is proved

And

$$\begin{aligned} CK &= \frac{1}{3}AM = \frac{1}{3}AC \\ AK &= AC - CK = \frac{2}{3}AC = \frac{2}{3}AM \end{aligned}$$

whence

$$KM = \frac{2}{3}AM$$

Thus

$$KM = \frac{2}{3}AK$$

Therefore

$$PV_{\text{or}} > \frac{2}{3}PI$$

It follows that

$$PI_{\text{or}} < 2IV$$

so that

Let F be the centre of gravity of the immersed portion of the paraboloid so that $PF = 2FV$. Produce FC to H the centre of gravity of the portion above the surface

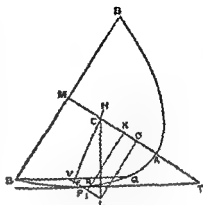
Draw OL perpendicular to PV

Then since $CO = \frac{1}{2}p$ CL must be perpendicular to PT and therefore to the surface of the fluid

And the forces acting on the immersed portion of the paraboloid and the portion above the surface act respectively upwards and downwards along lines through F and H parallel to CL

Hence the paraboloid cannot remain in the position in which B just touches the surface but must turn in the direction of increasing the angle PTM

The proof is the same in the case where the point I is not on VP but on VP produced as in the second figure



PROPOSITION 7

Given a right segment of a paraboloid of revolution lighter than a fluid and such that its axis AM is greater than $\frac{1}{2}p$, but $AM - \frac{1}{2}p < 15/4$, if the segment be placed in the fluid so that its base is entirely submerged, it will never rest in such a position that the base touches the surface of the fluid at one point only

Suppose the solid so placed that one point of the base only (B) touches the surface of the fluid. Let the plane through B and the axis AM cut the solid in the parabolic segment BAB and the plane of the surface of the fluid in the chord BQ of the parabola

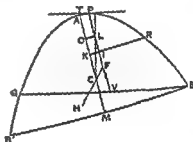
Let C be the centre of gravity of the segment so that $AC = 2CM$ and measure CA along CA such that

$$AM : CK = 15 : 4$$

It follows that $CK < \frac{1}{2}p$

Measure CO along CA equal to $\frac{1}{2}p$. Draw KR perpendicular to AM meeting the parabola in R

Let PT touching at P be the tangent to the parabola which is parallel to BQ , and PV the diameter bisecting BQ i.e. the axis of the portion of the paraboloid above the surface



Then as in the last proposition we prove that

$$PV > \frac{1}{2}PI$$

and

$$PI < 2IV$$

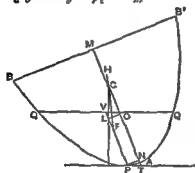
Let F be the centre of gravity of the portion of the solid above the surface, join FC and produce it to H the centre of gravity of the portion submerged

Draw OL perpendicular to PV and as before since $CO = \frac{1}{2}p$ CL is perpendicular to the tangent PT . And the lines through H F parallel to CL are perpendicular to the surface of the fluid thus the proposition is established as before

The proof is the same if the point I is not on VP but on VP produced

PROPOSITION 8

Given a solid in the form of a right segment of a paraboloid of revolution whose axis AM is greater than $\frac{1}{2}p$ but such that $AM - \frac{1}{2}p < 15/4$, and whose specific gravity bears to that of a fluid a ratio less than $(AM - \frac{1}{2}p)^2 / AM^2$ then, if the solid be placed in the fluid so that its base does not touch the fluid and its axis is inclined at an angle to the vertical the solid will not return to the position in which its axis is vertical and will not remain in any position except that in which its axis makes with the surface of the fluid a certain angle to be described



Let am be taken equal to the axis AM , and let a be a point on am such that $ac = 2cm$. Measure co along ca equal to $\frac{1}{2}p$ and or along oc equal to $\frac{1}{2}ao$

Let $X+Y$ be a straight line such that

(spec gr of solid) (spec gr of fluid) $= (Y+1)^2 am^2$, (α)
and suppose $X=2Y$

$$\begin{aligned} \text{Now } ar &= \frac{1}{2}ao = \frac{1}{2}(\frac{1}{2}am - \frac{1}{2}p) \\ &= am - \frac{1}{2}p \\ &= AM - \frac{1}{2}p \end{aligned}$$

Therefore by hypothesis

$(Y+Y)^2 am^2 < ar^2 am^2$,
whence $(Y+Y) < ar$, and therefore
 $\lambda < ao$

Measure ob along oa equal to X , and draw bd perpendicular to ab and of such length that

$$bd^2 = \frac{1}{2}co \cdot ab \quad (\beta)$$

Join ad

Now let the solid be placed in the fluid with its axis AM inclined at an angle to the vertical. Through AM draw a plane perpendicular to the surface of the fluid and let this plane cut the paraboloid in the parabola BAB' and the plane of the surface of the fluid in the chord QQ of the parabola.

Draw the tangent PT parallel to QQ touching at P , and let PV be the diameter bisecting QQ in V (or the axis of the immersed portion of the solid), and PN the ordinate from P .

Measure AO along AM equal to ao and OC along OM equal to oc and draw OL perpendicular to PV .

I Suppose the angle OTP greater than the angle dab

Thus

$$PN^2 \cdot NT > db^2 \cdot ba^2$$

But

$$PN^2 \cdot NT^2 = p \cdot 4AN$$

$$= co \cdot NT$$

and

$$db^2 \cdot ba^2 = \frac{1}{2}co \cdot ab \text{ by } (\beta)$$

Therefore

$$NT < 2ab$$

or

$$AN < ab$$

whence

$$NO > bo \text{ (since } ao = AO)$$

$$> 1$$

Now $(Y+1)^2 am^2 = (\text{spec gr of solid}) (\text{spec gr of fluid})$
 $= (\text{portion immersed}) (\text{rest of solid})$

$$= PV^2 \cdot AM^2,$$

so that

$$\lambda + \gamma = PV$$

But

$$PL (=NO) > \lambda$$

$$> \frac{2}{3}(\lambda + \gamma), \text{ since } \gamma = 2\gamma,$$

$$> \frac{2}{3}PV,$$

or

$$PV < \frac{3}{2}PL$$

and therefore

$$PL > 2LV$$

Take a point F on PV so that $PF = 2FV$, i.e. so that F is the centre of gravity of the immersed portion of the solid

Also $AC = ac = \frac{2}{3}am = \frac{2}{3}AM$, and therefore C is the centre of gravity of the whole solid

Join FC and produce it to H the centre of gravity of the portion of the solid above the surface

Now since $CO = \frac{1}{3}p$ CL is perpendicular to the surface of the fluid therefore so are the parallels to CL through F and H . But the force on the immersed portion acts upwards through F and that on the rest of the solid downwards through H

Therefore the solid will not rest but turn in the direction of diminishing the angle MTP

II Suppose the angle OTP less than the angle dab . In this case we shall have instead of the above results, the following,

$$AN > ab$$

$$NO < \lambda$$

Also

$$PV > \frac{3}{2}PL$$

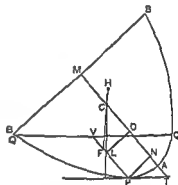
and therefore

$$PL < 2LV$$

Make PF equal to $2FV$ so that F is the centre of gravity of the immersed portion

And, proceeding as before we prove in this case that the solid will turn in the direction of increasing the angle MTP

III When the angle MTP is equal to the angle dab , equalities replace inequalities in the results obtained, and L is itself the centre of gravity of the immersed portion. Thus all the forces act in one straight line the perpendicular CL therefore there is equilibrium and the solid will rest in the position described



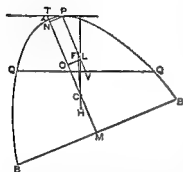
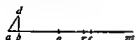
PROPOSITION 9

Given a solid in the form of a right segment of a paraboloid of revolution whose axis AM is greater than $\frac{1}{2}p$, but such that $1AM - \frac{1}{2}p < 15/4$ and whose specific gravity bears to that of a fluid a ratio greater than $\{AM^2 - (AM - \frac{1}{2}p)^2\} / AM^2$, then, if the solid be placed in the fluid with its axis inclined at an angle to the vertical but so that its base is entirely below the surface the solid will not return to the position in which its axis is vertical and will not remain in any position except that in which its axis makes with the surface of the fluid an angle equal to that described in the last proposition

Take am equal to AM and take m on am such that $ac = 2cm$. Measure co along ca equal to $\frac{1}{2}p$ and ar along ac such that $ar = \frac{2}{3}ao$

Let $\lambda + \gamma$ be such a line that

(spec gr of solid) (spec gr of fluid) = $\{am^2 - (\lambda + Y)^2\} am^2$,
and suppose $\lambda = 2Y$



Let PT be the tangent parallel to QQ PL the diameter bisecting QQ (or) the axis of the portion of the paraboloid above the surface), PN the ordinate from P

I Suppose the angle MTP greater than the angle dab Let AM be cut as before in C and O so that $AC = 2CM$, $OC = \frac{1}{2}p$ and accordingly AM , am are equally divided Draw OL perpendicular to PV

Then we have, as in the last proposition

$$PN^2 - NT^2 > db^2 \quad ba^2$$

whence

$$co \quad NT > \frac{1}{2}co \quad ab,$$

and therefore

$$AN < ab$$

It follows that

$$NO > bo$$

$$> X$$

Again since the specific gravity of the solid is to that of the fluid as the immersed portion of the solid to the whole

$$AM^2 - (Y + Y)^2 \quad AM^2 = AM^2 - PV^2 \quad AM^2,$$

$$\text{or} \quad (\lambda + Y)^2 \quad AM^2 = PV^2 \quad AM^2$$

$$\text{That is} \quad Y + Y = PV$$

$$\text{And} \quad PL \text{ (or } NO) > Y$$

$$> \frac{1}{2}PV$$

so that

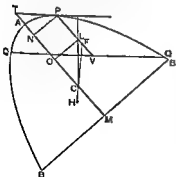
$$PL > 2LV$$

Take Γ on PL so that $PF = 2FV$ Then F is the centre of gravity of the portion of the solid above the surface

Also C is the centre of gravity of the whole solid Join FC and produce it to H , the centre of gravity of the immersed portion

Then since $CO = \frac{1}{2}p$ CL is perpendicular to PT and to the surface of the fluid, and

the force acting on the immersed portion of the solid acts upwards along the



parallel to CL through H , while the weight of the rest of the solid acts downwards along the parallel to CL through F

Hence the solid will not rest but turn in the direction of diminishing the angle MTP

II Exactly as in the last proposition we prove that, if the angle MTP be less than the angle dab the solid will not remain in its position but will turn in the direction of increasing the angle MTP

III If the angle MTP is equal to the angle dab , the solid will rest in that position because L and F will coincide and all the forces will act along the one line CL

PROPOSITION 10

Given a solid in the form of a right segment of a paraboloid of revolution in which the axis AM is of a length such that $AM \cdot \frac{1}{2}p > 15 \cdot 4$ and supposing the solid placed in a fluid of greater specific gravity so that its base is entirely above the surface of the fluid, to investigate the positions of rest

(PRELIMINARY)

Suppose the segment of the paraboloid to be cut by a plane through its axis AM in the parabolic segment BAB_1 of which BB_1 is the base

Divide AM at C so that $AC = 2CM$, and measure CK along CA so that

$$AM \cdot CK = 15 \cdot 4, \quad (\alpha)$$

whence by the hypothesis $CK > \frac{1}{2}p$

Suppose CO measured along CA equal to $\frac{1}{2}p$ and take a point R on AM such that

$$MR = \frac{1}{2}CO$$

$$\begin{aligned} \text{Thus } AR &= AM - MR \\ &= \frac{1}{2}(AC - CO) \\ &= \frac{1}{2}AO \end{aligned}$$

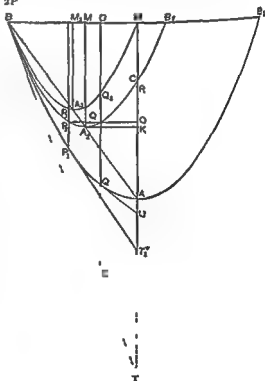
Join BA draw KA_2 perpendicular to AM meeting BA in A_2 , bisect BA in A_1 and draw A_1M_2 , A_1M_2 parallel to AM meeting BM in M_2 , M_2 respectively

On A_1M_2 , A_1M_2 as axes describe parabolic segments similar to the segment BAB_1 (It follows by similar triangles that BM will be the base of the segment whose axis is A_1M_2 and BB_2 the base of that whose axis is A_2M_2 where $BB_2 = 2BM_2$)

The parabola BA_2B_2 will then pass through C

[For

$$\begin{aligned} BM_2 \cdot M_2M &= BM_2 \cdot A_2K \\ &= KM \cdot AK \end{aligned}$$



$$\begin{aligned}
 &= CM + CK \quad AC - CK \\
 &= \left(\frac{1}{2} + \frac{1}{15}\right) AM \quad \left(\frac{1}{2} - \frac{1}{15}\right) AM \\
 &= 9 \quad 6 \\
 &= MA \quad AC
 \end{aligned}
 \tag{\beta}$$

Thus C is seen to be on the parabola BA_1B_2 by the converse of Prop 4 of the *Quadrature of the Parabola*]

Also if a perpendicular to AM be drawn from O it will meet the parabola BA_1B_2 in two points, as $Q_1 P$. Let Q_1Q Q_2D be drawn through Q_1 Q_2 parallel to AM meeting the parabolas BAB_1 BA_1M respectively in Q_1 Q_2 and BM in D , and let $P_1P_2P_3$ be the corresponding parallel to AM through P_2 . Let the tangents to the outer parabola at P_1 Q_1 meet MA produced in T_1 U respectively

Then, since the three parabolic segments are similar and similarly situated with their bases in the same straight line and having one common extremity, and since Q_1Q_2 Q_2Q_3 is a diameter common to all three segments, it follows that

$$Q_1Q_2 \quad Q_2Q_3 = (B_2B_1 \quad B_1B) \quad (BM \quad MB_2)$$

$$\begin{aligned}
 \text{Now} \quad & B_2B_1 \quad B_1B = MM_2 \quad BM && \text{(dividing by 2)} \\
 & = 2 \quad 5 && \text{by means of } (\beta) \text{ above}
 \end{aligned}$$

$$\begin{aligned}
 \text{And} \quad & BM \quad MB_2 = BM \quad (2B_1M_2 - BM) \\
 & = 5 \quad (6 - 5) && \text{by means of } (\beta), \\
 & = 5 \quad 1
 \end{aligned}$$

It follows that

$$Q_1Q \quad Q_2Q_3 = 2 \quad 1$$

$$\text{or} \quad \left. \begin{aligned} Q_1Q_2 &= 2Q_2Q_3 \\ P_1P_2 &= 2P_2P_3 \end{aligned} \right\}$$

Similarly

Also, since

$$MR = \frac{1}{2}CO = \frac{1}{2}p$$

$$AR = AM - MR$$

$$= AM - \frac{1}{2}p$$

(ENUNCIATION)

If the segment of the paraboloid be placed in the fluid with its base entirely above the surface then

(I) if

$$\begin{aligned}
 (\text{spec gr of solid}) (\text{spec gr of fluid}) &< AR^2 \quad AM^2 \\
 &[< (4M - \frac{1}{2}p)^2 \quad AM^2]
 \end{aligned}$$

the solid will rest in the position in which its axis AM is vertical,

(II) if

$$\begin{aligned}
 (\text{spec gr of solid}) (\text{spec gr of fluid}) &< AR^2 \quad AM^2 \\
 &\text{but} > Q_1Q_2^2 \quad AM^2
 \end{aligned}$$

the solid will not rest with its base touching the surface of the fluid in one point only but in such a position that its base does not touch the surface at any point and its axis makes with the surface an angle greater than U

(III a) if

$$(\text{spec gr of solid}) (\text{spec gr of fluid}) = Q_1Q_2^2 \quad AM^2,$$

the solid will rest and remain in the position in which the base touches the surface of the fluid at one point only and the axis makes with the surface an angle equal to U

(III b) if

$$(\text{spec gr of solid}) (\text{spec gr of fluid}) = P_1P_2^2 \quad AM^2,$$

the solid will rest with its base touching the surface of the fluid at one point only and with its axis inclined to the surface at an angle equal to T_1 ,

(IV) if

$$(\text{spec gr of solid}) (\text{spec gr of fluid}) > P_1 P_2 \cdot AM^2 \\ \text{but} < Q_1 Q_2 \cdot AM^2,$$

the solid will rest and remain in a position with its base more submerged,

(V) if

$$(\text{spec gr of solid}) (\text{spec gr of fluid}) < P_1 P_2 \cdot AM^2,$$

the solid will rest in a position in which its axis is inclined to the surface of the fluid at an angle less than T_1 , but so that the base does not even touch the surface at one point

(PROOF)

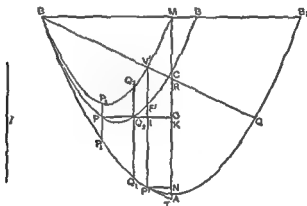
(I) Since $AM > \frac{2}{3}p$ and

$$(\text{spec gr of solid}) (\text{spec gr of fluid}) < (AM - \frac{2}{3}p)^2 \cdot AM^2$$

it follows by Prop 4 that the solid will be in stable equilibrium with its axis vertical

(II) In this case

$$(\text{spec gr of solid}) (\text{spec gr of fluid}) < AR^2 \cdot AM^2 \\ \text{but} > Q_1 Q_2 \cdot AM^2$$



Suppose the ratio of the specific gravities to be equal to $l^2 \cdot AM^2$, so that
 $l < AR$ but $> Q_1 Q_2$

Place PV' between the two parabolas BAB_1 BP_2Q_2M equal to l and parallel to AM and let PV' meet the intermediate parabola in F

Then by the same proof as before we obtain

$$P'F' = 2FV$$

Let PT the tangent at P' to the outer parabola meet MA in T , and let $P'N'$ be the ordinate at P

Join BV and produce it to meet the outer parabola in Q . Let OQ P_1 meet PV' in I

Now, since in two similar and similarly situated parabolic segments with bases BM BB_1 in the same straight line BV BQ are drawn making the same angle with the bases,

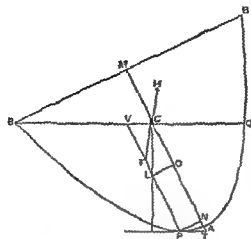
$$BV : BQ = BM : BB_1$$

$$= 1 : 2$$

$$BV : V'Q$$

so that

Suppose the segment of the paraboloid placed in the fluid as described with its axis inclined at an angle to the vertical, and with its base touching the surface at one point B only. Let the solid be cut by a plane through the axis and perpendicular to the surface of the fluid and let the plane intersect the solid in the parabolic segment BAB and the plane of the surface of the fluid in BQ .



Take the points C, O on AM as before described. Draw the tangent parallel to BQ touching the parabola in P and meeting AM in T and let PV be the diameter bisecting BQ (i.e. the axis of the immersed portion of the solid).

$$\begin{aligned} \text{Then} \quad P \cdot 4M^2 &= (\text{spec gr of solid}) (\text{spec gr of fluid}) \\ &= (\text{portion immersed}) (\text{whole solid}) \\ &= PV^2 \cdot AM^2 \end{aligned}$$

whence

$$PV = l = PV$$

Thus the segments in the two figures namely BPQ BPQ are equal and similar

Therefore

$$\angle PTV = \angle PTN$$

Also

$$AT = AT \quad AN = AN, \quad PV = PN$$

Now, in the first figure $PI < 2IV$

Therefore if OL be perpendicular to PV in the second figure

$$PL < 2LV$$

Take F on LV so that $PF = 2TV$ i.e. so that F is the centre of gravity of the immersed portion of the solid. And C is the centre of gravity of the whole solid. Join FC and produce it to H the centre of gravity of the portion above the surface.

Now since $CO = \frac{1}{2}P$ CL is perpendicular to the tangent at P and to the surface of the fluid. Thus as before we prove that the solid will not rest with B touching the surface but will turn in the direction of increasing the angle PTN .

Hence in the position of rest the axis AM must make with the surface of the fluid an angle greater than the angle U which the tangent at Q_1 makes with AM .

(III a) In this case

$$(\text{spec gr of solid}) (\text{pec gr of fluid}) = Q_1 Q_1^2 \cdot AM^2$$

Let the segment of the paraboloid be placed in the fluid so that its base nowhere touches the surface of the fluid and its axis is inclined at an angle to the vertical.

Let the plane through AM perpendicular to the surface of the fluid cut the paraboloid in the parabola BAB and the plane of the surface of the fluid QQ . Let PT be the tangent parallel to QQ PL the diameter bisecting QQ the ordinate at P .

Divide AM as before at C, O

In the other figure let Q_1N' be the ordinate at Q_1 . Join BQ_2 and produce it to meet the outer parabola in q . Then $BQ_2 = Q_2q$, and the tangent Q_1U is parallel to Bq . Now

$$\begin{aligned} Q_1Q_2^2 \cdot AM^2 &= (\text{spec gr of solid}) (\text{spec gr of fluid}) \\ &= (\text{portion immersed}) (\text{whole solid}) \\ &= PV^2 \cdot AM^2 \end{aligned}$$

Therefore $Q_1Q_2 = PV$, and the segments QPQ , BQ_2q of the paraboloid are equal in volume. And the base of one passes through B , while the base of the other passes through Q a point nearer to A than B is.

It follows that the angle between QQ' and BB is less than the angle B_1Bq .

Therefore

$$\angle U < \angle PTN,$$

whence $AN' > AN$,

and therefore

$$NO (\text{or } Q_1Q_2) < PL,$$

where OL is perpendicular to PV .

It follows since $Q_1Q_2 = 2Q_2Q_3$, that

$$PL > 2LV$$

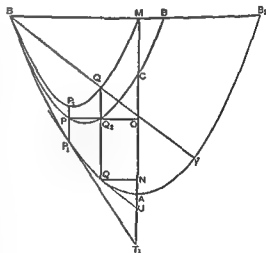
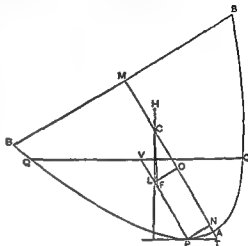
Therefore F , the centre of gravity of the immersed portion of the solid, is between P and L while as before, CL is perpendicular to the surface of the fluid.

Producing FC to H the centre of gravity of the portion of the solid above the surface we see that the solid must turn in the direction of diminishing the angle PTN until one point B of the base just touches the surface of the fluid.

When this is the case we shall have a segment BPQ equal and similar to the segment BQ_2q , the angle PTN will be equal to the angle U and AN will be equal to AN' .

Hence in this case $PL = 2LV$ and F, L coincide, so that F, C, H are all in one vertical straight line.

Thus the paraboloid will remain in the position in which one point B of the base touches the surface of the fluid, and the axis makes with the surface an angle equal to U .



(III b) In the case where

$$(\text{spec gr of solid}) (\text{spec gr of fluid}) = P_1 P_2^2 AM^2$$

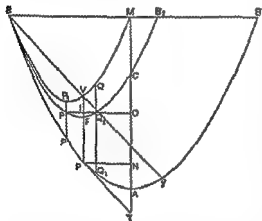
we can prove in the same way that if the solid be placed in the fluid so that its axis is inclined to the vertical and its base does not anywhere touch the surface of the fluid the solid will take up and rest in the position in which one point only of the base touches the surface and the axis is inclined to it at an angle equal to T_1 (in the figure on p. 552)

(IV) In this case

$$(\text{spec gr of solid}) (\text{spec gr of fluid}) > P_1 P_2 \cdot A^{1/2}$$

but $\angle Q_1 Q_2 P = AM^2$

Suppose the ratio to be equal to $l = AM^2$, so that l is greater than P_1P_2 but less than Q_1Q_2 .



Place PV' between the parabolas BP_1Q_1 , BP_2Q_2 so that PV is equal to l and parallel to AM , and let PV' meet the intermediate parabola in F and OQ_1P_1 in I .

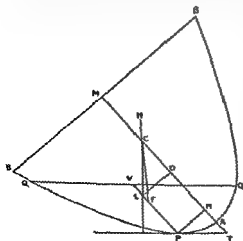
Join B_1 and produce it to meet the outer parabola in q .

Then as before $BV = Vq$ and accordingly the tangent PT at P' is parallel to Bq . Let PN be the ordinate of P

1 Now let the segment be placed in the fluid *first*, with its axis so inclined to the vertical that its base does not

anywhere touch the surface of the fluid

Let the plane through AM perpendicular to the surface of the fluid cut the paraboloid in the parabola BAB' and the plane of the surface of the fluid in



QQ Let PT be the tangent parallel to QQ PV the diameter bisecting QQ Divide AM at C O as before and draw OL perpendicular to PV

Then as before we have PI
 $= I = P V$

Thus the segments BP & QPQ of the paraboloid are equal in volume and it follows that the angle between QQ and BE is less than the angle B_1Bq .

Therefore

$$\angle PTN < \angle PTN$$

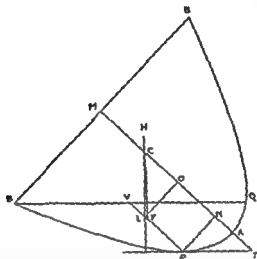
and hence $AN > AN$

so that $\Delta O > \Delta O$

 $PL > PI$

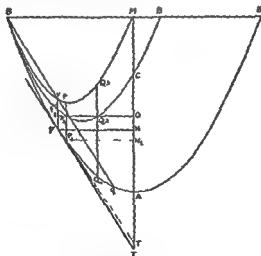
$\geq P F$ a fortiori

Thus $PL > 2LV$, so that F , the centre of gravity of the immersed portion of the solid is between L and P , while CL is perpendicular to the surface of the fluid



If then we produce FC to H the centre of gravity of the portion of the solid above the surface we prove that the solid will not rest but turn in the direction of diminishing the angle PTN

2 Next let the paraboloid be so placed in the fluid that its base touches the surface of the fluid at one point B only, and let the construction proceed as before



Then $PV = P'V'$, and the segments BPQ $BP'q$ are equal and similar, so that

$$\angle PTN = \angle P'T'N'$$

$$AN = AN' \quad NO = N'O$$

$$PI = PL$$

$$IL > 2LV$$

It follows that
and therefore
whence

Thus F again lies between P and L and as before the paraboloid will turn in the direction of diminishing the angle PTN , i.e. so that the base will be more submerged

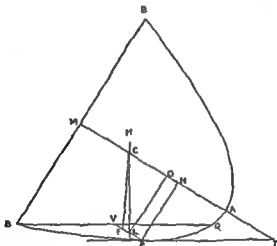
(V) In this case

$$(\text{spec gr of solid}) (\text{spec gr of fluid}) < P_1 P_2^2 AM^2$$

If then the ratio is equal to $P_1 P_2^2 AM^2$ Place PV' between the parabolas BP_1Q_1 and BP_2Q_2 equal in length to l and parallel to AM . Let PV' meet the intermediate parabola in F' and OP_2 in I .

Join BV' and produce it to meet the outer parabola in q . Then, as before, $BV' = Vq$ and the tangent PT' is parallel to Bq .

1 Let the paraboloid be so placed in the fluid that its base touches the surface at one point only



Let the plane through AM perpendicular to the surface of the fluid cut the paraboloid in the parabolic section BAB and the plane of the surface of the fluid in BQ .

Making the usual construction, we find

$$PV = l = PV'$$

and the segments BPQ BP_1q are equal and similar

Therefore

$$\angle PTN = \angle P'I N'$$

and

$$AN = AN \quad NO = NO$$

Therefore

$$PL \sim P'I$$

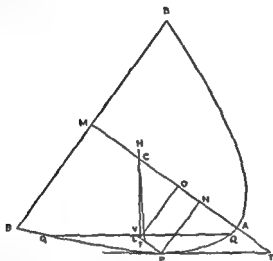
whence it follows that

$$PL < 2LV$$

Thus F the centre of gravity of the immersed portion of the solid, lies between L and V while CL is perpendicular to the surface of the fluid

Producing FC to H the centre of gravity of the portion above the surface we prove as usual that there will not be rest but the solid will turn in the direction of increasing the angle PTN so that the base will not anywhere touch the surface

2 The solid will however rest in a position where its axis makes with the surface of the fluid an angle less than T_1



For let it be placed so that the angle PTN is not less than T_1

Then with the same construction as before $PV = l = P'V'$

And since

$$\angle T < \angle T_1$$

$$AN > AN_1$$

and therefore $NO < N_1O$, where P_1N_1 is the ordinate of P_1

Hence

$$Pl < P_1P_2$$

But

$$P_1P_2 > P'F'$$

Therefore

$$Pl > \frac{1}{2}PV,$$

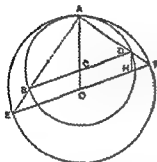
so that F the centre of gravity of the immersed portion of the solid lies between P and L

Thus the solid will turn in the direction of diminishing the angle PTN until that angle becomes less than T_1

BOOK OF LEMMAS

PROPOSITION 1

If two circles touch at A, and if BD EF be parallel diameters in them ADF is a straight line



Let O C be the centres of the circles and let OC be joined and produced to A . Draw DH parallel to AD meeting OF in H .

Then since $OH = CD = CA$,
and $OF = OA$
we have by subtraction $HF = CO = DH$
Therefore $\angle HDF = \angle HFD$

Thus both the triangles CAD HDF are isosceles and the third angles ACD DHF in each are equal. Therefore the equal angles in each are equal to one another and

$$\angle ADC = \angle DFH$$

Add to each the angle CDF and it follows that

$$\angle ADC + \angle CDF = \angle CDF + \angle DFH$$

= (two right angles)

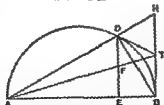
Hence ADF is a straight line

The same proof applies if the circles touch externally

PROPOSITION 2

Let AB be the diameter of a semicircle and let the tangents to it at B and at any other point D on it meet in T. If now DE be drawn perpendicular to AB and if AT DE meet in F,

$$DF = FE$$



Produce AD to meet BT produced in H . Then the angle ADB in the semicircle is right therefore the angle BDH is also right. And TB , TD are equal. Therefore T is the centre of the semicircle on BH as diameter, which passes through D .

Hence

$$HT = TB$$

And since $DE \parallel HB$ it follows that $DF = FE$

PROPOSITION 3

Let P be any point on a segment of a circle whose base is AB , and let PN be perpendicular to AB . Take D on AB so that $AN = ND$. If now PQ be an arc equal to the arc PA , and BQ be joined,

BQ BD shall be equal

Join PA , PQ , PD , DQ .

Then since the arcs PA , PQ are equal,

$$PA = PQ$$

But, since $AN = ND$, and the angles at N are right,

$$PA = PD$$

$$PQ = PD$$

Therefore and $\angle PQD = \angle PDQ$

Now, since A , P , Q , B are concyclic,

$$\angle PAD + \angle PQB = (\text{two right angles}),$$

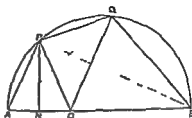
$$\begin{aligned} \angle PDA + \angle PQB &= (\text{two right angles}) \\ &= \angle PDA + \angle PDB \end{aligned}$$

$$\text{Therefore } \angle PQB = \angle PDB,$$

and since the parts, the angles PQD , PDQ are equal,

$$\angle BQD = \angle BDQ$$

and $BQ = BD$



PROPOSITION 4

If AB be the diameter of a semicircle and N any point on AB , and if semicircles be described within the first semicircle and having AN , NB as diameters respectively, the figure included between the circumference of the three semicircles is what Archimedes called an $\alpha\rho\beta\eta\lambda\omicron\varsigma$ ¹ and its area is equal to the circle on PN as diameter, where PN is perpendicular to AB and meets the original semicircle in P .

For

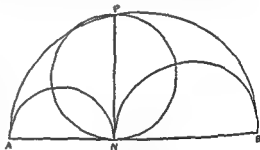
$$\begin{aligned} AB^2 &= AN^2 + NB^2 + 2AN \cdot NB \\ &= AN^2 + NB^2 + 2PN^2 \end{aligned}$$

But circles (or semicircles) are to one another as the squares of their radii (or diameters)

Hence

$$\begin{aligned} (\text{semicircle on } AB) &= (\text{sum of} \\ &\text{semicircles on } AN, NB) \\ &\quad + 2(\text{semicircle on } PN) \end{aligned}$$

That is the circle on PN as diameter is equal to the difference between the semicircle on AB and the sum of the semicircles on AN , NB , i.e. is equal to the area of the $\alpha\rho\beta\eta\lambda\omicron\varsigma$.



PROPOSITION 5

Let AB be the diameter of a semicircle C any point on AB and CD perpendicular to it and let semicircles be described within the first semicircle and having AC , CB as diameters. Then if two circles be drawn touching CD on different sides and each touching two of the semicircles the circles so drawn will be equal.

Let one of the circles touch CD at E the semicircle on AB in F , and the semicircle on AC in G .

¹ $\alpha\rho\beta\eta\lambda\omicron\varsigma$ is literally a shoemaker's knife

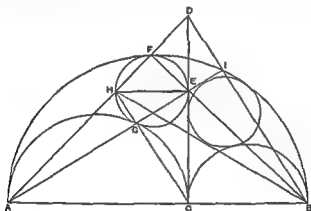
Draw the diameter EH of the circle which will accordingly be perpendicular to CD and therefore parallel to AB

Join FH, HA , and FE, EB Then, by Prop 1, FHA, FEB are both straight lines, since EH, AB are parallel

For the same reason AGE, CGH are straight lines

Let AF produced meet CD in D , and let AE produced meet the outer semi circle in I Join BI, ID

Then since the angles AFB, ACD are right, the straight lines AD, AB are such that the perpendiculars on each from the extremity of the other meet in the point E Therefore by the properties of triangles AE is perpendicular to the line joining B to D



But AE is perpendicular to BI

Therefore BID is a straight line

Now, since the angles at G, I are right CH is parallel to BD

Therefore

$$\begin{aligned} AB \cdot BC &= AD \cdot DH \\ &= AC \cdot HE \end{aligned}$$

so that

$$AC \cdot CB = AB \cdot HE$$

In like manner if d is the diameter of the other circle, we can prove that

$$AC \cdot CB = AB \cdot d$$

Therefore $d = HE$ and the circles are equal

PROPOSITION 6

Let AB the diameter of a semicircle be divided at C so that $AC = \frac{1}{2}CB$ [or in any ratio] Describe semicircles within the first semicircle and on AC, CB as diameters and suppose a circle drawn touching all three semicircles If GH be the diameter of this circle to find the relation between GH and AB

Let GH be that diameter of the circle which is parallel to AB , and let the circle touch the semicircles on AB, AC, CB in D, E, F respectively

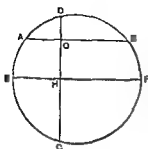
Join AG, GD and BH, HD Then, by Prop 1, AGD, BHD are straight lines

For a like reason AEH, BFG are straight lines as also are CEG, CFH

Let AD meet the semicircle on AC in I and let BD meet the semicircle on CB in K Join CI, CK meeting AE, BF respectively in L, M , and let GL, HM produced meet AB in N, P respectively

$$(\text{arc } AD) + (\text{arc } CB) = (\text{arc } AC) + (\text{arc } DB)$$

Let the chords intersect at O , and draw the diameter EF parallel to AB intersecting CD in H . EF will thus bisect CD at right angles in H , and



$$(\text{arc } ED) = (\text{arc } EC)$$

Also EDF ECF are semicircles while

$$(\text{arc } ED) = (\text{arc } EA) + (\text{arc } AD)$$

Therefore

$$(\text{sum of arcs } CF, EA, AD) = (\text{arc of a semicircle})$$

And the arcs AE, BF are equal

Therefore

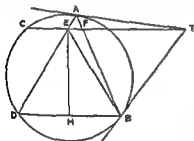
$$(\text{arc } CB) + (\text{arc } AD) = (\text{arc of a semicircle})$$

Hence the remainder of the circumference the sum of the arcs AC, DB is also equal to a semicircle and the proposition is proved :

PROPOSITION 10

Suppose that TA, TB are two tangents to a circle, while TC cuts it. Let BD be the chord through B parallel to TC and let AD meet TC in E . Then, if EH be drawn perpendicular to BD it will bisect it in H .

Let AB meet TC in F , and join BE .



Now the angle TAB is equal to the angle in the alternate segment i.e.

$$\angle TAB = \angle ADB$$

$$= \angle AET, \text{ by parallels}$$

Hence the triangles EAT, AFT have one angle equal and another (at T) common. They are therefore similar and

$$FT : AT = AT : ET$$

$$\text{Therefore } ET : TT = TA^2 = TB^2$$

It follows that the triangles EBT, BFT are similar

$$\angle TEB = \angle TBF$$

$$= \angle TAB$$

But the angle TEB is equal to the angle EBD and the angle TAB was proved equal to the angle EDB .

Therefore

$$\angle EDB = \angle EBD$$

And the angles at H are right angles

It follows that

$$BH = HD$$

PROPOSITION 11

If two chords AB, CD in a circle intersect at right angles in a point O not being the centre then

$$AO^2 + BO^2 + CO^2 + DO^2 = (\text{diameter})^2$$

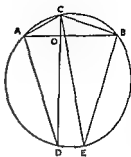
Draw the diameter CE and join AC, CB, AD, BE

Then the angle CAO is equal to the angle CEB in the same segment and the angles AOC , EBC are right, therefore the triangles AOC , EBC are similar and

$$\angle ACO = \angle ECB$$

It follows that the subtended arcs and therefore the chords AD , BE are equal

$$\begin{aligned}\text{Thus } (AO^2 + DO^2) + (BO^2 + CO^2) &= AD^2 + BC^2 \\ &= BE^2 + BC^2 \\ &= CE^2\end{aligned}$$



PROPOSITION 12

If AB be the diameter of a semicircle and TP , TQ the tangents to it from any point T and if AQ , BP be joined meeting in R , then TR is perpendicular to AB

Let TR produced meet AB in M and join PA , QB

Since the angle APB is right,

$$\begin{aligned}\angle PAB + \angle PBA &= (\text{a right angle}) \\ &= \angle AQB\end{aligned}$$

Add to each side the angle RBQ and

$$\angle PAB + \angle QBA = (\text{exterior}) \angle PRQ$$

But

$\angle TPR = \angle PAB$, and $\angle TQR = \angle QBA$,
in the alternate segments

therefore $\angle TPR + \angle TQR = \angle PRQ$

It follows from this that

$$TP = TQ = TR$$

[For if PT be produced to O so that $TO = TQ$ we have

$$\angle TOQ = \angle TQO$$

And by hypothesis,

$$\angle PRQ = \angle TPR + \angle TQR$$

By addition

$$\angle POQ + \angle PRQ = \angle TPR + \angle OQR$$

It follows that, in the quadrilateral $OPRQ$ the opposite angles are together equal to two right angles. Therefore a circle will go round $OPQR$ and T is its centre because $TP = TO = TQ$. Therefore $TR = TP$]

Thus

$$\angle TRP = \angle TPR = \angle PAM$$

Adding to each the angle PRM ,

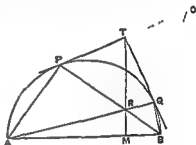
$$\begin{aligned}\angle PAM + \angle PRM &= \angle TRP + \angle PRM \\ &= (\text{two right angles})\end{aligned}$$

Therefore

$$\angle APR + \angle AMR = (\text{two right angles}),$$

whence

$$\angle AMR = (\text{a right angle})$$



PROPOSITION 13

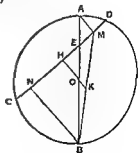
If a diameter AB of a circle meet any chord CD not a diameter in E and if AM , BN be drawn perpendicular to CD then

$$CN = DM$$

Let O be the centre of the circle and OH perpendicular to CD . Join BM and produce HO to meet BM in K

Then

$$CH = HD$$



And by parallels, since

$$BO = OA,$$

$$BK = KM$$

Therefore

$$NH = HM$$

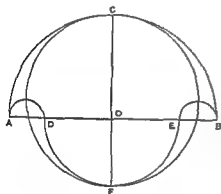
Accordingly

$$CN = DM$$

PROPOSITION 14

Let ACB be a semicircle on AB as diameter and let AD, BE be equal lengths measured along AB from A, B respectively. On AD, BE as diameters describe semicircles on the side towards C , and on DE as diameter a semicircle on the opposite side. Let the perpendicular to AB through O , the centre of the first semicircle, meet the opposite semicircles in C, F respectively.

Then shall the area of the figure bounded by the circumferences of all the semicircles be equal to the area of the circle on CF as diameter.



By Eucl. II. 10 since ED is bisected at O and produced to A

$$EA^2 + AD^2 = 2(EO^2 + OA^2)$$

and $CF = OA + OE = EA$

Therefore

$$AB^2 + DE^2 = 4(EO^2 + OA^2) = 2(CF^2 + 4D^2)$$

But circles (and therefore semicircles) are to one another as the squares on their radii (or diameters)

Therefore

$$\begin{aligned} & (\text{sum of semicircles on } AB, DE) \\ &= (\text{circle on } CF) + (\text{sum of semicircles on } AD, BE) \end{aligned}$$

Therefore

$$(\text{area of 'sahnon'}) = (\text{area of circle on } CF \text{ as diam.})$$

PROPOSITION 15

Let AB be the diameter of a circle. AC a side of an inscribed regular pentagon. D the middle point of the arc AC . Join CD and produce it to meet BA produced in E . Join AC, DB meeting in F and draw FM perpendicular to AB . Then

$$EM = (\text{radius of circle})$$

Let O be the centre of the circle and join DA, DM, DO, CB

Now

$$\angle ABC = \frac{1}{2}(\text{right angle}),$$

and

$$\angle ABD = \angle DBC = \frac{1}{3}(\text{right angle}),$$

whence

$$\angle AOD = \frac{1}{5}(\text{right angle})$$

Further the triangles FCB, FMB are equal in all respects

Therefore in the triangles DCB, DMB the sides CB, MB being equal and BD common while the angles CBD, MBD are equal

$$\angle BCD = \angle BMD = \frac{2}{5}(\text{right angle})$$

But

$$\begin{aligned} \angle BCD + \angle BAD &= (\text{two right angles}) \\ &= \angle BAD + \angle DAE \\ &= \angle BMD + \angle DMA \end{aligned}$$

so that

$$\angle DAE = \angle BCD$$

and

$$\angle BAD = \angle AMD$$

Therefore

$$AD = MD$$

THE METHOD TREATING OF MECHANICAL PROBLEMS

"Archimedes to Eratosthenes greeting

"I sent you on a former occasion some of the theorems discovered by me merely writing out the enunciations and inviting you to discover the proofs which at the moment I did not give. The enunciations of the theorems which I sent were as follows

1 'If in a right prism with a parallelogrammic base a cylinder be inscribed which has its bases in the opposite parallelograms,¹ and its sides [i.e. four generators] on the remaining planes (faces) of the prism, and if through the centre of the circle which is the base of the cylinder and (through) one side of the square in the plane opposite to it a plane be drawn the plane so drawn will cut off from the cylinder a segment which is bounded by two planes and the surface of the cylinder, one of the two planes being the plane which has been drawn and the other the plane in which the base of the cylinder is and the surface being that which is between the said planes and the segment cut off from the cylinder is one sixth part of the whole prism.

2 "If in a cube a cylinder be inscribed which has its bases in the opposite parallelograms² and touches with its surface the remaining four planes (faces), and if there also be inscribed in the same cube another cylinder which has its bases in other parallelograms and touches with its surface the remaining four planes (faces), then the figure bounded by the surfaces of the cylinders which is within both cylinders is two-thirds of the whole cube.

Now these theorems differ in character from those communicated before for we compared the figures then in question conoids and spheroids and segments of them in respect to size, with figures of cones and cylinders but none of those figures have yet been found to be equal to a solid figure bounded by planes whereas each of the present figures bounded by two planes and surfaces of cylinders is found to be equal to one of the solid figures which are bounded by planes. The proofs then of these theorems I have written in this book and now send to you. Seeing moreover in you as I say an earnest student, a man of considerable eminence in philosophy and an admirer [of mathematical inquiry] I thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is I am persuaded no less useful even for the proof of the theorems themselves for certain things first became clear to me by a mechanical method although they had to be demonstrated by geom-

¹The parallelograms are apparently squares

²i.e. squares

etry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method some knowledge of the questions to supply the proof than it is to find it without any previous knowledge. This is a reason why in the case of the theorems the proof of which Eudoxus was the first to discover, namely that the cone is a third part of the cylinder and the pyramid of the prism, having the same base and equal height, we should give no small share of the credit to Democritus who was the first to make the assertion with regard to the said figure though he did not prove it. I am myself in the position of having first made the discovery of the theorem now to be published [by the method indicated], and I deem it necessary to expound the method partly because I have already spoken of it and I do not want to be thought to have uttered vain words but equally because I am persuaded that it will be of no little service to mathematics for I apprehend that some either of my contemporaries or of my successors, will by means of the method when once established be able to discover other theorems in addition which have not yet occurred to me.

First then I will set out the very first theorem which became known to me by means of mechanics, namely that

Any segment of a section of a right-angled cone (i.e. a parabola) is four thirds of the triangle which has the same base and equal height

and after this I will give each of the other theorems investigated by the same method. Then at the end of the book, I will give the geometrical [proofs of the propositions]

[I premise the following propositions which I shall use in the course of the work]

1 If from [one magnitude] another magnitude be subtracted which has not the same centre of gravity the centre of gravity of the remainder is found by] producing [the straight line joining the centres of gravity of the whole magnitude and of the subtracted part in the direction of the centre of gravity of the whole] and cutting off from it a length which has to the distance between the said centres of gravity the ratio which the weight of the subtracted magnitude has to the weight of the remainder. [*On the Equilibrium of Planes* : 8]

2 'If the centres of gravity of any number of magnitudes whatever be on the same straight line the centre of gravity of the magnitude made up of all of them will be on the same straight line' [*Cf Ibid* : 5]

3 The centre of gravity of any straight line is the point of bisection of the straight line [*Cf Ibid* : 4]

4 'The centre of gravity of any triangle is the point in which the straight lines drawn from the angular points of the triangle to the middle points of the (opposite) sides cut one another' [*Ibid* : 13 14]

5 The centre of gravity of any parallelogram is the point in which the diagonals meet' [*Ibid* : 10]

6 The centre of gravity of a circle is the point which is also the centre [of the circle]

7 'The centre of gravity of any cylinder is the point of bisection of the axis'

8 The centre of gravity of any cone is [the point which divides its axis so that] the portion [adjacent to the vertex is] triple [of the portion adjacent to the base]

[All the e propositions have already been] proved¹ [Besides these I require also the following proposition which is easily proved]

If in two series of magnitudes those of the first series are in order proportional to those of the second series and further, "the magnitudes [of the first series] either all or some of them, are in any ratio whatever [to those of a third series] and if the magnitudes of the second series are in the same ratio to the corresponding magnitudes [of a fourth series], then the sum of the magnitudes of the first series has to the sum of the selected magnitudes of the third series the same ratio which the sum of the magnitudes of the second series has to the sum of the (correspondingly) selected magnitudes of the fourth series"² [*On Conoids and Spheroids* Prop 1]

PROPOSITION I

Let ABC be a segment of a parabola bounded by the straight line AC and the parabola ABC , and let D be the middle point of AC . Draw the straight line DBE parallel to the axis of the parabola and join AB BC .

Then shall the segment ABC be $\frac{2}{3}$ of the triangle ABC .

From A draw AKF parallel to DE and let the tangent to the parabola at C meet DBE in E and AKF in F . Produce CB to meet AF in K and again produce CK to H making KH equal to CK .

Consider CH as the bar of a balance K being its middle point.

Let MO be any straight line parallel to ED and let it meet CF CK AC in M N O and the curve in P .

Now since CE is a tangent to the parabola and CD the semi ordinate
 $EB = BD$,

"for this is proved in the Elements [of Conics]"³

Since FA MO are parallel to ED it follows that
 $FA = KA$ $MN = NO$

Now by the property of the parabola proved in a lemma⁴

$MO \cdot OP = CA \cdot AO$ [*Of Quadrature of Parabola* Prop 5]

$= CK \cdot KN$

[*Eucl* vi 2]

$= HK \cdot KN$

Take a straight line TG equal to OP and place it with its centre of gravity at H , so that $TH = HG$ then since N is the centre of gravity of the straight line VO and

$MO \cdot TG = HK \cdot KN$

it follows that TG at H and MO at N will be in equilibrium about K [*On the Equilibrium of Planes* I § 7]

¹The problem of finding the centre of gravity of a cone is not solved in any extant work of Archimedes

² i.e. the works on conics by Aristaeus and Euclid

Similarly, for all other straight lines parallel to DE and meeting the arc of the parabola, (1) the portion intercepted between FC , AC with its middle point on AC and (2) a length equal to the intercept between the curve and AC placed with its centre of gravity at H will be in equilibrium about K

Therefore K is the centre of gravity of the whole system consisting (1) of all the straight lines as MO intercepted between FC , AC and placed as they actually are in the figure and (2) of all the straight lines placed at H equal to the straight lines as PO intercepted between the curve and AC

And, since the triangle CFA is made up of all the parallel lines like MO , and the segment CBA is made up of all the straight lines like PO within the curve

it follows that the triangle placed where it is in the figure, is in equilibrium about K with the segment CBA placed with its centre of gravity at H

Divide AC at W so that $CH = 3AW$, then W is the centre of gravity of the triangle ACF , "for this is proved in the books on equilibrium (*ἐν τοῖς ἰσορροπικοῖς*)"

[Cf *On the Equilibrium of Planes* 1 15]

Therefore $\triangle ACF$ (segment ABC) $= HA \cdot AW$
 $= 3 \cdot 1$

Therefore segment $ABC = \frac{1}{3} \triangle ACF$

But $\triangle ACF = 4 \triangle ABC$

Therefore segment $ABC = \frac{1}{4} \triangle ABC$

"Now the fact here stated is not actually demonstrated by the argument used but that argument has given a sort of indication that the conclusion is true. Seeing then that the theorem is not demonstrated but at the same time suspecting that the conclusion is true, we shall have recourse to the geometrical demonstration which I myself discovered and have already published"

PROPOSITION 2

We can investigate by the same method the propositions that

(1) Any sphere is (in respect of solid content) four times the cone with base equal to a great circle of the sphere and height equal to its radius and

(2) the cylinder with base equal to a great circle of the sphere and height equal to the diameter is $1\frac{1}{2}$ times the sphere

(1) Let $ABCD$ be a great circle of a sphere and AC , BD diameters at right angles to one another

Let a circle be drawn about BD as diameter and in a plane perpendicular to AC and on this circle as base let a cone be described with A as vertex. Let the surface of this cone be produced and then cut by a plane through C parallel to its base the section will be a circle on EF as diameter. On this circle as base let a cylinder be erected with height and axis AC , and produce CA to H , making AH equal to CA

Let CH be regarded as the bar of a balance A being its middle point

Draw any straight line MN in the plane of the circle $ABCD$ and parallel to BD . Let MN meet the circle in O , P the diameter AC in S , and the straight lines AE , AF in Q , R respectively. Join AO

Through MN draw a plane at right angles to AC ,

this plane will cut the cylinder in a circle with diameter MN , the sphere in a circle with diameter OP , and the cone in a circle with diameter QR

Now, since $MS=AC$, and $QS=AS$

$$\begin{aligned} MS \cdot SQ &= CA \cdot AS \\ &= 10 \\ &= OS + SQ^2 \end{aligned}$$

And, since $HA=AC$,

$$\begin{aligned} HA \cdot AS &= CA \cdot AS \\ &= MS \cdot SQ \\ &= MS \cdot MS \cdot SQ \\ &= MS \cdot (OS^2 + SQ^2), \\ &\quad \text{from above} \\ &= MN^2 \cdot (OP + QR^2) \\ &= (\text{circle, diam } MN) \cdot (\text{circle diam } OP \\ &\quad + \text{circle, diam } QR) \end{aligned}$$

That is

$$HA \cdot AS = (\text{circle in cylinder}) \cdot (\text{circle in sphere} + \text{circle in cone})$$

Therefore the circle in the cylinder placed where it is is in equilibrium about A , with the circle in the sphere together with the circle in the cone if both the latter circles are placed with their centres of gravity at H

Similarly for the three corresponding sections made by a plane perpendicular to AC and passing through any other straight line in the parallelogram LF parallel to EF

If we deal in the same way with all the sets of three circles in which planes perpendicular to AC cut the cylinder the sphere and the cone, and which make up those solids respectively it follows that the cylinder, in the place where it is will be in equilibrium about A with the sphere and the cone together

when both are placed with their centres of gravity at H

Therefore since K is the centre of gravity of the cylinder

$$HA \cdot AK = (\text{cylinder}) \cdot (\text{sphere} + \text{cone } AEF)$$

$$\text{But } HA = 2AK$$

$$\text{therefore cylinder} = 2(\text{sphere} + \text{cone } AEF)$$

$$\text{Now cylinder} = 3(\text{cone } AEF),$$

$$\text{therefore cone } AEF = 2(\text{sphere})$$

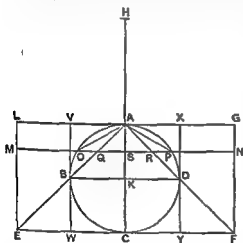
$$\text{But since } EF = 2BD$$

$$\text{cone } AEF = 8(\text{cone } ABD)$$

$$\text{therefore sphere} = 4(\text{cone } ABD)$$

(2) Through B D draw $\downarrow BIV$ $\downarrow DYC$ parallel to AC

and imagine a cylinder which has AC for axis and the circles on VX , WY as diameters for bases



[Eucl vii 10]

Then

$$\begin{aligned}
 \text{cylinder } VY &= 2(\text{cylinder } VD) \\
 &= 6(\text{cone } ABD) \\
 &= \frac{3}{2}(\text{sphere}), \text{ from above}
 \end{aligned}$$

[Eucl XII 10]

Q E D

"From this theorem to the effect that a sphere is four times as great as the cone with a great circle of the sphere as base and with height equal to the radius of the sphere I conceived the notion that the surface of any sphere is four times as great as a great circle in it, for, judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius

PROPOSITION 3

By this method we can also investigate the theorem that

A cylinder with base equal to the greatest circle in a spheroid and height equal to the axis of the spheroid is $1\frac{1}{2}$ times the spheroid,
and when this is established, it is plain that

If any spheroid be cut by a plane through the centre and at right angles to the axis the half of the spheroid is double of the cone which has the same base and the same axis as the segment (i.e. the half of the spheroid)

Let a plane through the axis of a spheroid cut its surface in the ellipse $ABCD$ the diameters (i.e. axes) of which are AC BD , and let A be the centre

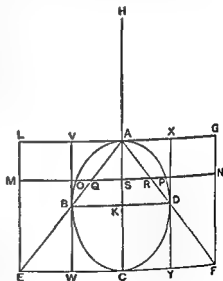
Draw a circle about BD as diameter and in a plane perpendicular to AC imagine a cone with this circle as base and A as vertex produced and cut by a plane through C parallel to its base the section will be a circle in a plane at right angles to AC and about EF as diameter

Imagine a cylinder with the latter circle as base and axis AC , produce CA to H making AH equal to CA

Let HC be regarded as the bar of a balance A being its middle point

In the parallelogram LF draw any straight line MN parallel to EF meeting the ellipse in O P and AE AF , AC in Q R , S respectively

If now a plane be drawn through MN at right angles to AC , it will cut the cylinder in a circle with diameter MN the spheroid in a circle with diameter OP and the cone in a circle with diameter QR



Since $HA = AC$

$$\begin{aligned} HA \quad AS &= CA \quad AS \\ &= EA \quad AQ \\ &= MS \quad SQ \end{aligned}$$

Therefore

$$HA \quad AS = MS \quad MS \quad SQ$$

But, by the property of the ellipse,

$$\begin{aligned} AS \quad SC \quad SO &= AK^2 \quad KB^2 \\ &= AS^2 \quad SQ^2 \end{aligned}$$

therefore

$$\begin{aligned} SQ \quad SO^2 &= AS^2 \quad AS \quad SC \\ &= SQ \quad SQ \quad QM, \end{aligned}$$

and accordingly

$$SO^2 = SQ \quad QV$$

Add SQ^2 to each side and we have

$$SO + SQ^2 = SQ \quad SM$$

Therefore from above we have

$$\begin{aligned} HA \quad AS &= MS^2 \quad (SO + SQ^2) \\ &= MN^2 \quad (OP^2 + QR^2) \\ &= (\text{circle diam } MN) \quad (\text{circle diam } OP + \text{circle diam } QR) \end{aligned}$$

That is

$$HA \quad AS = (\text{circle in cylinder}) \quad (\text{circle in spheroid} + \text{circle in cone})$$

Therefore the circle in the cylinder in the place where it is in equilibrium about A , with the circle in the spheroid and the circle in the cone together if both the latter circles are placed with their centres of gravity at H

Similarly for the three corresponding sections made by a plane perpendicular to AC and passing through any other straight line in the parallelogram LP parallel to EF

If we deal in the same way with all the sets of three circles in which planes perpendicular to AC cut the cylinder the spheroid and the cone and which make up those figures respectively, it follows that the cylinder in the place where it is will be in equilibrium about A with the spheroid and the cone together when both are placed with their centres of gravity at H

Therefore since K is the centre of gravity of the cylinder

$$HA \quad 4K = (\text{cylinder}) \quad (\text{spheroid} + \text{cone } AEF)$$

But $HA = 2AK$

$$\text{therefore} \quad \text{cylinder} = 2(\text{spheroid} + \text{cone } AEF)$$

$$\text{And} \quad \text{cylinder} = 3(\text{cone } AEF)$$

$$\text{therefore} \quad \text{cone } AEF = 2(\text{spheroid})$$

[Eucl viii 10]

But since $ET = 2BD$

$$\text{cone } AEF = 8(\text{cone } ABD)$$

$$\text{therefore} \quad \text{spheroid} = 4(\text{cone } ABD)$$

$$\text{and} \quad \text{half the spheroid} = 2(\text{cone } ABD)$$

Through $B \quad D$ draw $VBW \quad \perp DY$ parallel to AC and imagine a cylinder which has AC for axis and the circles on VY, WY as diameters for bases

$$\text{Then} \quad \text{cylinder } \perp Y = 2(\text{cylinder } \perp D)$$

$$= 6(\text{cone } ABD)$$

$$= \frac{2}{3}(\text{spheroid}) \text{ from above}$$

Q E D

PROPOSITION 4

Any segment of a right-angled conoid (i.e. a paraboloid of revolution) cut off by a plane at right angles to the axis is $1\frac{1}{2}$ times the cone which has the same base and the same axis as the segment

This can be investigated by our method, as follows

Let a paraboloid of revolution be cut by a plane through the axis in the parabola BAC , and let it also be cut by another plane at right angles to the axis and intersecting the former plane in BC . Produce DA , the axis of the segment to H , making HA equal to AD

Imagine that HD is the bar of a balance, A being its middle point

The base of the segment being the circle on BC as diameter and in a plane perpendicular to AD ,

imagine (1) a cone drawn with the latter circle as base and A as vertex and (2) a cylinder with the same circle as base and AD as axis

In the parallelogram EC let any straight line MN be drawn parallel to BC , and through MN let a plane be drawn at right angles to AD . This plane will cut the cylinder in a circle with diameter MN and the paraboloid in a circle with diameter OP

Now BAC being a parabola and BD , OS ordinates

$$\begin{array}{l} DA \quad AS = BD^2 \quad OS^2 \\ \text{or} \quad HA \quad AS = MS^2 \quad SO^2 \end{array}$$

Therefore

$$\begin{aligned} HA \quad AS &= (\text{circle, rad } MS) \quad (\text{circle rad } OS) \\ &= (\text{circle in cylinder}) \quad (\text{circle in paraboloid}) \end{aligned}$$

Therefore the circle in the cylinder in the place where it is, will be in equilibrium about A with the circle in the paraboloid, if the latter is placed with its centre of gravity at H

Similarly for the two corresponding circular sections made by a plane perpendicular to AD and passing through any other straight line in the parallelogram which is parallel to BC

Therefore as usual if we take all the circles making up the whole cylinder and the whole segment and treat them in the same way we find that the cylinder in the place where it is is in equilibrium about A with the segment placed with its centre of gravity at H

If A is the middle point of AD K is the centre of gravity of the cylinder, therefore

$$HA \quad AK = (\text{cylinder}) \quad (\text{segment})$$

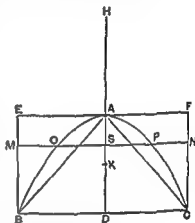
$$\text{cylinder} = 2(\text{segment})$$

$$\text{cylinder} = 3(\text{cone } ABC),$$

therefore

$$\text{segment} = \frac{2}{3}(\text{cone } ABC)$$

[Eucl XII 10]



PROPOSITION III

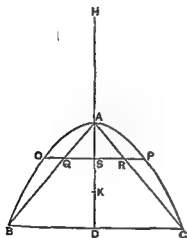
The centre of gravity of a segment of a right-angled conoid (i.e. a paraboloid of revolution) cut off by a plane at right angles to the axis is on the straight line which is the axis of the segment and divides the said straight line in such a way that the portion of it adjacent to the vertex is double of the remaining portion

This can be investigated by the method as follows

Let a paraboloid of revolution be cut by a plane through the axis in the parabola BAC , and let it also be cut by another plane at right angles to the axis and intersecting the former plane in BC

Produce DA the axis of the segment, to H making HA equal to AD , and imagine DH to be the bar of a balance its middle point being A

The base of the segment being the circle on BC as diameter and in a plane perpendicular to AD , imagine a cone with this circle as base and A as vertex so that AB AC are generators of the cone



In the parabola let any double ordinate OP be drawn meeting AB , AD AC in Q , S , R respectively

Now, from the property of the parabola,

$$\begin{aligned} BD^2 \cdot OS &= DA \cdot AS \\ &= BD \cdot QS \\ &= BD^2 \cdot BD \cdot QS \end{aligned}$$

Therefore $OS^2 = BD \cdot QS$,

or $BD \cdot OS = OS^2 \cdot QS$,

whence $BD \cdot QS = OS^2 \cdot QS$

$$\begin{aligned} \text{But } BD \cdot QS &= AD \cdot AS \\ &= HA \cdot AS \end{aligned}$$

$$\begin{aligned} \text{Therefore } HA \cdot AS &= OS^2 \cdot QS^2 \\ &= OP^2 \cdot QR^2 \end{aligned}$$

If now through OP a plane be drawn at right angles to AD , this plane cuts the paraboloid in a circle with diameter OP and the cone in a circle with diameter QR

We see therefore that $HA \cdot AS = (\text{circle, diam } OP) \cdot (\text{circle diam } QR)$
 $= (\text{circle in paraboloid}) \cdot (\text{circle in cone}),$

and the circle in the paraboloid in the place where it is in equilibrium about A with the circle in the cone placed with its centre of gravity at H

Similarly for the two corresponding circular sections made by a plane perpendicular to AD and passing through any other ordinate of the parabola

Dealing therefore in the same way with all the circular sections which make up the whole of the segment of the paraboloid and the cone respectively we see that the segment of the paraboloid in the place where it is in equilibrium about A with the cone placed with its centre of gravity at H

Now since A is the centre of gravity of the whole system as placed and the centre of gravity of part of it namely the cone as placed is at H , the centre of gravity of the rest namely the segment is at a point K on HA produced such that

$$HA \cdot AK = (\text{segment}) \cdot (\text{cone})$$

But

$$\text{segment} = \frac{1}{2}(\text{cone})$$

[Prop 4]

Therefore $HA = \frac{3}{2}AK$,
that is K divides AD in such a way that $AK = 2KD$

PROPOSITION 6

The centre of gravity of any hemisphere [is on the straight line which] is its axis, and divides the said straight line in such a way that the portion of it adjacent to the surface of the hemisphere has to the remaining portion the ratio which 3 has to 2

Let a sphere be cut by a plane through its centre in the circle $ABCD$, let AC , BD be perpendicular diameters of this circle, and through BD let a plane be drawn at right angles to AC

The latter plane will cut the sphere in a circle on BD as diameter

Imagine a cone with the latter circle as base and A as vertex

Produce CA to H making AH equal to CA , and let HC be regarded as the bar of a balance, A being its middle point

In the semicircle BAD , let any straight line OP be drawn parallel to BD and cutting AC in E and the two generators AB AD of the cone in Q R respectively Join AO

Through OP let a plane be drawn at right angles to AC

this plane will cut the hemisphere in a circle with diameter OP and the cone in a circle with diameter QR

Now

$$\begin{aligned} HA \cdot AE &= AC \cdot AE \\ &= AO \cdot AE^2 \\ &= (OE^2 + AE^2) \cdot AE^2 \\ &= (OE^2 + QE^2) \cdot QE^2 \\ &= (\text{circle diam } OP + \text{circle diam } QR) \\ &\quad (\text{circle diam } QR) \end{aligned}$$

Therefore the circles with diameters OP , QR , in the places where they are are in equilibrium about A with the circle with diameter QR if the latter is placed with its centre of gravity at H

And since the centre of gravity of the two circles with diameters OP , QR taken together in the place where they are is

[There is a lacuna here but the proof can easily be completed on the lines of the corresponding but more difficult case in Prop 8

We proceed thus from the point where the circles with diameters OP , QR in the place where they are balance about A , the circle with diameter QR placed with its centre of gravity at H

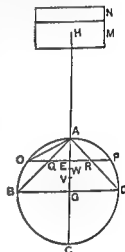
A similar relation holds for all the other sets of circular sections made by other planes passing through points on AG and at right angles to AG

Taking then all the circles which fill up the hemisphere BAD and the cone ABD respectively, we find that

the hemisphere BAD and the cone ABD in the places where they are, together balance about A as cone equal to ABD placed with its centre of gravity at H

Let the cylinder $M+N$ be equal to the cone ABD

Then since the cylinder $M+N$ placed with its centre of gravity at H balances the hemisphere BAD and the cone ABD in the places where they are suppose that the portion M of the cylinder placed with its centre of gravity at



H , balances the cone ABD (alone) in the place where it is therefore the portion N of the cylinder placed with its centre of gravity at H balances the hemisphere (alone) in the place where it is

Now the centre of gravity of the cone is at a point V such that $AG=4GV$, therefore, since M at H is in equilibrium with the cone

$$M \text{ (cone)} = \frac{3}{4}AG \quad HA = \frac{3}{4}AC \quad AC,$$

whence $M = \frac{3}{8}(\text{cone})$

But $M+N=(\text{cone})$ therefore $N=\frac{5}{8}(\text{cone})$

Now let the centre of gravity of the hemisphere be at W which is somewhere on AG

Then, since λ at H balances the hemisphere alone,

$$(\text{hemisphere}) \quad N=HA \quad AW$$

But the hemisphere BAD =twice the cone ABD ,

[On the Sphere and Cylinder I 34 and Prop III above]

and $N=\frac{5}{8}(\text{cone})$, from above

$$\text{Therefore} \quad 2 \frac{5}{8} = HA \quad AW \\ = 2AG \quad AW$$

whence $AW=\frac{4}{5}AG$, so that W divides AG in such a way that

$$AW \quad WG=5 \quad 3]$$

PROPOSITION 7

We can also investigate by the same method the theorem that

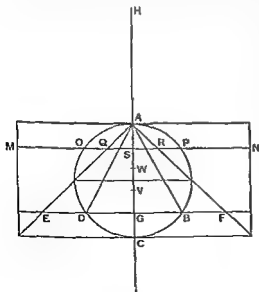
[Any segment of a sphere has] to the cone [with the same base and height the ratio which the sum of the radius of the sphere and the height of the complementary segment has to the height of the complementary segment]

[There is a lacuna here but all that is missing is the construction and the construction is easily understood by means of the figure BAD is of course the

segment of the sphere the volume of which is to be compared with the volume of a cone with the same base and height]

The plane drawn through MN and at right angles to AC will cut the cylinder in a circle with diameter MN the segment of the sphere in a circle with diameter OP and the cone on the base EF in a circle with diameter QR

In the same way as before [cf Prop 2] we can prove that the circle with diameter MN in the place where it is in equilibrium about A with the two circles with diameters $OP \quad QR$ if these circles are both moved and placed with their centres of gravity at H



The same thing can be proved of all sets of three circles in which the cylinder

der, the segment of the sphere, and the cone with the common height $4G$ are all cut by any plane perpendicular to AC

Since then the sets of circles make up the whole cylinder, the whole segment of the sphere and the whole cone respectively, it follows that the cylinder in the place where it is, is in equilibrium about A with the sum of the segment of the sphere and the cone if both are placed with their centres of gravity at H

Divide AG at W , V in such a way that

$$AW = WG, AV = 3VG$$

Therefore W will be the centre of gravity of the cylinder, and V will be the centre of gravity of the cone

Since, now the bodies are in equilibrium as described,

$$(\text{cylinder}) (\text{cone } AEF + \text{segment } BAD \text{ of sphere}) = HA \cdot AW$$

[The rest of the proof is lost, but it can easily be supplied thus

We have

$$\begin{aligned} (\text{cone } AEF + \text{segmt } BAD) (\text{cylinder}) &= AW \cdot AC \\ &= AW \cdot AC \cdot AC^2 \end{aligned}$$

But

$$\begin{aligned} (\text{cylinder}) (\text{cone } AEF) &= 4C^2 \cdot \frac{1}{2}EG^2 \\ &= AC^2 \cdot \frac{1}{2}AG^2 \end{aligned}$$

Therefore, *ex aequali*

$$\begin{aligned} (\text{cone } AEF + \text{segment } BAD) (\text{cone } AEF) &= AW \cdot AC \cdot \frac{1}{2}AG^2 \\ &= \frac{1}{2}AC \cdot \frac{1}{2}AG, \end{aligned}$$

whence

$$\begin{aligned} (\text{segment } BAD) (\text{cone } AEF) &= (\frac{1}{2}AC - \frac{1}{2}AG) \cdot \frac{1}{2}AG \\ (\text{cone } AEF) (\text{cone } ABD) &= EG^2 \cdot DG^2 \\ &= AG^2 \cdot AG \cdot GC \\ &= AG \cdot GC \\ &= \frac{1}{2}AG \cdot \frac{1}{2}GC \end{aligned}$$

Therefore, *ex aequali*

$$\begin{aligned} (\text{segment } BAD) (\text{cone } ABD) &= (\frac{1}{2}AC - \frac{1}{2}AG) \cdot \frac{1}{2}GC \\ &= (\frac{1}{2}AC - AG) \cdot GC \\ &= (\frac{1}{2}AC + GC) \cdot GC \quad \text{Q.E.D.} \end{aligned}$$

PROPOSITION 8

[The enunciation the setting out, and a few words of the construction are missing

The enunciation however can be supplied from that of Prop III with which it must be identical except that it cannot refer to any segment, and the presumption therefore is that the proposition was enunciated with reference to one kind of segment only i.e. either a segment greater than a hemisphere or a segment less than a hemisphere

Heiberg's figure corresponds to the case of a segment greater than a hemisphere The segment investigated is of course the segment BAD The setting-out and construction are self-evident from the figure.]

Produce AC to H , O , making HA equal to AC and CO equal to the radius of the sphere.

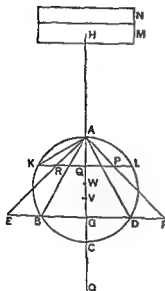
and let HBC be regarded as the bar of a balance the middle point being A

In the plane cutting off the segment describe a circle with G as centre and radius (GE) equal to $4G$ and on this circle as base and with A as vertex, let a cone be described AE AF are generators of this cone

Draw KL through any point Q on AG , parallel to EF and cutting the segment in K, L and AE, AF in R, P respectively Join AK

Now

$$\begin{aligned} HA \cdot AQ &= CA \cdot AQ \\ &= AK^2 - AQ^2 \\ &= (KQ^2 + QA^2) - QA^2 \\ &= (KQ^2 + PQ^2) - PQ^2 \\ &= (\text{circle, diam } KL - \text{circle diam } PR) \quad (\text{circle diam } PR) \end{aligned}$$



Imagine a circle equal to the circle with diameter PR placed with its centre of gravity at H therefore the circles on diameters KL, PR in the places where they are are in equilibrium about A with the circle with diameter PR placed with its centre of gravity at H

Similarly for the corresponding circular sections made by any other plane perpendicular to AG

Therefore, taking all the circular sections which make up the segment ABD of the sphere and the cone AEF respectively, we find that the segment ABD of the sphere and the cone AEF in the places where they are are in equilibrium with the cone AEF assumed to be placed with its centre of gravity at H

Let the cylinder $M+N$ be equal to the cone AEF which has A for vertex and the circle on EF as diameter for base

Divide AG at V so that

$$4G = 4VG,$$

therefore V is the centre of gravity of the cone AEF , for this has been proved before

Let the cylinder $M+N$ be cut by a plane perpendicular to the axis in such a way that the cylinder M (alone) placed with its centre of gravity at H is in equilibrium with the cone AEF

Since $M+N$ suspended at H is in equilibrium with the segment ABD of the sphere and the cone AEF in the places where they are while M , also at H is in equilibrium with the cone AEF in the place where it is, it follows that

N at H is in equilibrium with the segment ABD of the sphere in the place where it is

Now (segment ABD of sphere) (cone ABD) = $OG \cdot GC$
for this is already proved [Cf *On the Sphere and Cylinder* II Cor as well as Prop 7 ante]

And

$$\begin{aligned} &(\text{cone } ABD) \quad (\text{cone } AEF) \\ &= (\text{circle diam } BD) \quad (\text{circle diam } EF) \\ &= BD^2 \cdot EF^2 \\ &= BG^2 \cdot GE^2 \\ &= CG \cdot GA \cdot GA^2 \\ &= CG \cdot GA \end{aligned}$$

Therefore, *ex aequali*,

$$(\text{segment } ABD \text{ of sphere}) (\text{cone } AEF) = OG \cdot GA$$

Take a point W on AG such that

$$AW \cdot WG = (GA + 4GC) (GA + 2GC)$$

We have then inversely,

$$GW \cdot WA = (2GC + GA) (4GC + GA),$$

and componendo,

$$GA \cdot AW = (6GC + 2GA) (4GC + GA)$$

$$\text{But } GO = \frac{1}{2}(6GC + 2GA),$$

$$[\text{for } GO - GC = \frac{1}{2}(CG + G)]$$

$$\text{and } CV = \frac{1}{2}(4GC + GA),$$

therefore

$$GA \cdot AW = OG \cdot CV,$$

and alternately and inversely,

$$OG \cdot GA = CV \cdot WA$$

It follows, from above that

$$(\text{segment } ABD \text{ of sphere}) (\text{cone } AEF) = CV \cdot WA$$

Now since the cylinder M with its centre of gravity at H is in equilibrium about A with the cone AEF with its centre of gravity at V ,

$$(\text{cone } AEF) (\text{cylinder } M) = HA \cdot AV$$

$$= CA \cdot AV,$$

and since the cone AEF = the cylinder $M + N$ we have dividendo and intertendo,

$$(\text{cylinder } M) (\text{cylinder } N) = AV \cdot CV$$

Hence componendo

$$(\text{cone } AEF) (\text{cylinder } N) = CA \cdot CV$$

$$= HA \cdot CV$$

But it was proved that

$$(\text{segment } ABD \text{ of sphere}) (\text{cone } AEF) = CV \cdot WA,$$

therefore, *ex aequali*,

$$(\text{segment } ABD \text{ of sphere}) (\text{cylinder } N) = HA \cdot AW$$

And it was above proved that the cylinder N at H is in equilibrium about A with the segment ABD in the place where it is

therefore since H is the centre of gravity of the cylinder N , W is the centre of gravity of the segment ABD of the sphere

PROPOSITION 9

In the same way we can investigate the theorem that

The centre of gravity of any segment of a sphere is on the straight line which is the axis of the segment and divides this straight line in such a way that the part of it adjacent to the vertex of the segment has to the remaining part the ratio which the sum of the axis of the segment and four times the axis of the complementary segment has to the sum of the axis of the segment and double the axis of the complementary segment

[As this theorem relates to "any segment" but states the same result as that proved in the preceding proposition, it follows that Prop. 8 must have related to one kind of segment either a segment greater than a semicircle (as in Heiberg's figure of Prop. 8) or a segment less than a semicircle, and the present proposition completed the proof for both kinds of segments. It would only require a slight change in the figure in any case.]

PROPOSITION 10

By this method too we can investigate the theorem that

[A segment of an obtuse angled conoid (i.e. a hyperboloid of revolution) has to the cone which has] the same base [as the segment and equal height the same ratio as the sum of the axis of the segment and three times] the "annex to the axis" (i.e. half the transverse axis of the hyperbolic section through the axis of the hyperboloid, or in other words the distance between the vertex of the segment and the vertex of the enveloping cone) has to the sum of the axis of the segment and double of the 'annex' [this is the theorem proved in *On Conoids and Spheroids*, Prop 25] "and also many other theorems which as the method has been made clear by means of the foregoing examples I will omit in order that I may now proceed to compass the proofs of the theorems mentioned above."

PROPOSITION 11

If in a right prism with square bases a cylinder be inscribed having its bases in opposite square faces and touching with its surface the remaining four parallelogrammic faces and if through the centre of the circle which is the base of the cylinder and one side of the opposite square face a plane be drawn, the figure cut off by the plane so drawn is one sixth part of the whole prism

'This can be investigated by the method and when it is set out I will go back to the proof of it by geometrical considerations'

[The investigation by the mechanical method is contained in the two Propositions, 11, 12 Prop 13 gives another solution which, although it contains no mechanics is still of the character which Archimedes regards as inconclusive since it assumes that the solid is actually made up of parallel plane sections

and that an auxiliary parabola is actually made up of parallel straight lines in it Prop 14 added the conclusive geometrical proof]

Let there be a right prism with a cylinder inscribed as stated

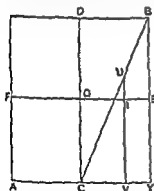
Let the prism be cut through the axis of the prism and cylinder by a plane perpendicular to the plane which cuts off the portion of the cylinder let this plane make as section the parallelogram AB , and let it cut the plane cutting off the portion of the cylinder (which plane is perpendicular to AB) in the straight line BC

Let CD be the axis of the prism and cylinder let EF bisect it at right angles and through EF

let a plane be drawn at right angles to CD this plane will cut the prism in a square and the cylinder in a circle

Let MN be the square and $OPQR$ the circle and let the circle touch the sides of the square in $O P Q R$ [$F E$ in the first figure are identical with $O Q$ respectively] Let H be the centre of the circle

Let KL be the intersection of the plane through EF perpendicular to the axis of the cylinder and the plane cutting off the portion of the cylinder KL is bisected by OHQ [and passes through the middle point of HQ]

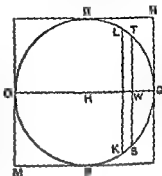


Let any chord of the circle, as ST , be drawn perpendicular to HQ , meeting HQ in W ,

and through ST let a plane be drawn at right angles to OQ and produced on both sides of the plane of the circle $OPQR$

The plane so drawn will cut the half cylinder having the semicircle PQR for section and the axis of the prism for height in a parallelogram, one side of which is equal to ST and another is a generator of the cylinder, and it will also cut the portion of the cylinder cut off in a parallelogram, one side of which is equal to ST and the other is equal and parallel to UV (in the first figure)

UV will be parallel to BY and will cut off, along EG in the parallelogram DE , the segment EI equal to QW



Now, since EC is a parallelogram, and VI is parallel to GC ,

$$EG \quad GI = IC \quad CV$$

$$= BY \quad UV$$

$$= (\square \text{ in half cyl}) \quad (\square \text{ in portion of cyl})$$

And $EG = HQ$ $GI = HW$, $QH = OH$,

therefore $OH \quad HW = (\square \text{ in half cyl}) \quad (\square \text{ in portion})$

Imagine that the parallelogram in the portion of the cylinder is moved and placed at O so that O is the centre of gravity, and that OQ is the bar of a balance H being its middle point

Then, since W is the centre of gravity of the parallelogram in the half cylinder, it follows from the above that the parallelogram in the half cylinder, in the place where it is, with its centre of gravity at W is in equilibrium about H with the parallelogram in the portion of the cylinder when placed with its centre of gravity at O

Similarly for the other parallelogrammic sections made by any plane perpendicular to OQ and passing through any other chord in the semicircle PQR perpendicular to OQ

If then we take all the parallelograms making up the half cylinder and the portion of the cylinder respectively, it follows that the half cylinder, in the place where it is is in equilibrium about H with the portion of the cylinder cut off when the latter is placed with its centre of gravity at O

PROPOSITION 12

Let the parallelogram (square) MN perpendicular to the axis with the circle $OPQR$ and its diameters $OQ \quad PR$ be drawn separately

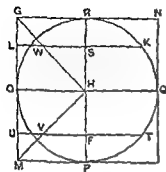
Join $HG \quad HM$, and through them draw planes at right angles to the plane of the circle, producing them on both sides of that plane

This produces a prism with triangular section GHM and height equal to the axis of the cylinder, this prism is $\frac{1}{2}$ of the original prism circumscribing the cylinder

Let LK , UT be drawn parallel to OQ and equidistant from it, cutting the circle in K , T , RP in $S \quad T$ and $GH \quad HM$ in $W \quad V$ respectively

Through LK , UT draw planes at right angles to PR , producing them on both sides of the plane of the circle,

these planes produce as sections in the half cylinder PQR and in the prism GHI four parallelograms in which the heights are equal to the axis of the cylinder, and the other sides are equal to KS , TF , LW , UV respectively



[The rest of the proof is missing but, as Zeuthen says the result obtained and the method of arriving at it are plainly indicated by the above

Archimedes wishes to prove that the half cylinder PQR , in the place where it is, balances the prism GHI , in the place where it is, about H as fixed point

He has first to prove that the elements (1) the parallelogram with side = KS and (2) the parallelogram with side = LW , in the places where they are, balance about S , or, in other words that the straight lines SK , LW , in the places where they are, balance about S

$$(\text{radius of circle } OPQR)^2 = SK^2 + SH^2,$$

$$SL^2 = SK^2 + SW^2$$

$$LS^2 - SW^2 = SK^2,$$

$$(LS + SW) LW = SK^2,$$

$$\frac{1}{2}(LS + SW) \cdot \frac{1}{2}SK = SK \cdot LW$$

Now
or

Therefore
and accordingly
whence

And $\frac{1}{2}(LS + SW)$ is the distance of the centre of gravity of LW from S while $\frac{1}{2}SK$ is the distance of the centre of gravity of SK from S

Therefore SK and LW , in the places where they are, balance about S

Similarly for the corresponding parallelograms

Taking all the parallelogrammic elements in the half cylinder and prism respectively we find that the half cylinder PQR and the prism GHI in the places where they are respectively, balance about H

From this result and that of Prop 11 we can at once deduce the volume of the portion cut off from the cylinder. For in Prop 11 the portion of the cylinder, placed with its centre of gravity at O , is shown to balance (about H) the half cylinder in the place where it is. By Prop 12 we may substitute for the half cylinder in the place where it is the prism GHI of that proposition turned the opposite way relatively to RP . The centre of gravity of the prism as thus placed is at a point (say Z) on HQ such that $HZ = \frac{1}{3}HQ$

Therefore assuming the prism to be applied at its centre of gravity we have

$$(\text{portion of cylinder}) (\text{prism}) = \frac{1}{3}HQ \cdot OH$$

$$= \frac{2}{3}HQ^2$$

therefore

$$(\text{portion of cylinder}) = \frac{2}{3}(\text{prism } GHI)$$

$$= \frac{2}{3}(\text{original prism})$$

PROPOSITION 13

Let there be a right prism with square bases one of which is $ABCD$ in the prism let a cylinder be inscribed the base of which is the circle $EFGH$ touching the sides of the square $ABCD$ in E , F , G , H

Through the centre and through the side corresponding to CD in the square

face *opposite* to $ABCD$ let a plane be drawn, this will cut off a prism equal to $\frac{1}{2}$ of the original prism and formed by three parallelograms and two triangles, the triangles forming opposite faces

In the semicircle EPG describe the parabola which has FA for axis and passes through E, G , draw MN parallel to AF meeting GE in M , the parabola in L , the semicircle in O and CD in N

Then $MN \cdot NL = NF^2$,

"for this is clear"

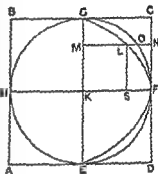
[Cf Apollonius, *Conics* I 11]

[The parameter is of course equal to GK or AF]

Therefore $MN \cdot NL = GK^2 = LS^2$

Through MN draw a plane at right angles to EG ,

this will produce as sections (1) in the prism cut off from the whole prism a right angled triangle, the base of which is MN , while the perpendicular is perpendicular at N to the plane $ABCD$ and equal to the axis of the cylinder, and the hypotenuse is in the plane cutting the cylinder and (2) in the portion of the cylinder cut off a right angled triangle the base of which is MO while the perpendicular is the generator of the cylinder perpendicular at O to the plane AN and the hypotenuse is



[There is a lacuna here to be supplied as follows

$$\begin{aligned} \text{Since } MN \cdot NL &= GK^2 = LS^2 \\ &= MN^2 \cdot LS, \\ \text{it follows that } MN \cdot ML &= MN^2 \cdot (MN^2 - LS^2) \\ &= MN^2 \cdot (MN^2 - MK^2) \\ &= MN^2 \cdot MO^2 \end{aligned}$$

But the triangle (1) in the prism is to the triangle (2) in the portion of the cylinder in the ratio of $MN^2 : MO^2$

Therefore $(\Delta \text{ in prism}) : (\Delta \text{ in portion of cylinder})$

$$= MN : ML$$

$$= (\text{straight line in rect } DG) : (\text{straight line in parabola})$$

We now take all the corresponding elements in the prism, the portion of the cylinder the rectangle DG and the parabola EFG respectively], and it will follow that

$$(\text{all the } \Delta \text{ s in prism}) : (\text{all the } \Delta \text{ s in portion of cylinder})$$

$$= (\text{all the str. lines in } \square DG) : (\text{all the straight lines between parabola and } EG)$$

But the prism is made up of the triangles in the prism [the portion of the cylinder is made up of the triangles in it] the parallelogram DG of the straight lines in it parallel to AF and the parabolic segment of the straight lines parallel to AF intercepted between its circumference and EG , therefore

$$(\text{prism}) : (\text{portion of cylinder})$$

$$= (\square DG) : (\text{parabolic segment } EFG)$$

But $\square DG = 2(\text{parabolic segment } EFG)$,

"for this is proved in my earlier treatise

[*Quadrature of Parabola*]

Therefore prism = $\frac{1}{2}$ (portion of cylinder)

If then we denote the portion of the cylinder by 2 the prism is 3 and the original prism circumscribing the cylinder is 12 (being 4 times the other prism), therefore the portion of the cylinder = $\frac{1}{4}$ (original prism) \square E D

[The above proposition and the next are peculiarly interesting for the fact that the parabola is an auxiliary curve introduced for the sole purpose of analytically reducing the required cubature to the known quadrature of the parabola.]

PROPOSITION 14

Let there be a right prism with square bases [and a cylinder inscribed therein having its base in the square $ABCD$ and touching its sides at E, F, G, H let the cylinder be cut by a plane through EG and the side corresponding to CD in the square face opposite to $ABCD$]

This plane cuts off from the prism a prism, and from the cylinder a portion of it

It can be proved that the portion of the cylinder cut off by the plane is $\frac{1}{2}$ of the whole prism

But we will first prove that it is possible to inscribe in the portion cut off from the cylinder, and to circumscribe about it solid figures made up of prisms which have equal height and similar triangular bases in such a way that the circumscribed figure exceeds the inscribed by less than any assigned magnitude

But it was proved that

(prism cut off by oblique plane) $< \frac{1}{2}$ (figure inscribed in portion of cylinder)

Now (prism cut off) (inscribed figure)

= $\square DG$ (\square s inscribed in parabolic segment)

therefore $\square DG < \frac{1}{2}$ (\square s in parabolic segment)

which is impossible since it has been proved elsewhere' that the parallelogram DG is $\frac{1}{2}$ of the parabolic segment

Consequently

not greater

And (all the prisms in prism cut off)

(all prisms in circumscr figure)

= (all \square s in $\square DG$) (all \square s in fig circumscr about parabolic segmt), therefore

(prism cut off) (figure circumscr about portion of cylinder)

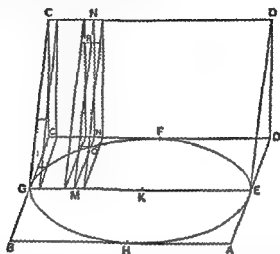
= ($\square DG$) (figure circumscr about parabolic segment)

But the prism cut off by the oblique plane is $> \frac{1}{2}$ of the solid figure circumscribed about the portion of the cylinder

[There are large gaps in the exposition of this geometrical proof but the way in which the method of exhaustion was applied and the parallelism between this and other applications of it are clear. The first fragment shows that solid figures made up of prisms were circumscribed and inscribed to the portion of the cylinder. The parallel triangular faces of these prisms were perpendicular to GE in the figure of Prop 13 they divided GE into equal portions of the requisite smallness each section of the portion of the cylinder by such a

plane was a triangular face common to an inscribed and a circumscribed right prism. The planes also produced prisms in the prism cut off by the same oblique plane as cuts off the portion of the cylinder and standing on GD as base.

The number of parts into which the parallel planes divided GE was made great enough to secure that the circumscribed figure exceeded the inscribed figure by less than a small assigned magnitude.



The second part of the proof began with the assumption that the portion of the cylinder is $> \frac{1}{2}$ of the prism cut off, and this was proved to be impossible, by means of the use of the auxiliary parabola and the proportion

$$MN \cdot ML = MN^2 \cdot MO^2$$

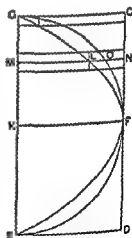
which are employed in Prop. 13.

We may supply the missing proof as follows:

In the accompanying figure are represented (1) the first element-prism circumscribed to the portion of the cylinder (2) two element prisms adjacent to the ordinate OM of which that on the left is circumscribed and that on the right (equal to the other) inscribed (3) the corresponding element-prisms forming part of the prism cut off ($CC'GEDD'$) which is $\frac{1}{2}$ of the original prism.

In the second figure are shown element rectangles circumscribed and inscribed to the auxiliary parabola, which rectangles correspond exactly to the circumscribed and inscribed element prisms represented in the first figure (the length of GM is the same in both figures and the breadths of the element-rectangles are the same as the heights of the element-prisms), the corresponding element-rectangles forming part of the rectangle GD are similarly shown.

For convenience we suppose that GE is divided into an even number of equal parts so that GA contains an integral number of these parts.



For the sake of brevity we will call each of the two element prisms of which OM is an edge 'el prism (O)' and each of the element-prisms of which MNN' is a common face 'el prism (N)'. Similarly we will use the corresponding abbreviations 'el rect (L)' and 'el rect (N)' for the corresponding elements in relation to the auxiliary parabola as shown in the second figure

Now it is easy to see that the figure made up of all the inscribed prisms is less than the figure made up of the circumscribed prisms by twice the final circumscribed prism adjacent to FK , i.e. by twice 'el prism (N)' and as the height of this prism may be made as small as we please by dividing GK into sufficiently small parts it follows that inscribed and circumscribed solid figures made up of element prisms can be drawn differing by less than any assigned solid figure

(1) Suppose if possible that

$$(\text{portion of cylinder}) > \frac{3}{2}(\text{prism cut off}),$$

or $(\text{prism cut off}) < \frac{2}{3}(\text{portion of cylinder})$

Let $(\text{prism cut off}) = \frac{2}{3}(\text{portion of cylinder} - Y)$, say

Construct circumscribed and inscribed figures made up of element-prisms such that

$$(\text{circumscr fig}) - (\text{inscr fig}) < Y$$

Therefore $(\text{inscr fig}) > (\text{circumscr fig} - Y)$
and a fortiori $> (\text{portion of cyl} - X)$

It follows that

$$(\text{prism cut off}) < \frac{2}{3}(\text{inscribed figure})$$

Considering now the element-prisms in the prism cut off and those in the inscribed figure respectively we have

$$\begin{aligned} \text{el prism (N)} - \text{el prism (O)} &= MN^2 - MO^2 \\ &= MN \cdot ML \quad [\text{as in Prop 13}] \\ &= \text{el rect (N)} - \text{el rect (L)} \end{aligned}$$

It follows that

$$\Sigma\{\text{el prism (N)}\} - \Sigma\{\text{el prism (O)}\} = \Sigma\{\text{el rect (N)}\} - \Sigma\{\text{el rect (L)}\}$$

(There are really two more prisms and rectangles in the first and third than there are in the second and fourth terms respectively but this makes no difference because the first and third terms may be multiplied by a common factor $n/(n-2)$ without affecting the truth of the proposition Cf the proposition from *On Conoids and Spheroids* quoted on p 571 above)

Therefore

$$\begin{aligned} (\text{prism cut off}) - (\text{figure inscr in portion of cyl}) \\ = (\text{rect GD}) - (\text{fig inscr in parabola}) \end{aligned}$$

But it was proved above that

$$(\text{prism cut off}) < \frac{2}{3}(\text{fig inscr in portion of cyl})$$

therefore $(\text{rect GD}) < \frac{2}{3}(\text{fig inscr in parabola})$,

and a fortiori $(\text{rect GD}) < \frac{2}{3}(\text{parabolic segmt})$

which is impossible since

$$(\text{rect GD}) = \frac{2}{3}(\text{parabolic segmt})$$

Therefore (portion of cyl) is *not* greater than $\frac{3}{2}(\text{prism cut off})$

(2) In the second lacuna must have come the beginning of the next *reductio ad absurdum* demolishing the other possible assumption that the portion of the cylinder $\leq \frac{2}{3}$ of the prism cut off

In this case our assumption is that

(prism cut off) $> \frac{2}{3}$ (portion of cylinder),
and we circumscribe and inscribe figures made up of element prisms, such that
(prism cut off) $> \frac{2}{3}$ (fig circumscrib about portion of cyl)

We now consider the element prisms in the prism cut off and in the circumscribed figure respectively, and the same argument as above gives
(prism cut off) (fig circumscrib about portion of cyl)
= (rect GD) (fig circumscrib about parabola),

whence it follows that

$$(\text{rect } GD) > \frac{2}{3}(\text{fig circumscribed about parabola}),$$

and, *a fortiori*,

$$(\text{rect } GD) > \frac{2}{3}(\text{parabolic segment})$$

which is impossible since

$$(\text{rect } GD) = \frac{2}{3}(\text{parabolic segmt})$$

Therefore

$$(\text{portion of cyl}) \text{ is not less than } \frac{2}{3}(\text{prism cut off})$$

But it was also proved that neither is it greater,

$$\text{therefore} \quad (\text{portion of cyl}) = \frac{2}{3}(\text{prism cut off}) \\ = \frac{2}{3}(\text{original prism})$$

[PROPOSITION 15]

[This proposition, which is lost would be the mechanical investigation of the second of the two special problems mentioned in the preface to the treatise, namely that of the cubature of the figure included between two cylinders, each of which is inscribed in one and the same cube so that its opposite bases are in two opposite faces of the cube and its surface touches the other four faces]

Zeuthen has shown how the mechanical method can be applied to this case
In the accompanying figure $VWYX$ is a section of the cube by a plane (that of the paper) passing through the axis BD of one of the cylinders inscribed in the cube and parallel to two opposite faces

The same plane gives the circle $ABCD$ as the section of the other inscribed cylinder with axis perpendicular to the plane of the paper and extending on each side of the plane to a distance equal to the radius of the circle or half the side of the cube

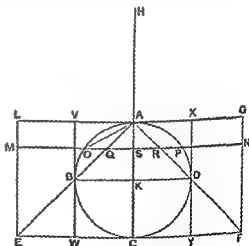
AC is the diameter of the circle which is perpendicular to BD

Join AB AD and produce them to meet the tangent at C to the circle in E F

Then $EC = CF = CA$

Let LG be the tangent at A and complete the rectangle $EFGL$

Draw straight lines from A to the four corners of the section in which the plane through BD perpendicular to AK cuts the cube These straight lines if



produced, will meet the plane of the face of the cube opposite to A in four points forming the four corners of a square in that plane with sides equal to EF or double of the side of the cube, and we thus have a pyramid with A for vertex and the latter square for base

Complete the prism (parallelepiped) with the same base and height as the pyramid

Draw in the parallelogram LF any straight line MN parallel to EF and through MN draw a plane at right angles to AC

This plane cuts—

- (1) the solid included by the two cylinders in a square with side equal to OP ,
- (2) the prism in a square with side equal to MN , and
- (3) the pyramid in a square with side equal to QR

Produce CA to H , making HA equal to AC , and imagine HC to be the bar of a balance

Now, as in Prop 2 since $MS=AC$, $QS=AS$,

$$\begin{aligned} MS \cdot SQ &= CA \cdot AS \\ &= AO^2 \\ &= OS^2 + SQ^2 \end{aligned}$$

$$\begin{aligned} \text{Also } HA \cdot AS &= CA \cdot AS \\ &= MS \cdot SQ \\ &= MS \cdot MS \cdot SQ \\ &= MS^2 (OS + SQ) \text{ from above,} \\ &= MN^2 (OP^2 + QR^2) \\ &= (\text{square side } MN) (\text{sq, side } OP + \text{sq, side } QR) \end{aligned}$$

Therefore the square with side equal to MN , in the place where it is in equilibrium about A with the squares with sides equal to OP , QR respectively placed with their centres of gravity at H

Proceeding in the same way with the square sections produced by other planes perpendicular to AC we finally prove that the prism in the place where it is in equilibrium about A with the solid included by the two cylinders and the pyramid, both placed with their centres of gravity at H

Now the centre of gravity of the prism is at K

$$\begin{aligned} \text{Therefore } HA \cdot AK &= (\text{prism}) (\text{solid} + \text{pyramid}) \\ \text{or } 2 \cdot 1 &= (\text{prism}) (\text{solid} + \frac{1}{3} \text{ prism}) \end{aligned}$$

$$\text{Therefore } 2 (\text{solid}) + \frac{2}{3} (\text{prism}) = (\text{prism})$$

It follows that

$$\begin{aligned} (\text{solid included by cylinders}) &= \frac{1}{3} (\text{prism}) \\ &= \frac{1}{3} (\text{cube}) \end{aligned} \quad Q.E.D.$$

There is no doubt that Archimedes proceeded to and completed the rigorous geometrical proof by the method of exhaustion

As observed by Prof C Juel (Zeuthen *l.c.*) the solid in the present proposition is made up of 8 pieces of cylinders of the type of that treated in the preceding proposition. As however the two propositions are separately stated, there is no doubt that Archimedes' proofs of them were distinct

In this case AC would be divided into a very large number of equal parts and planes would be drawn through the points of division perpendicular to AC . These planes cut the solid and also the cube VY in square sections. Thus we can inscribe and circumscribe to the solid the requisite solid figures made up of element prisms and differing by less than any assigned solid magnitude the

prisms have square bases and their heights are the small segments of AC . The element prism in the inscribed and circumscribed figures which has the square equal to OP^2 for base corresponds to an element prism in the cube which has for base a square with side equal to that of the cube, and as the ratio of the element prisms is the ratio $OS^2 : BK^2$, we can use the same auxiliary parabola, and work out the proof in exactly the same way, as in Prop. 14.]

CONICS

BIOGRAPHICAL NOTE

APOLLONIUS, \approx 262-c 200 \approx c

APOLLONIUS was born at Perga in Pamphylia Asia Minor some twenty five years after the birth of Archimedes which would place his birth around the year 262 B C. He seems to have gone when quite young to Alexandria where, according to Pappus the fourth century mathematician he was attracted by the reputation of the astronomer, Aristarchus of Samos. Apollonius studied under the successors of Euclid at Alexandria and continued to reside there during the reigns of Ptolemy Euergetes and of Ptolemy Philopator (247-203 \approx c). He was also for some time in Pergamum, where he made the acquaintance of the mathematician, Eudemus to whom he dedicated the first three books of his *Conics* and of King Attalus I (269-197 B \approx), to whom the remaining five books of the *Conics* were dedicated.

Apollonius appears to have been associated with the leading mathematicians of his day. In the dedicatory epistles of the *Conics* he records that he met Philonides while on a trip to Ephesus and that he undertook the composition of this work in the first instance for Naucrates, who was staying in Alexandria. Speaking in the same place of the preceding writers on conics Apollonius points out their limitations and inadequacies in such a way that some of his readers such as Pappus have considered him boastful and envious but it would seem that Apollonius is only trying to explain the appearance of a new text book on the elements of conics (Books I-IV) and the publication of his own original and more advanced investigations (Books V-VIII).

The *Conics* were at once recognized as the authoritative treatise on the subject winning for their author the name of the great geometer. They are regularly cited by later writers. Pappus added a group of lemmas and Eutocius (fl. 500 A D) edited and commented on the first four books. These books are extant in the original Greek the fifth sixth and seventh books exist in an Arabic translation, the eighth book is known only indirectly.

Although the titles and a general indication of the contents of other works by Apollonius are given by later writers especially by Pappus only one the *Cutting of a Ratio* has survived and that like parts of the *Conics* only in an Arabic version. All of the original work with the exception of the second half of the *Conics* has perished. Books not extant but known through Pappus are *Cutting of an Area*, *Determinate Section*, *Tangencies*, *Inclinations* and *Plane Loci*. He wrote on irrationals and like Archimedes devised a system of multiplication for counting large numbers and calculated an approximate value for the ratio of the circumference of a circle to the diameter. The ancient writers also record that Apollonius wrote *On the Burning Glass* in which he probably treated the properties of the parabola a work comparing the dodecahedron and the icosahedron inscribed in the same sphere and a book perhaps on the general principles of mathematics in which he criticized and suggested improvements for Euclid's *Elements*. Lastly in astronomy he is credited by Ptol-

emy with an explanation of the motion of the planets by means of epicycles and eccentric circles. He seems to have been especially interested in the theory of the moon, and the Alexandrians are said to have called him Epsilon from the resemblance of that Greek letter to the lunar crescent.

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TRANSLATOR'S NOTE

If on first appearance this treatise should seem to the reader a jumble of propositions rigorous indeed, but without much rhyme or reason in their sequence, then he can be sure he has not read aright, and as with the planets, he must look further to save the appearances. There are one or two hypotheses at least that can order the apparent wanderings of parabolas, hyperbolas, and ellipses through the first four books. Such hypotheses are the analogies between the three sections, and especially the development of the analogy between the hyperbola and the ellipse reaching its culmination, in the first book, with the final theorem, the construction of conjugate opposite sections.

In First Definitions I 5, Apollonius innocently defines two kinds of diameters, the transverse and the upright. Each one, in a conic section, bisects all the straight lines parallel to the other. But the upright diameter, defined here only as to position, has, in the case of the ellipse, natural bounds fixed by the section itself, and in Proposition I 15 we find it is the mean proportional between the corresponding transverse diameter (or conjugate diameter) and its parameter. The transverse diameter, in turn, is the mean proportional between the upright diameter (or conjugate) and its parameter, so "upright" and "transverse" become meaningless terms in the case of the ellipse, for something better expressed by the symmetrical relation "conjugate" (First Def I 6). Immediately in Proposition I 16, as if arbitrarily, the upright diameter of the hyperbola is bounded in the same way, given a definite magnitude and becomes "the second diameter." But so far transverse and upright diameters, or transverse and second diameters, are distinct things in the case of the hyperbola and there seems to be little reason for giving this second diameter in magnitude *νομος* has not yet become *φύσις*. That the upright diameter should be given even in position for the hyperbola becomes only very significant with two pairs of propositions—Propositions I 37 and 38 and I 39 and 40—where it is shown that certain properties holding for ordinates to the transverse diameter of the hyperbola and ellipse hold also for the ordinates to their conjugates. But it is only with the final proposition of the first book (I 60) that the magnitude of the hyperbola's second diameter is justified in magnitude as well as position. It is the corresponding diameter of the opposite sections conjugate to the first. And this analogy between the hyperbol-

and ellipse now stands on the threshold of a vast development. For this theorem, coming as a climax to the first book, makes possible the main theme of the second book—the asymptotes, those strange lines all but touching each opposite section (II 2, 13, 14) and forming a single bound between each adjacent pair (II 15, 17), so making the hyperbola an all but closed section, a puckered ellipse, a mouth turned inside out. And in the third book, the fruits of this analogy are gathered as in the especially nice case of Proposition III 15.

Although this translation is literal, we have not hesitated to use such symbols and abbreviations as, without prejudicing any Greek number theory or introducing any modern theory of symbols, would yet make the reading and the mechanic of study easier and at the same time preserve all the rigor of Greek mathematics.

As for the Greek text, we have used Heiberg, and have constantly referred to the *editio princeps* of Halley. In certain instances we have been glad to consult the very excellent French translation of Paul Ver Eecke (Desclée de Brouwer, Bruges, 1923). We have also deferred, at all relevant points, to the English usage of T. L. Heath's translation of Euclid's *Elements*.

EXAMPLES OF ABBREVIATIONS AND SYMBOLS USED

$A=B$ for A is equal to B

$A+B$ for A added to B

$A-B$ for B subtracted from A

$A \ B \ C \ D$ for A is to B as C is to D

rect AB, BC for rectangle AB, BC

sq AB for square on AB

ar for area

pllg for parallelogram

trgl for triangle

quadr for quadrilateral

rect $AB \ BC$ rect $CD \ DE$ comp $AB \ CD \ BC \ DE$
for ratio of rectangle $AB \ BC$ to rectangle CD, DE ■
compounded of the ratio of AB to CD and of BC to
 DE

ratio comp $AB \ BC, CD \ DE$ = ratio comp $XY \ YZ,$
 $ZW \ WV$ for ratio compounded of AB to BC and of
 CD to DE is the same as the ratio compounded of XY
to YZ and of ZW to WV

$A > B$ for A is greater than B

$A < B$ for A is less than B

rt angle for right angle

BOOK ONE

APOLLONIUS to EUDEMUS greetings

If you are restored in body and other things go with you to your mind well and good and we too fare pretty well At the time I was with you in Pergamum I observed you were quite eager to be kept informed of the work I was doing in conics And so I have sent you this first book revised and we shall dispatch the others when we are satisfied with them For I don't believe you have forgotten hearing from me how I worked out the plan for these conics at the request of Naucrates the geometer at the time he was with us in Alexandria lecturing and how on arranging them in eight books we immediately communicated them in great haste because of his near departure not revising them but putting down whatever came to us with the intention of a final going over And so finding now the occasion of correcting them one book after another we publish them And since it happened that some others among those frequenting us got acquainted with the first and second books before the revision don't be surprised if you come upon them in a different form

Of the eight books the first four belong to a course in the elements The first book contains the generation of the three sections and of the opposite branches and the principal properties (τα αρχικα συμμαματα) in them worked out more fully and universally than in the writings of others The second book contains the properties (τα συμβαιοντα) having to do with the diameters and axes and also the asymptotes and other things of a general and necessary use for limits of possibility (προς τους δυνατους) And what I call diameters and what I call axes you will know from this book The third book contains many incredible theorems of use for the construction of solid loci and for limits of possibility of which the greatest part and the most beautiful are new And when we had grasped these we knew that the three line and four line locus had not been constructed by Euclid but only a chance part of it and that not very happily For it was not possible for this construction to be completed without the additional things found by us The fourth book shows in how many ways the sections of a cone intersect with each other and with the circumference of a circle and contains other things in addition none of which has been written up by our predecessors that is in how many points the section of a cone or the circumference of a circle and the opposite branches meet the opposite branches The rest of the books are fuller in treatment For there is one dealing more fully with maxima and minima and one with equal and similar sections of a cone and one with limiting theorems and one with determinate conic problems And so indeed with all of them published those happening upon them can judge them as they see fit Good bye

FIRST DEFINITIONS

1 If from a point a straight line \equiv joined to the circumference of a circle which \equiv not in the same plane with the point, and the line is produced in both directions, and if, with the point remaining fixed, the straight line being rotated about the circumference of the circle returns to the same place from which it began then the generated surface composed of the two surfaces lying vertically opposite one another, each of which increases indefinitely as the generating straight line \equiv produced indefinitely, I call \equiv conic surface, and I call the fixed point the vertex, and the straight line drawn from the vertex to the center of the circle the axis

2 And the figure contained by the circle and by the conic surface between the vertex and the circumference of the circle I call a cone, and the point which is also the vertex of the surface I call the vertex of the cone, and the straight line drawn from the vertex to the center of the circle the axis and the circle the base of the cone

3 I call right cones those having axes perpendicular to their bases and oblique those not having axes perpendicular to their bases

4 Of any curved line which is in one plane I call that straight line the diameter which drawn from the curved line, bisects all straight lines drawn to this curved line parallel to some straight line and I call the end of that straight line (the diameter) situated on the curved line the vertex of the curved line, and I say that each of these parallels is drawn ordinately to the diameter (*τεταγμένως ἐπὶ τὴν διάμετρον κατὰ χεῖρας*)¹

5 Likewise of any two curved lines lying in one plane I call that straight line the transverse diameter (*διάμετρος πλάγια*) which cuts the two curved lines and bisects all the straight lines drawn to either of the curved lines parallel to some straight line and I call the ends of the diameter situated on the curved lines the vertices of the curved lines and I call that straight line the upright diameter (*διάμετρος ὀρθία*) which lying between the two curved lines, bisects all the straight lines intercepted between the curved lines and drawn parallel to some straight line and I say that each of the parallels is drawn ordinately to the diameter

6 The two straight lines each of which being a diameter bisects the straight lines parallel to the other I call the conjugate diameters (*συνυγείε διαίτηροι*) of a curved line and of two curved lines

7 And I call that straight line the axis of a curved line and of two curved lines which being a diameter of the curved line or lines cuts the parallel straight lines at right angles

8 And I call those straight lines the conjugate axes of a curved line and of two curved lines which being conjugate diameters cut the straight lines parallel to each other at right angles

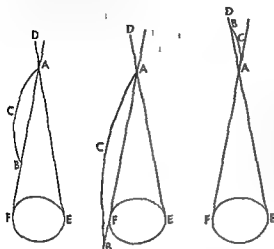
PROPOSITION 1

The straight lines drawn from the vertex of the conic surface to points on the surface are on that surface

Let there be a conic surface whose vertex is the point Δ and let there be

¹We shall follow modern usage and generally call these parallels ordinates

taken some point B on the conic surface and let \equiv straight line ACB be joined
 I say that the straight line ACB is on the conic surface



For if possible, let it not be, and let the straight line DE be the line generating the surface, and EF be the circle along which ED is moved. Then if, the point A remaining fixed, the straight line DE \equiv moved along the circumference of the circle EF it will also go through the point B (Def 1), and two straight lines will have the same ends. And this \equiv absurd.

Therefore the straight line joined from A to B cannot not be on the surface. Therefore it is on the surface.

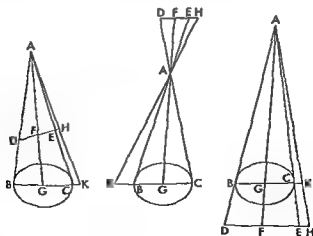
PORISM

It is also evident that if a straight line is joined from the vertex to some point among those within the surface it will fall within the conic surface, and if it is joined to some point among those without it will be outside the surface.

PROPOSITION 2

If on either one of the two vertically opposite surfaces two points are taken and the straight line joining the points does not verge to the vertex then it will fall within the surface and produced it will fall outside.

Let there be a conic surface whose vertex is the point A , and a circle BC



along whose circumference the generating straight line is moved and let two points D and E be taken on either one of the two vertically opposite

and let the joining straight line DE not verge to the point A

I say that the straight line DE will be within the surface, and produced will be without

Let AF and AD be joined and produced. Then they will fall on the circumference of the circle (1. 1). Let them fall to the points B and C , and let BC be joined. Therefore the straight line BC will be within the circle, and so too within the conic surface.

Then let a point F be taken at random on DE , and let the straight line AF be joined and produced. Then it will fall on the straight line BC , for the triangle BCA is in one plane (Eucl. 1. 2). Let it fall to the point G . Since then the point G is within the conic surface, therefore the straight line AG is also within the conic surface (1. 1, porism), and so too the point F is within the conic surface. Then likewise it will be shown that all the points on the straight line DE are within the surface. Therefore the straight line DE is within the surface.

Then let DE be produced to H . I say then it will fall outside the conic surface.

For if possible let there be some point H of it not outside the conic surface and let AH be joined and produced. Then it will fall either on the circumference of the circle or within (1. 1 and porism). And this is impossible, for it falls on BC produced as for example to the point K . Therefore the straight line EH is outside the surface.

Therefore the straight line DE is within the conic surface and produced is outside.

PROPOSITION 3

If a cone is cut by a plane through the vertex the section is a triangle.

Let there be a cone whose vertex is the point A and whose base is the circle BC and let it be cut by some plane through the point A and let it make as sections lines AB and AC on the surface and the straight line BC in the base.

I say that ABC is a triangle.

For since the line joined from A to B is the common section of the cutting plane and of the surface of the cone therefore AB is a straight line. And likewise also AC . And BC is also a straight line. Therefore ABC is a triangle.

If then a cone is cut by some plane through the vertex, the section is a triangle.



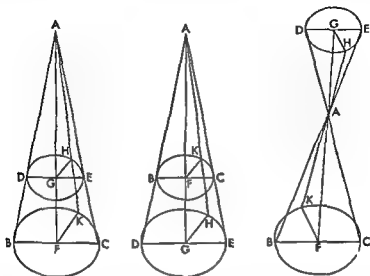
PROPOSITION 4

If either one of the vertically opposite surfaces is cut by some plane parallel to the circle along which the straight line generating the surface is moved, the plane cut off within the surface will be a circle having its center on the axis, and the figure contained by the circle and the conic surface intercepted by the cutting plane on the side of the vertex will be a cone.

Let there be a conic surface whose vertex is the point A and whose circle along which the straight line generating the surface is moved is BC and let it be cut by some plane parallel to the circle BC , and let it make on the surface as a section the line DE .

I say that the line DE is a circle having its center on the axis

For let the point F be taken as the center of the circle BC , and let AF be joined. Therefore AF is in the axis (Def. 1) and meets the cutting plane. Let it meet it at the point G , and let some plane be produced through AF . Then the



section will be the triangle ABC (1. 3). And since the points D, G, E are points in the cutting plane, and are also in the plane of the triangle ABC , therefore DGE is a straight line (Eucl. xi. 3).

Then let some point H be taken on the line DE and let AH be joined and produced. Then it falls on the circumference BC (1. 1). Let it meet it at K , and let GH and FK be joined. And since two parallel planes DE and BC are cut by a plane AHK , their common sections are parallel (Eucl. xi. 16). Therefore the straight line DE is parallel to the straight line BC . Then for the same reason the straight line GH is also parallel to the straight line FK . Therefore

$$FA : AG = FB : DG = FC : GE = FK : GH \quad (\text{Eucl. vi. 4})$$

And

$$BF = AF = FC$$

Therefore also

$$DG = GH = GE \quad (\text{Eucl. v. 9})$$

Then likewise we could show also that all the straight lines falling from the point G on the line DE are equal to each other.

Therefore the line DE is a circle having its center on the axis.

And it is evident that the figure contained by the circle DE and the conic surface cut off by it on the side of the point A is a cone.

And it is therewith proved that the common section of the cutting plane and of the axial triangle (triangle through the axis) is a diameter of the circle.

PROPOSITION 5

If an oblique cone is cut by a plane through the axis at right angles to the base, and is also cut by another plane on the one hand at right angles to the axial and on the other cutting off on the side of the vertex a triangle similar to the

triangle and lying subcontrariwise then the section is a circle, and let such a section be called subcontrary

Let there be an oblique cone whose vertex is the point A and whose base is the circle BC , and let it be cut by a plane through the axis perpendicular to the circle BC , and let it make as a section the triangle ABC (13). Then let it also be cut by another plane perpendicular to the triangle ABC and cutting off on the side of the point A the triangle AHG similar to the triangle ABC and lying subcontrariwise, that is, so that the angle AAG is equal to the angle ABC . And let it make as a section on the surface, the line GHA .

I say that the line GHA is a circle.

For let any points H and L be taken on the lines GHA and BC , and from the points H and L let perpendiculars be dropped to the plane through the triangle ABC . Then they will fall to the common sections of the planes (Eucl xi def 6). Let them fall as for example FH and LM . Therefore FH is parallel to LM (Eucl xi 6).

Then let the straight line DFE be drawn through F parallel to BC , and FH is also parallel to LM . Therefore the plane through FH and DE is parallel to the base of the cone (Eucl xi 15). Therefore it is a circle whose diameter is the straight line DE (11).

Therefore

$$\text{rect } DF \cdot FF = \text{sq } FH \text{ (Eucl iii 31 and vi 8, porism)}$$

And since ED is parallel to BC angle ADE is equal to angle ABC . And angle AAG is supposed equal to angle ABC . And therefore angle AKG is equal to angle ADE . And the vertical angles at the point F are also equal. Therefore triangle DFG is similar to triangle AFE and therefore

$$DF : FK :: GF : FD \text{ (Eucl vi 4)}$$

Therefore

$$\text{rect } EF \cdot FD = \text{rect } AF \cdot FG \text{ (Eucl vi 16)}$$

But it has been shown that

$$\text{sq } FH = \text{rect } FF \cdot FD,$$

and therefore

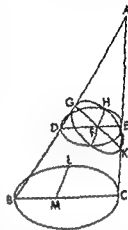
$$\text{rect } AF \cdot FC = \text{sq } FH$$

Likewise then all the perpendiculars drawn from the line GHA to the straight line GA could also be shown to be equal in square to the rectangle, in each case contained by the segments of the straight line GK .

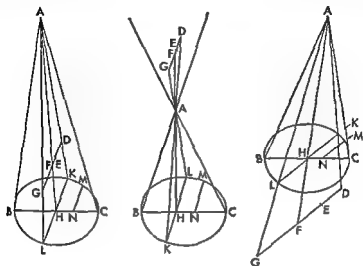
Therefore the section is a circle whose diameter is the straight line GA .

PROPOSITION 6

If a cone is cut by a plane through the axis and some point is taken on the surface of the cone which is not on a side of the axial triangle and from it is drawn a straight line parallel to some straight line which is a perpendicular from the circumference of the circle to the base of the triangle then it meets the axial triangle, and on being produced to the other side of the surface it will be bisected by the triangle.



Let there be a cone whose vertex is the point A and whose base is the circle BC , and let the cone be cut by a plane through the axis and let it make a common section the triangle ABC (1 3), and from some point M of those on the



circumference let the straight line MN be drawn perpendicular to the straight line BC . Then let some point D be taken on the surface of the cone and through D let the straight line DE be drawn parallel to MN .

I say that the straight line DE produced will meet the plane of the triangle ABC and, if further produced toward the other side of the cone until it meet its surface, will be bisected by the triangle ABC .

Let the straight line AD be joined and be produced. Therefore it will meet the circumference of the circle BC (1 1). Let it meet it at K and from K let the straight line KHL be drawn perpendicular to the straight line BC . Therefore KH is parallel to MN , and therefore to DE (Eucl. 1 9).

Let the straight line AH be joined from A to H . Since then in the triangle AHK the straight line DE is parallel to the straight line HK , therefore DE produced will meet AH . But AH is in the plane of ABC , therefore DE will meet the plane of the triangle ABC .

For the same reasons it also meets AH ; let it meet it at F and let DF be produced in a straight line until it meet the surface of the cone. Let it meet it at G .

I say that DF is equal to FG .

For since A, G, L are points on the surface of the cone but also in the plane extended through the straight lines AH, AK, DG, KL which is a triangle through the vertex of the cone (1 3), therefore A, C, L are points on the common section of the cone's surface and of the triangle. Therefore the line through A, G, L is a straight line. Since then in the triangle ALK the straight line DG has been drawn parallel to the base KHL and some straight line AFH has been drawn across them from the point A , therefore

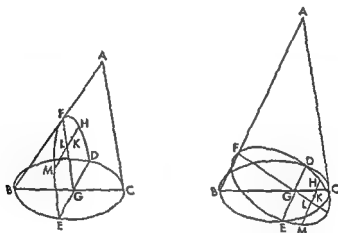
KH, HL, DF, FG (Eucl. 1 2)

But KH is equal to HL since KL is a chord in circle BC perpendicular to the diameter (Eucl. III. 3). Therefore DF is equal to FG .

PROPOSITION 7

If a cone is cut by a plane through the axis and if it is also cut by another plane cutting the plane the base of the cone is in, in a straight line perpendicular either to the base of the axial triangle or to it produced, then the straight lines drawn from the resulting section on the cone's surface, made by the cutting plane, parallel to the straight line perpendicular to the base of the triangle will fall on the common section of the cutting plane and of the axial triangle and further produced to the other side of the section, are bisected by the common section, and if it is a right cone the straight line in the base will be perpendicular to the common section of the cutting plane and of the axial triangle and if oblique, it will not always be perpendicular, but whenever the plane through the axis is perpendicular to the base of the cone

Let there be a cone whose vertex is the point A and whose base is the circle BC and let it be cut by a plane through the axis and let it make as a section

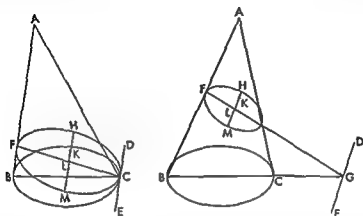


the triangle ABC (1. 3). And let it also be cut by another plane cutting the plane the circle BC is in, in the straight line DE perpendicular either to the straight line BC or to it produced and let it make as a section on the surface of the cone the line DFE . Then the straight line FG is the common section of the cutting plane and of the triangle ABC . And let any point H be taken on the section DFE and let the straight line HK be drawn through H parallel to the straight line DE .

I say that the straight line HK meets the straight line FG , and on being produced to the other side of the section DFE , will be bisected by FG .

For since a cone whose vertex is the point A and whose base is the circle BC has been cut by a plane through its axis, and makes as a section the triangle ABC and some point H on the surface, not on a side of the triangle ABC has been taken and since the straight line DG is perpendicular to the straight line BC therefore the straight line drawn through H parallel to DG that is HK , meets the triangle ABC and if further produced to the other side of the surface, will be bisected by the triangle (1. 6).

Then since the straight line drawn through H parallel to the straight line DE meets the triangle ABC and is in the plane of the section DFE , therefore



it will fall on the common section of the cutting plane and of the triangle ABC . But the straight line FG is the common section of the planes. Therefore the straight line drawn through H parallel to DE will fall on FG , and, if further produced to the other side of the section DFE , will be bisected by the straight line FG .

Then either the cone is a right cone or the axial triangle ABC is perpendicular to the circle BC , or neither.

First let the cone be a right cone. Then the triangle ABC would be perpendicular to the circle BC (Def 3 Eucl xi 18). Since then the plane ABC is perpendicular to the plane BC and the straight line DE has been drawn in one of the planes BC perpendicular to their common section the straight line BC therefore the straight line DE is perpendicular to the triangle ABC (Eucl xi def 4) and therefore to all the straight lines touching it and in the triangle ABC (Eucl xi def 3). And so it is also perpendicular to the straight line FG .

Then let the cone not be a right cone. If now the axial triangle is perpendicular to the circle BC , we could likewise show that DE is perpendicular to FG .

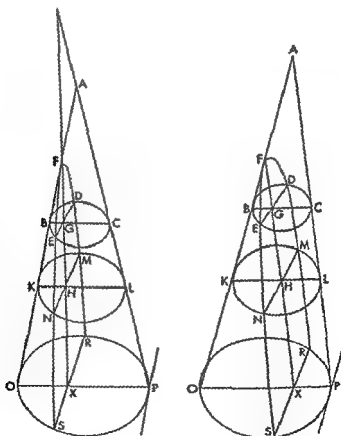
Then let the axial triangle ABC not be perpendicular to the circle BC — I say that DE is not perpendicular to FG . For if possible let it be. And it is also perpendicular to the straight line BC . Therefore DE is perpendicular to both BC and FG , and therefore it will be perpendicular to the plane through BC and FG . But the plane through BC and GF is the triangle ABC and therefore DE is perpendicular to the triangle ABC . And therefore all the planes through it are perpendicular to the triangle ABC . But one of the planes through DE is the circle BC therefore the circle BC is perpendicular to the triangle ABC . And so the triangle ABC will also be perpendicular to the circle BC . And this is not supposed. Therefore the straight line DE is not perpendicular to the straight line FG .

PORISM

Then from this it is evident that the straight line FG is the diameter of the section DFE , since it bisects the straight lines drawn parallel to some straight line DE and that it is possible for some parallels to be bisected by the diameter FG and not be perpendicular

PROPOSITION 8

If a cone is cut by a plane through its axis and is cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if the diameter of the resulting section on the surface is either parallel to one of the sides of the triangle or meets one of them beyond the vertex of the cone and the surface of the cone and the cutting plane are produced indefinitely then the section will also increase indefinitely, and some straight line drawn from the section of the cone parallel to the straight line in the base of the cone will cut off from the diameter on the side of the vertex a straight line equal to any given straight line



Let there be a cone whose vertex is the point A and whose base is the circle BC and let it be cut by a plane through its axis and let it make as a section the triangle ABC (1. 3). And let it be cut also by another plane cutting the

circle BC in a straight line DE perpendicular to the straight line BC , and let it make as a section on the surface the line DFE . And let the diameter FG of the section DFE be either parallel to the straight line AC or on being produced meet it beyond the point A (r 7 and porism)

I say that, if both the surface of the cone and the cutting plane are produced indefinitely, the section DFE also will increase indefinitely

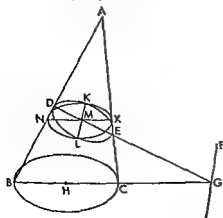
For let both the surface of the cone and the cutting plane be produced. Then it is evident that also the straight lines AB , AC , FG will be therewith produced. Since the straight line FG is either parallel to AC or produced meets it beyond the point A , therefore the straight lines FG and AC on being produced in the direction of C and G will never meet. Then let them be produced and let some point H be taken at random on the straight line FG , and let the straight line KHL be drawn through the point H parallel to the straight line BC and MHN parallel to DE . Therefore the plane through KL and MN is parallel to the plane through BC and DE (Eucl. VI. 15). Therefore the plane $KLMN$ is a circle (r 4).

And since the points D , E , M , N are in the cutting plane and also on the surface of the cone, therefore they are on the common section. Therefore the section DFE has increased to the points M and N . Therefore, with the surface of the cone and the cutting plane increased to the circle $KLMN$, the section DFE has also increased to the points M and N . Then likewise we could show also that if the surface of the cone and the cutting plane are extended indefinitely, the section $MDFEN$ will also increase indefinitely.

And it is evident that some straight line will cut off on straight line FH on the side of point F a straight line equal to any given straight line. For if we lay down the straight line FY equal to the given straight line and draw a parallel to DE through X , it will meet the section just as the straight line through H was also proved to meet the section in the points M and N . And so some straight line is drawn meeting the section parallel to DE , and cutting off on FG on the side of point H a straight line equal to the given straight line.

PROPOSITION 9

If a cone is cut by a plane meeting both sides of the axial triangle and neither parallel to the base nor situated subcontrariwise then the section will not be a circle.



Let there be a cone whose vertex is the point A and whose base is the circle BC and let it be cut by some plane neither parallel to the base nor situated subcontrariwise, and let it make as a section on the surface the line DKE .

I say that the line DKE will not be a circle.

For if possible let it be and let the cutting plane meet the base and let the straight line FG be the common section of the planes, and let the point H be the center of the circle BC and let the straight line HG be

drawn from it perpendicular to the straight line FG . And let a plane be extended through GHI and the axis and let it make as sections on the conic surface the straight lines BA and AC (1 1). Since then D, E, G are points in the plane through the line $D\Lambda E$ and also in the plane through the points A, B, C , therefore D, E, G are points on the common section of the planes. Therefore GED is a straight line (Eucl 11 3).

Then let some point Λ be taken on the line $D\Lambda E$, and through Λ let the straight line KL be drawn parallel to the straight line FG , then KM will be equal to ML (1 7). Therefore the straight line DE is the diameter of the circle $D\Lambda LE$ (Def 4). Then let the straight line $NM\Lambda$ be drawn through M parallel to the straight line BC . But ΛL is also parallel to FG . And so the plane through the straight lines NV and ΛM is parallel to the plane through the straight lines BC and FG , that is to the base (Eucl 11 15), and the section will be a circle (1 4). Let it be the circle NKX .

And since the straight line FG is perpendicular to the straight line BG , the straight line KM is also perpendicular to the straight line NX (Eucl 11 10). And so

$$\text{rect } NM \cdot M\Lambda = \text{sq } KM \text{ (Eucl III 31 VI 8, porism)}$$

But

$$\text{rect } DM \cdot ME = \text{sq } KM,$$

for the line $D\Lambda EL$ is supposed a circle, and the straight line DE is its diameter. Therefore

$$\text{rect } NM \cdot M\Lambda = \text{rect } DM \cdot ME$$

Therefore

$$MN \cdot MD = EM \cdot M\Lambda \text{ (Eucl VI 16)}$$

Therefore triangle DMN is similar to triangle ΛME (Eucl VI 6, VI def 1) and angle DNM is equal to angle $ME\Lambda$. But angle DNM is equal to angle ABC , for the straight line $N\Lambda$ is parallel to the straight line BC . And therefore angle ABC is equal to angle $ME\Lambda$. Therefore the section is subcontrary (1 5). And this is not supposed. Therefore the line $D\Lambda E$ is not a circle.

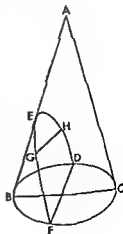
PROPOSITION 10

If two points are taken on the section of a cone the straight line joining the two points will fall within the section, and produced in a straight line it will fall outside.

Let there be a cone whose vertex is the point A and whose base is the circle BC , and let it be cut by a plane through the axis and let it make as a section the triangle ABC (1 3). Then let it also be cut by another plane, and let it make as a section on the surface of the cone the line DEF , and let two points G and H be taken on the line DEF .

I say that the straight line joining the two points G and H will fall within the line DEF , and produced in a straight line it will fall outside.

For since a cone, whose vertex is the point A and whose base is the circle BC , has been cut by a plane through the axis and some points G and H have been taken on its surface which are not on a side of the axial



triangle and since the straight line joining G and H does not verge to the point A therefore the straight line joining G and H will fall within the cone and produced in a straight line it will fall outside (1 2), consequently also outside the section DFE

PROPOSITION 11

If a cone is cut by a plane through its axis and also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if further the diameter of the section is parallel to one side of the axial triangle then any straight line which is drawn from the section of the cone to its diameter parallel to the common section of the cutting plane and of the cone's base will equal in square the rectangle contained by the straight line cut off by it on the diameter beginning from the section's vertex and by another straight line which has the ratio to the straight line between the angle of the cone and the vertex of the section that the square on the base of the axial triangle has to the rectangle contained by the remaining two sides of the triangle And let such a section be called a parabola ($\kappa\alpha\pi\alpha\beta\omicron\lambda\eta$)

Let there be a cone whose vertex is the point A , and whose base is the circle BC , and let it be cut by a plane through its axis and let it make as a section the triangle ABC (1 3) And let it also be cut by another plane cutting the base of the cone in the straight line DE perpendicular to the straight line BC and let it make as a section on the surface of the cone the line DFE , and let the diameter of the section FG (1 7 and def 4) be parallel to one side AC of the axial triangle And let the straight line FH be drawn from the point F perpendicular to the straight line FG and let it be contrived that

$$\text{sq } BC \text{ rect } BA \cdot AC = FH \cdot FA$$

And let some point K be taken at random on the section and through K let the straight line KL be drawn parallel to the straight line DE

I say that $\text{sq } KL = \text{rect } HF \cdot FL$

For let the straight line MN be drawn through L parallel to the straight line BC And the straight line DE is also parallel to the straight line KL Therefore the plane through KL and MN is parallel to the plane through BC and DE (Eucl xi 15) that is to the base of the cone Therefore the plane through KL and MN is a circle whose diameter is MN (1 4) And KL is perpendicular to MN since DE is also perpendicular to BC (Eucl xi 10) Therefore

$$\text{rect } ML \cdot LN = \text{sq } KL \text{ (Eucl iii 31 vi 8 porism)}$$

And since

$$\text{sq } BC \text{ rect } BA \cdot AC = HF \cdot FA$$

and

$$\text{sq } BC \text{ rect } BA \cdot AC \text{ comp } BC \cdot CA = BC \cdot BA \text{ (Eucl vi 23),}$$

therefore

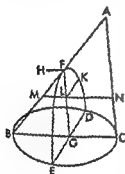
$$HF \cdot FA \text{ comp } BC \cdot CA = BC \cdot BA$$

But

$$BC \cdot CA = MN \cdot NA = ML \cdot LF \text{ (Eucl vi 4),}$$

and

$$BC \cdot BA = MN \cdot MA = LM \cdot MF = NL \cdot FL \text{ (Eucl vi 2)}$$



Therefore

$$HF \cdot FA \text{ comp } ML \cdot LF, NL \cdot FA$$

But

$$\text{rect } ML, LN \text{ rect } LF, FA \text{ comp } ML \cdot LF, LN \cdot FA \text{ (Eucl vi 23)}$$

Therefore

$$HF \cdot FA \text{ rect } ML, LN \text{ rect } LF, FA$$

But with the straight line FL taken as common height

$$HF \cdot FA \text{ rect } HF, FL \text{ rect } LF, FA \text{ (Eucl vi 1),}$$

therefore

$$\text{rect } ML, LN \text{ rect } LF, FA \text{ rect } HF, FL \text{ rect } LF, FA \text{ (Eucl v 11)}$$

Therefore

$$\text{rect } ML, LN = \text{rect } HF, FL \text{ (Eucl v 9)}$$

But

$$\text{rect } ML \cdot LN = \text{sq } KL$$

therefore also

$$\text{sq } KL = \text{rect } HF \cdot FL$$

And let such a section be called a parabola, and let HF be called the straight line to which the straight lines drawn ordinatewise to the diameter FG are applied in square ($\pi\alpha\rho\ \eta\gamma\ \delta\upsilon\alpha\tau\alpha\iota\ \alpha\lambda\ \kappa\alpha\tau\alpha\gamma\omicron\mu\epsilon\upsilon\alpha\iota\ \tau\epsilon\tau\alpha\gamma\mu\acute{\epsilon}\nu\omega\iota\ \epsilon\pi\iota\ \tau\eta\gamma\ \textit{ZH} \delta\iota\acute{\alpha}\mu\epsilon\tau\epsilon\rho\omega\varsigma$), and let it also be called the upright side ($\delta\omicron\theta\epsilon\iota\alpha$)¹

PROPOSITION 12

If a cone is cut by a plane through its axis and also by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if the diameter of the section produced meets one side of the axial triangle beyond the vertex of the cone then any straight line which is drawn from the section to its diameter parallel to the common section of the cutting plane and of the cone's base will equal in square some area applied to a straight line to which the straight line added along the diameter of the section and subtending the exterior angle of the triangle has the ratio that the square on the straight line drawn from the cone's vertex to the triangle's base parallel to the section's diameter has to the rectangle contained by the sections of the base which this straight line makes when drawn, this area having as breadth the straight line cut off on the diameter beginning from the section's vertex by this straight line from the section to the diameter and exceeding ($\nu\pi\epsilon\rho\beta\acute{\alpha}\lambda\lambda\omicron\upsilon\omega\varsigma$) by a figure ($\epsilon\iota\delta\omicron\varsigma$) similar and similarly situated to the rectangle contained by the straight line subtending the exterior angle of the triangle and by the parameter. And let such a section be called an hyperbola ($\nu\pi\epsilon\rho\beta\omicron\lambda\eta$).

Let there be a cone whose vertex is the point A and whose base is the circle BC , and let it be cut by a plane through its axis, and let it make as a section the triangle ABC (i 3). And let it also be cut by another plane cutting the base of the cone in the straight line DE perpendicular to BC the base of the triangle ABC , and let it make as a section on the surface of the cone the line DFE , and

¹The Greek of the phrase the straight line to which the straight lines drawn ordinatewise to the diameter are applied in square that is $\eta\ \pi\alpha\theta\ \eta\ \delta\ \alpha\upsilon\tau\alpha\iota\ \alpha\lambda\ \kappa\alpha\tau\alpha\gamma\omicron\mu\epsilon\upsilon\alpha\iota\ \tau\epsilon\tau\alpha\gamma\mu\acute{\epsilon}\nu\omega\iota\ \epsilon\pi\iota\ \tau\eta\gamma\ \delta\ \acute{\alpha}\mu\epsilon\tau\epsilon\rho\omega\varsigma$ soon becomes abbreviated to $\eta\ \pi\alpha\theta\ \eta\gamma\ \delta\epsilon\ \alpha\upsilon\tau\alpha\ \alpha\lambda\ \kappa\alpha\tau\alpha\gamma\omicron\mu\epsilon\upsilon\alpha\iota$ and to $\eta\ \pi\alpha\theta\ \eta\gamma\ \delta\epsilon\ \alpha\upsilon\tau\alpha$. We shall translate these abbreviations by the word parameter. And we shall later on after proposition XIV shorten the long expression to the parameter of the ordinates to the diameter.

The Latin translation of $\delta\omicron\theta\epsilon\iota\alpha$ ($\nu\lambda\epsilon\upsilon\phi\acute{\alpha}$) is *latus rectum* which has become an English term too.

let FG the diameter of the section (1 7 and def 4) when produced meet AC one side of the triangle ABC beyond the vertex of the cone at the point H . And let the straight line AA be drawn through A parallel to the diameter of the section FG , and let it cut BC . And let the straight line FL be drawn from F perpendicular to FG , and let it be contrived that

$$\text{sq } AA \text{ rect } BK \text{ } KC \text{ } FH \text{ } FL$$

And let some point M be taken at random on the section, and through M let the straight line MN be drawn parallel to DE , and through N let the straight line VOX be drawn parallel to FL . And let the straight line HL be joined and produced to Y , and let the straight lines LO and XP be drawn through L and X parallel to FN .

I say that MN is equal in square to the parallelogram FA which is applied to FL having FN as breadth and exceeding by a figure LY similar to the rectangle contained by HF and FL .

For let the straight line RNS be drawn through N parallel to BC and NM is also parallel to DE . Therefore the plane through MN and RS is parallel to the plane through BC and DE that is to the base of the cone (Eucl xi 15). Therefore if the plane is produced through MN and RS , the section will be a circle whose diameter is the straight line RNS (1 4). And MN is perpendicular to it. Therefore

$$\text{rect } RN, NS = \text{sq } MN$$

And since

$$\text{sq } AA \text{ rect } BA \text{ } KC \text{ } FH \text{ } FL,$$

and

$$\text{sq } AA \text{ rect } BA \text{ } KC \text{ comp } AA \text{ } KC \text{ } AK \text{ } KB \text{ (Eucl vi 23),}$$

therefore also

$$FH \text{ } FL \text{ comp } AA \text{ } KC \text{ } AK \text{ } KB$$

But

$$AK \text{ } KC \text{ } HG \text{ } GC \text{ } HN \text{ } NS \text{ (Eucl vi 4),}$$

and

$$AA \text{ } KB \text{ } FG \text{ } GB \text{ } FN \text{ } NR$$

Therefore

$$HF \text{ } FL \text{ comp } HN \text{ } NS \text{ } FN \text{ } NR$$

And

$$\text{rect } HN \text{ } NF \text{ rect } SN \text{ } NR \text{ comp } HN \text{ } NS \text{ } FN \text{ } NR \text{ (Eucl vi 23)}$$

Therefore also

$$\text{rect } HN \text{ } NF \text{ rect } SN \text{ } NR \text{ } HF \text{ } FL \text{ } HN \text{ } NY \text{ (Eucl vi 4)}$$

But, with the straight line FN taken as common height

$$HN \text{ } NY \text{ rect } HN \text{ } NF \text{ rect } FN \text{ } NY \text{ (Eucl vi 1)}$$

Therefore also

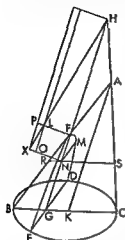
$$\text{rect } HN \text{ } NF \text{ rect } SN \text{ } NR \text{ rect } HN \text{ } NY \text{ rect } XN \text{ } NF \text{ (Eucl v 11)}$$

Therefore

$$\text{rect } SN \text{ } NR = \text{rect } XN \text{ } NF \text{ (Eucl v 9)}$$

But it was shown

$$\text{sq } MN = \text{rect } SN \text{ } NR,$$



therefore also

$$\text{sq } MN = \text{rect } XN \cdot NF$$

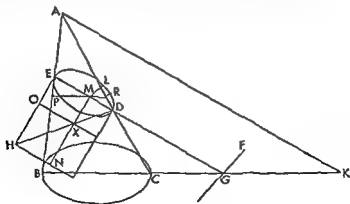
But the rectangle contained by YN and NF is the parallelogram YF . Therefore the straight line MN is equal in square to YF which is applied to the straight line FL , having FN as breadth and exceeding by the parallelogram LX similar to the rectangle contained by HF and FL (Eucl vi 24)

And let such a section be called an hyperbola, and let LF be called the straight line to which the straight lines drawn ordinatewise to FG are applied in square, and let the same straight line also be called the upright side, and the straight line FH the transverse side

PROPOSITION 13

If a cone is cut by a plane through its axis, and is also cut by another plane on the one hand meeting both sides of the axial triangle and on the other extended neither parallel to the base nor subcontrariwise, and if the plane the base of the cone is in and the cutting plane meet in a straight line perpendicular either to the base of the axial triangle or to it produced then any straight line which is drawn from the section of the cone to the diameter of the section parallel to the common section of the planes will equal in square some area applied to a straight line to which the diameter of the section has the ratio that the square on the straight line drawn from the cone's vertex to the triangle's base parallel to the section's diameter has to the rectangle contained by the intercepts of this straight line (on the base) from the sides of the triangle an area having as breadth the straight line cut off on the diameter beginning from the section's vertex by this straight line from the section to the diameter and deficient ($\epsilonλλειπov$) by a figure similar and similarly situated to the rectangle contained by the diameter and parameter. And let such a section be called an ellipse ($\epsilonλλειψις$)

Let there be a cone whose vertex is the point A and whose base is the circle BC and let it be cut by a plane through its axis, and let it make as a section



the triangle ABC . And let it also be cut by another plane on the one hand meeting both sides of the axial triangle and on the other extended neither parallel to the base of the cone nor subcontrariwise, and let it make as a section on the surface of the cone the line DE . And let the common section of the cut

ting plane and of the plane the base of the cone is in, be the straight line FG perpendicular to the straight line BC , and let the diameter of the section be the straight line ED (i 7 and Def 4) And let the straight line EH be drawn from E perpendicular to ED and let the straight line IA be drawn through A parallel to ED and let it be contrived that

$$\text{sq } AA \text{ rect } BK \ AC \ DE \ EH$$

And let some point L be taken on the section and let the straight line LM be drawn through L parallel to FG

I say that the straight line LM is equal in square to some area which is applied to EH , having EM as breadth and deficient by a figure similar to the rectangle contained by DE and EH

For let the straight line DH be joined and on the one hand let the straight line UV be drawn through V parallel to HE and on the other let the straight lines HN and VO be drawn through H and X parallel to EM and let the straight line PMR be drawn through V parallel to BC

Since then PR is parallel to BC and LM is also parallel to FG therefore the plane through LM and PR is parallel to the plane through FG and BC that is to the base of the cone (Eucl xi 15) If therefore a plane is extended through LM and PR the section will be a circle whose diameter is PR (i 4) And LM is perpendicular to it Therefore

$$\text{rect } PV \ MR = \text{sq } LM$$

And since

$$\text{sq } AA \text{ rect } BA \ AC \ DE \ EH$$

and

$$\text{sq } AK \text{ rect } BK \ AC \text{ comp } AA \ AB \ AA \ AC \text{ (Eucl vi 23),}$$

but

$$AA \ KB \ EG \ GB \ EM \ MP \text{ (Eucl vi 4)}$$

and

$$AK \ AC \ DG \ GC \ DV \ MR$$

therefore

$$DE \ EH \text{ comp } EM \ MP \ DV \ MR$$

But

$$\text{rect } EM \ MD \text{ rect } PV \ MR \text{ comp } EV \ VP \ DV \ MR \text{ (Eucl vi 23)}$$

Therefore

$$\text{rect } EM \ MD \text{ rect } PV \ MR \ DE \ EH \ DV \ MA \text{ (Eucl vi 4)}$$

And with the straight line ME taken as common height

$$DV \ MA \text{ rect } DV \ ME \text{ rect } XM \ ME \text{ (Eucl vi 1)}$$

Therefore also

$$\text{rect } DV \ ME \text{ rect } PM \ MR \text{ rect } DV \ ME \text{ rect } XM \ ME \text{ (Eucl v 11)}$$

Therefore

$$\text{rect } PV \ MR = \text{rect } VM \ ME \text{ (Eucl v 9)}$$

But it was shown

$$\text{rect } PM \ MR = \text{sq } LM$$

therefore also

$$\text{rect } VM \ ME = \text{sq } LM$$

Therefore the straight line LM is equal in square to the parallelogram MO which is applied to the straight line HE having EM as breadth and deficient by the figure OV similar to the rectangle contained by DE and EH (Eucl vi 24)

And let such \equiv section be called an ellipse, and let EH be called the straight line to which the straight lines drawn ordinatewise to DE are applied in square, and let the same straight line also be called the upright side, and the straight line ED the transverse side

PROPOSITION 14

If the vertically opposite surfaces are cut by a plane not through the vertex the section on each of the two surfaces will be that which is called the hyperbola and the diameter of the two sections will be the same straight line and the straight lines to which the straight lines drawn to the diameter parallel to the straight line in the cone's base are applied in square, are equal and the transverse side of the figure, that between the vertices of the sections, is common. And let such sections be called opposite (ἀντικείμεναι)

Let there be the vertically opposite surfaces whose vertex is the point A and let them be cut by a plane not through the vertex and let it make as sections on the surface the lines DEF and GHK

I say that each of the two sections DEF and GHK is the so-called hyperbola. For let there be the circle $BDCF$ along which the line generating the surface moves and let the plane λGOK be extended parallel to it on the vertically opposite surface and the straight lines FD and GK are common sections of the sections GHK and FED , and of the circles (1 4). Then they will be parallel (Eucl xi 16). And let the straight line LAU be the axis of the conic surface and the points L and U be the centers of the circles and let a straight line drawn from L perpendicular to the straight line FD be produced to the points B and C and let a plane be produced through the straight line BC and the axis. Then it will make as sections in the circles the parallel straight lines λO and BC (Eucl xi 16) and on the surface the straight lines BAO and $CA\gamma$ (1 1 and Def 4).

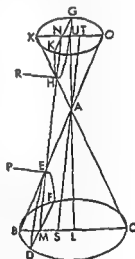
Then the straight line λO will be perpendicular to the straight line GK , since the straight line BC is also perpendicular to the straight line FD and each of the two is parallel to the other (Eucl xi 10). And since the plane through the axis meets the sections in the points M and N within the lines it is clear that the plane also cuts the lines. Let it cut them at H and E , therefore $M E H$ and N are points on the plane through the axis and in the plane the lines are in, therefore the line $MEHN$ is a straight line (Eucl xi 3). It is also evident both that $\lambda H A$, and C are in \equiv straight line and B, E, A , and O also. For they are both on the conic surface and in the plane through the axis (1 1).

Let then the straight lines HR and EP be drawn from H and E perpendicular to HE and let the straight line SAT be drawn through A parallel to $MEHN$ and let it be contrived that

$$HE \cdot EP = \text{sq } AS = \text{rect } BS \cdot SC,$$

and

$$EH \cdot HR = \text{sq } AT = \text{rect } OT \cdot T\lambda$$



Since then a cone whose vertex is the point A and whose base is the circle BC , has been cut by a plane through its axis and it has made as a section the triangle ABC and it has also been cut by another plane cutting the base of the cone in the straight line DUF perpendicular to the straight line BC and it has made as a section on the surface the line DEF , and the diameter ME produced has met one side of the axial triangle beyond the vertex of the cone and through the point A the straight line AS has been drawn parallel to the diameter of the section EM , and from E the straight line EP has been drawn perpendicular to the straight line EM , and

$$EH \cdot EP = \text{sq } AS \text{ rect } BS, SC,$$

therefore the section DEF is an hyperbola (I 12) and EP is the straight line to which the straight lines drawn ordinatewise to EM are applied in square and the straight line HE is the transverse side of the figure. And likewise GHA is also an hyperbola whose diameter is the straight line HN and whose straight line to which the straight lines drawn ordinatewise to HN are applied is HR , and the transverse side of whose figure is HE

I say that the straight line HR is equal to the straight line EP

For since BC is parallel to AO ,

$$AS : SC = AT : TX$$

and

$$AS : SB = AT : TO$$

But

$$\text{sq } AS \text{ rect } BS, SC \text{ comp } AS : SC = AS : SB \text{ (Eucl vi 23)}$$

and

$$\text{sq } AT \text{ rect } XT, TO \text{ comp } AT : TX = AT : TO,$$

therefore

$$\text{sq } AS \text{ rect } BS : SC = \text{sq } AT \text{ rect } XT, TO$$

Also

$$\text{sq } AS \text{ rect } BS : SC = HE : EP,$$

and

$$\text{sq } AT \text{ rect } XT : TO = HE : HR$$

Therefore also

$$HE : EP = EH : HR \text{ (Eucl v 11)}$$

Therefore

$$EP = HR \text{ (Eucl v 9)}$$

PROPOSITION 15

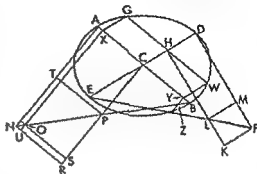
If in an ellipse a straight line drawn ordinatewise from the midpoint of the diameter, is produced both ways to the section and if it is contrived that as the straight line so produced is to the diameter so is the diameter to some straight line then any straight line which is drawn, from the section to the straight line produced parallel to the diameter, will equal in square the area applied to this third proportional and having as breadth the straight line cut off by it beginning from the section and deficient by a figure similar to the rectangle contained by the straight line to which the straight lines are drawn and by the parameter and if further produced to the other side of the section will be bisected by the straight line to which it has been drawn

Let there be an ellipse whose diameter is the straight line AB , and let AB be bisected at the point C and through C let the straight line DCE be drawn

ordinatewise and produced both ways to the section and from the point D let the straight line DF be drawn perpendicular to DE And let it be contrived that

$DE \cdot AB = AB \cdot DF$

And let some point G be taken on the section, and through G let the straight line GH be drawn parallel to AB , and let FF be joined and through H let the straight line HL be drawn parallel to DF , and through F and L let the straight lines FA and LM be drawn parallel to HD



I say that the straight line GH is equal in square to the area DL

which is applied to the straight line DF having as breadth the straight line DH and deficient by a figure LF similar to the rectangle contained by FD and DF

For let AN be the parameter of the ordinates to AB , and let BN be joined, and through G let the straight line GY be drawn parallel to DE , and through Y and C let the straight lines YO and CP be drawn parallel to AN , and through N , O and P let the straight lines NU , OS and TP be drawn parallel to AB Therefore

$$\text{sq } DC = \text{ar } AP, \text{ sq } GY = \text{ar } AO \text{ (I 13)}$$

And since

$$BA \cdot AN = BC \cdot CP = PT \cdot TN \text{ (I ucl vi 4),}$$

and

$$BC = CA = TP,$$

and

$$CP = TA,$$

therefore

$$\text{ar } AP = \text{ar } TR,$$

and

$$\text{ar } YT = \text{ar } TU$$

Since also

$$\text{ar } OT = \text{ar } OR \text{ (Eucl 1 43),}$$

and area NO is common therefore

$$\text{ar } TU = \text{ar } NS$$

But

$$\text{ar } TU = \text{ar } TY$$

and TS is common Therefore

$$\text{ar } NP = \text{ar } PA = \text{ar } AO + \text{ar } PO$$

and so

$$\text{ar } PA - \text{ar } AO = \text{ar } PO$$

Also

$$\text{ar } AP = \text{sq } CD, \text{ ar } NO = \text{sq } YG,$$

and

$$\text{ar } OP = \text{rect } OS, SP$$

therefore

$$\text{sq } CD - \text{sq } GY = \text{rect } OS, SP$$

Since also the straight line DE has been cut into equal parts at C , and into unequal parts at H therefore

$$\text{rect } EH, HD + \text{sq } CH = \text{sq } CD \text{ (Eucl II 5),}$$

or

$$\text{rect } EH, HD + \text{sq } XG = \text{sq } CD$$

Therefore

$$\text{sq } CD - \text{sq } XG = \text{rect } EH, HD,$$

but

$$\text{sq } CD - \text{sq } XG = \text{rect } OS, SP,$$

therefore

$$\text{rect } EH, HD = \text{rect } OS, SP$$

And since

$$DE \text{ } 4B \text{ } 4B \text{ } DF,$$

therefore

$$DE \text{ } DF \text{ } \text{sq } DE \text{ } \text{sq } AB \text{ (Eucl VI 20),}$$

that is

$$DE \text{ } DF \text{ } \text{sq } CD \text{ } \text{sq } CB \text{ (Eucl V 15),}$$

And

$$\text{rect } PC, CA = \text{rect } PC, CB = \text{sq } CD \text{ (I 13),}$$

and since

$$DE \text{ } DF \text{ } EH \text{ } HL \text{ (Eucl VI 4),}$$

or

$$DE \text{ } DF \text{ } \text{rect } EH, HD \text{ } \text{rect } DH, HL \text{ (Eucl VI 1),}$$

and since

$$DE \text{ } DF \text{ } \text{rect } PC, CB \text{ } \text{sq } CB,$$

and

$$\text{rect } PC, CB \text{ } \text{sq } CB \text{ } \text{rect } OS, SP \text{ } \text{sq } OS^1$$

therefore also

$$\text{rect } EH, HD \text{ } \text{rect } DH, HL \text{ } \text{rect } OS, SP \text{ } \text{sq } OS$$

And

$$\text{rect } EH, HD = \text{rect } OS, SP$$

therefore

$$\text{rect } DH, HL = \text{sq } OS = \text{sq } GH$$

Therefore the straight line GH is equal in square to the area DL which is applied to the straight line DF deficient by a figure FL similar to the rectangle contained by ED and DF (Eucl VI 24)

I say then that also if produced to the other side of the section the straight line GH will be bisected by the straight line DE

For let it be produced and let it meet the section at W , and let the straight line WY be drawn through Y parallel to GX , and through Y let the straight line YZ be drawn parallel to AN . And since

$$GX = WY,$$

therefore also

$$\text{sq } GX = \text{sq } WY$$

¹This follows from the proportions

and $\frac{PC}{PS} = \frac{CB}{OS}$ $\frac{PS}{OS} = \frac{OS}{OS}$ (Eucl VI 4)
and $\frac{PC}{PS} = \frac{CB}{OS}$ $\text{rect } PC, CB = \text{sq } CB$
 $\frac{PS}{OS} = \frac{OS}{OS}$ $\text{rect } PS, OS = \text{sq } OS$ (Eucl VI 1)

But

$$\text{sq } GX = \text{rect } A\lambda, \lambda O \text{ (I 13),}$$

and

$$\text{sq } W\lambda = \text{rect } AY, \lambda Z \text{ (I 13)}$$

Therefore

$$OY \cdot Z\lambda = YA \cdot A\lambda \text{ (Eucl VI 16)}$$

And

$$OX \cdot Z\lambda = \lambda B \cdot BY \text{ (Eucl VI 4),}$$

therefore also

$$YA \cdot A\lambda = \lambda B \cdot BY$$

And *separando*

$$YA \cdot A\lambda = \lambda X \cdot BY \text{ (Eucl V 17)}$$

Therefore

$$A\lambda = \lambda B$$

And also

$$AC = CB,$$

therefore also the remainders

$$XC = CY,$$

and so also

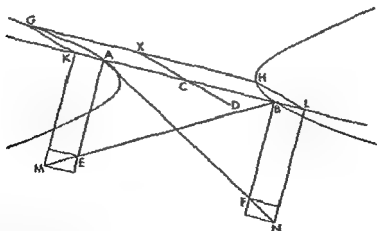
$$GH = HW$$

Therefore the straight line HG , produced to the other side of the section, is bisected by the straight line DH

PROPOSITION 16

If through the midpoint of the transverse side of the opposite sections a straight line be drawn parallel to a straight line drawn ordinatewise it will be a diameter of the opposite sections conjugate to the diameter just mentioned

Let there be the opposite sections whose diameter is the straight line AB ,



and let AB be bisected at C and through C let the straight line CD be drawn parallel to a straight line drawn ordinatewise

I say that the straight line CD is a diameter conjugate to AB

For let the straight lines AE and BF be the parameters and let the straight

lines AF and BL be joined and produced, and let some point G be taken at random on either section and through G let the straight line GH be drawn parallel to AB , and from G and H let the straight lines GK and HL be drawn ordinatewise, and through K and L let the straight lines KM and LN be drawn parallel to AE and BF . Since then

$$GK = HL \text{ (Eucl I 34),}$$

therefore also

$$\text{sq } GK = \text{sq } HL,$$

But

$$\text{sq } GK = \text{rect } AK, KM \text{ (I 12),}$$

and

$$\text{sq } HL = \text{rect } BL, LN \text{ (I 12),}$$

therefore

$$\text{rect } AK, KM = \text{rect } BL, LN$$

And since

$$AE = BF,$$

therefore

$$AE : AB :: BF : BA \text{ (Eucl V 7)}$$

But

$$AE : AB :: MK : KB \text{ (Eucl VI 4),}$$

and as

$$BF : BA :: NL : LA \text{ (Eucl VI 4)}$$

And therefore

$$MK : KB :: NL : LA$$

But, with KA taken as common height,

$$MK : KB :: \text{rect } MK, KA : \text{rect } BK, KA,$$

and, with BL taken as common height

$$NL : LA :: \text{rect } NL, LB : \text{rect } AL, LB$$

And therefore

$$\text{rect } MK, KA : \text{rect } BK, KA :: \text{rect } NL, LB : \text{rect } AL, LB$$

And alternately

$$\text{rect } MK, KA : \text{rect } NL, LB :: \text{rect } BK, KA : \text{rect } AL, LB \text{ (Eucl V 16)}$$

And

$$\text{rect } AK, KM = \text{rect } BL, LN,$$

therefore

$$\text{rect } BK, KA = \text{rect } AL, LB,$$

therefore

$$AK = LB^1$$

But also

$$AC = CB,$$

and therefore

$$KC = CL,$$

and so also

$$GX = XH$$

¹The intermediary steps to this conclusion are as follows. If

$$\text{rect } BK, KA = \text{rect } AL, LB$$

then

$$BK : LB :: AL : KA$$

or

$$BA + AK : LB :: BA + LB : AK$$

and componendo

$$BA + AK + LB : LB :: BA + LB + AK : AK$$

Therefore the straight line GH has been bisected by the straight line XCD and is parallel to the straight line AB . Therefore the straight line XCD is a diameter and conjugate to the straight line AB (Def. 4, 6)

SECOND DEFINITIONS

9 Let the midpoint of the diameter of both the hyperbola and the ellipse be called the center of the section, and let the straight line drawn from the center to meet the section be called the radius of the section.

10 And likewise let the midpoint of the transverse side of the opposite sections be called the center.

11 And let the straight line drawn from the center parallel to an ordinate, being a mean proportional to the sides of the figure ($\tau\delta\epsilon\iota\sigma\iota\varsigma$) and bisected by the center, be called the second diameter.

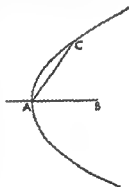
PROPOSITION 17

If in a section of a cone a straight line is drawn from the vertex of the line, and parallel to an ordinate, it will fall outside the section (Cf. Eucl. III 16)

Let there be a section of a cone whose diameter is the straight line AB .

I say that the straight line drawn from the vertex that is from the point A parallel to an ordinate will fall outside the section.

For if possible let it fall within as AC . Since then a point C has been taken at random on a section of a cone, therefore the straight line drawn from the point C within the section parallel to an ordinate will meet the diameter AB and will be bisected by it (17). Therefore the straight line AC produced will be bisected by the straight line AB . And this is absurd. For the straight line AC if produced will fall outside the section (10). Therefore the straight line drawn from the point A parallel to an ordinate will not fall within the line; therefore it will fall outside and therefore it is tangent to the section.



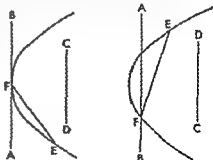
PROPOSITION 18

If a straight line meeting a section of a cone and produced both ways falls outside the section and some point is taken within the section and through it a parallel to the straight line meeting the section is drawn, the parallel so drawn if produced both ways will meet the section.

Let there be a section of a cone and the straight line AFB meeting it and let it fall when produced both ways outside the section. And let some point C be taken within the section and through C let the straight line CD be drawn parallel to the straight line AB .

I say that the straight line CD produced both ways will meet the section.

For let some point E be taken on the



ection and let the straight line EF be joined. And since the straight line AB is parallel to CD and some straight line EF meets AB , therefore CD produced will also meet EF . And if it meets EF between the points E and F , it is evident that it also meets the section, but if beyond the point E , that it will first meet the section. Therefore CD produced to the side of points D and E meets the section. Then likewise we could show that, produced to the side of points F and B it also meets it. Therefore the straight line CD produced both ways will meet the section.

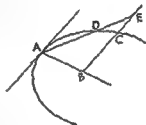
PROPOSITION 19

In every section of a cone, any straight line drawn from the diameter parallel to the ordinate, will meet the section.

Let there be a section of a cone whose diameter is the straight line AB , and let some point B be taken on the diameter, and through B let the straight line BC be drawn parallel to an ordinate.

I say that the straight line BC produced will meet the section.

For let some point D be taken on the section. But A is also on the section, therefore the straight line joined from A to D will fall within the section (I. 10). And since the straight line drawn from A parallel to an ordinate falls outside the section (I. 17), and the straight line AD meets it and the straight line BC is parallel to the ordinate therefore BC will also meet AD . And if it meets AD between the points A and D it is evident that it will also meet the section, but if beyond point D as at E that it will first meet the section. Therefore the straight line drawn from B parallel to an ordinate will meet the section.



PROPOSITION 20

If in a parabola two straight lines are dropped ordinatewise to the diameter, the squares on them will be to each other as the straight lines cut off by them on the diameter beginning from the vertex¹ are to each other.

Let there be a parabola whose diameter is the straight line AB and let some points C and D be taken on it and from the points C and D let the straight lines CE and DF be dropped ordinatewise to AB .

I say that

$$\text{sq } DF : \text{sq } CE :: FA : AE$$

For let AG be the parameter therefore

$$\text{sq } DF = \text{rect } FA, AG,$$

and

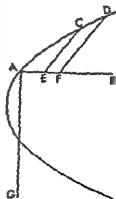
$$\text{sq } CE = \text{rect } EA, AG \text{ (I. 11)}$$

Therefore

$$\text{sq } DF : \text{sq } CE :: \text{rect } FA, AG : \text{rect } EA, AG$$

But

$$\text{rect } FA, AG : \text{rect } EA, AG :: FA : AE \text{ (Eucl. VI. 1),}$$



¹These are usually called *abscissas* from the Latin *abscindere*, to cut off

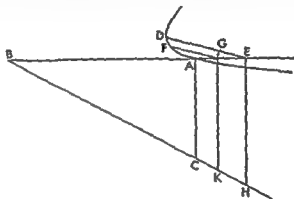
and therefore

$$\text{sq } DF \text{ sq } CE \quad FA \quad AE$$

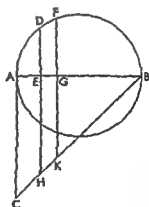
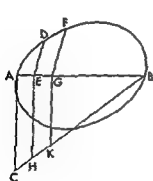
PROPOSITION 21

If in an hyperbola or ellipse or in the circumference of a circle straight lines are dropped ordinatewise to the diameter the squares on them will be to the areas contained by the straight lines cut off by them beginning from the ends of the transverse side of the figure, as the upright side of the figure is to the transverse and to each other as the areas contained by the straight lines cut off (abscissas), as we have said

Let there be an hyperbola or ellipse or circumference of a circle whose diam



eter is AB and whose parameter is the straight line AC and let the straight lines DE and FG be dropped ordinatewise to the diameter



I say that

$$\text{sq } FG \text{ rect } AG \text{ GB} \quad AC \quad AB$$

and

$$\text{sq } FG \text{ sq } DE \text{ rect } AG \text{ GB} \text{ rect } AE \text{ EB}$$

For let the straight line BC determining the figure be joined and through E and G let the straight lines EH and GK be drawn parallel to the straight line AC . Therefore

$$\begin{aligned} \text{sq } FG &= \text{rect } KG, GA \\ \text{sq } DE &= \text{rect } HE, EA \quad (\text{r}^1 12, 13) \end{aligned}$$

And since

$$AG, GB, CA, AB,$$

and with AG taken as common height

$$AG, GB, \text{rect } AG, GA, \text{rect } BG, GA,$$

therefore

$$CA, AB, \text{rect } AG, GA, \text{rect } BG, GA$$

or

$$CA, AB, \text{sq } FG, \text{rect } BG, GA$$

Then also for the same reasons

$$CA, AB, \text{sq } DE, \text{rect } BE, EA$$

And therefore

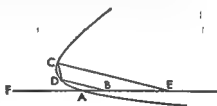
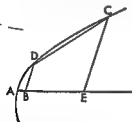
$$\text{sq } FG, \text{rect } BG, GA, \text{sq } DE, \text{rect } BE, EA$$

alternately

$$\text{sq } FG, \text{sq } DE, \text{rect } BG, GA, \text{rect } BE, EA^1$$

PROPOSITION 22

If a straight line cuts a parabola or hyperbola in two points not meeting the diameter inside it will, if produced, meet the diameter of the section outside the section. Let there be a parabola or hyperbola whose diameter is the straight line AB , and let some straight line cut the section in two points C and D .



I say that the straight line DC , if produced, will meet the straight line AB outside the section.

For let the straight lines CE and DB be dropped ordinately from C and D , and first let the section be a parabola. Since then in the parabola

$$\text{sq } CE = \text{sq } DB, EA, AB \quad (\text{r}^1 20)$$

and

$$EA > AB,$$

¹Eutocius commenting says It is to be noted that the parameter that is the upright side in the case of the circle is equal to the diameter. For if

$$\text{sq } DE = \text{rect } AE, EB, CA, AB^1$$

and only in the case of the circle

$$\text{sq } DE = \text{rect } AE, EB$$

therefore also

$$CA = AB$$

And this must also be noted that the ordinates on the circumference of the circle are in every case perpendicular to the diameter and are in a straight line with the parallels to AC (Eucl. III 3 4)

therefore also

$$\text{sq } CE > \text{sq } DB \text{ (Eucl v 14)}$$

And so also

$$CE > DB$$

And they are parallel, therefore CD produced will meet the diameter AB outside the section (I 10, Eucl I 33)

But then let it be an hyperbola. Since then in the hyperbola

$$\text{sq } CE \text{ sq } DB \text{ rect } FE, EA \text{ rect } FB, BA \text{ (I 21),}$$

therefore also

$$\text{sq } CE > \text{sq } DB$$

And they are parallel, therefore the straight line CD produced will meet the diameter of the section outside the section

PROPOSITION 23

If a straight line lying between the two (conjugate) diameters¹ cuts the ellipse, it will when produced, meet each of the diameters outside the section.

Let there be an ellipse whose diameters are the straight lines AB and CD (I 15), and let some straight line EF lying between the diameters AB and CD cut the section.

I say that the straight line EF , when produced will meet each of the straight lines AB and CD outside the section.

For let the straight lines GE and FH be dropped ordinately from E and F to AB and the straight lines EK and FL ordinately to CD . Therefore

$$\text{sq } EG \text{ sq } FH \text{ rect } BG, GA \text{ rect } BH, HA \text{ (I 21)}$$

and

$$\text{sq } FL \text{ sq } EK \text{ rect } DL, LC \text{ rect } DK, KC \text{ (I 21)}$$

And

$$\text{rect } BG, GA > \text{rect } BH, HA$$

for the point G is nearer the midpoint (Eucl VI 27 II 5), and

$$\text{rect } DL, LC > \text{rect } DK, KC,$$

therefore also

$$\text{sq } GE > \text{sq } FH$$

and

$$\text{sq } FL > \text{sq } EK,$$

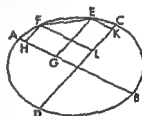
therefore also

$$GE > FH,$$

and

$$FI > EK.$$

And GE is parallel to FH and FL to EK therefore the straight line EF produced will meet each of the diameters AB and CD outside the section (I 10 Eucl I 33)



¹So far Apollonius by theorems I 6-13 III has shown for every ellipse the existence of at least one diameter and of one set of conjugate diameters but of no more. He can therefore now speak of the two diameters. Later on he will show the existence of an infinite number of such sets. The same is true of hyperbolas.

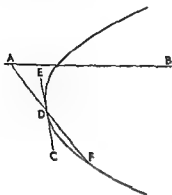
PROPOSITION 24

If a straight line, meeting a parabola or hyperbola at a point when produced both ways, falls outside the section, then it will meet the diameter

Let there be a parabola or hyperbola whose diameter is the straight line AB , and let the straight line CDE meet it at D and when produced both ways, let it fall outside the section

I say that it will meet the diameter AB

For let some point F be taken on the section, and let the straight line DF be joined therefore DF produced will meet the diameter of the section (1 22) Let it meet it at A and the straight line CDE lies between the section and the straight line FDA And therefore the line CDE produced will meet the diameter outside the section



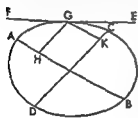
PROPOSITION 25

If a straight line, meeting an ellipse between the two (conjugate) diameters and produced both ways, falls outside the section, it will meet each of the diameters

Let there be an ellipse whose diameters are the straight lines AB and CD (1 15) and let EF , some straight line between the two diameters meet it at G , and produced both ways fall outside the section

I say that the straight line EF will meet each of the straight lines AB and CD

Let the straight lines GH and GK be dropped ordinatewise to the straight lines AB and CD respectively Since GK is parallel to AB (1 15) and some straight line GF has met GK therefore it will also meet AB Then likewise EF will also meet CD



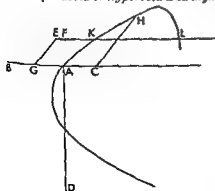
PROPOSITION 26

If in a parabola or hyperbola a straight line is drawn parallel to the diameter of the section it will meet the section in one point only

Let there first be a parabola whose diameter is the straight line ABC , and whose upright side is the straight line AD and let the straight line LE be drawn parallel to AB

I say that the straight line LE produced will meet the section

For let some point F be taken on LE and from F let the straight line IG be drawn parallel to an ordinate, and let



$$\text{rect } DA, AC > \text{sq } GE,$$

and from C let CH be erected ordinatewise (1 19) Therefore
 $\text{sq } HC = \text{rect } DA, AC$ (1 11)

But

$$\text{rect } DA, AC > \text{sq } EG,$$

therefore

$$\text{sq } HC > \text{sq } EG,$$

therefore

$$HC > EG$$

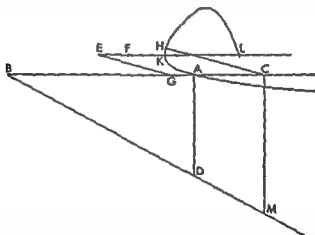
And they are parallel, therefore the straight line EF produced cuts the straight line HC , and so it will also meet the section

Let it meet it at the point K

Then I say also that it will meet it in the one point K only

For if possible let it also meet it in the point L Since then a straight line cuts a parabola in two points, if produced it will meet the diameter of the section (1 22) And this is absurd for it is supposed parallel Therefore the straight line EF produced meets the section in only one point

Next let the section be an hyperbola and the straight line AB the transverse



side of the figure and the straight line AD the upright side and let the straight line DB be joined and produced Then with the same things being constructed, let the straight line CM be drawn from C parallel to AD Since then

$$\text{rect } MC, CA > \text{rect } DA, AC,$$

and

$$\text{sq } CH = \text{rect } MC, CA,$$

and

$$\text{rect } DA, AC > \text{sq } GE,$$

therefore also

$$\text{sq } CH > \text{sq } GE$$

And so also

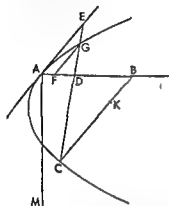
$$CH > GE$$

and the same things as in the first case will come to pass

PROPOSITION 27

If a straight line cuts the diameter of a parabola, then produced both ways it will meet the section

Let there be a parabola whose diameter is the straight line AB , and let some straight line CD cut it within the section



I say that the straight line CD produced both ways will meet the section

For let some straight line AE be drawn from A parallel to an ordinate, therefore the straight line AE will fall outside the section (I 17)

Then either the straight line CD is parallel to the straight line AE or not

If now it is parallel to it it has been dropped ordinatewise so that produced both ways it will meet the section (I 18)

Next let it not be parallel to AE but produced let it meet AE at E . Then it is evident that it meets the section the side the point E is on, for if it meets AE , *a fortiori* it cuts the section

I say that, produced the other way it also meets the section. For let the straight line MA be the parameter and the straight line GF an ordinate and let

$$\text{sq } AD = \text{rect } BA \cdot AF \text{ (Eucl vi 11)}$$

and let the straight line BK , parallel to the ordinate, meet the straight line DC at C . Since

$$\text{rect } BA \cdot AF = \text{sq } AD$$

hence

$$AB \cdot AD = AD \cdot AF$$

and therefore

$$BD \cdot DF = AB \cdot AD \text{ (Eucl v 19)}$$

Therefore also

$$\text{sq } BD + \text{sq } DF = \text{sq } AB + \text{sq } AD$$

But since

$$\text{sq } AD = \text{rect } BA \cdot AF$$

hence

$$AB \cdot AF + \text{sq } AB + \text{sq } AD = \text{sq } BD + \text{sq } FD$$

But

$$\text{sq } BD + \text{sq } DF = \text{sq } BC + \text{sq } FG$$

and

$$AB \cdot AF = \text{rect } BA \cdot AM = \text{rect } FA \cdot 4M$$

Therefore

$$\text{sq } BC + \text{sq } FG = \text{rect } BA \cdot AM + \text{rect } FA \cdot 4M$$

and alternately

$$\text{sq } BC + \text{rect } BA \cdot AM = \text{sq } FG + \text{rect } FA \cdot 4M$$

But

$$\text{sq } FG = \text{rect } FA \cdot 4M$$

because of the section (1 11) Therefore also

$$\text{sq } BC = \text{rect } BA, AM$$

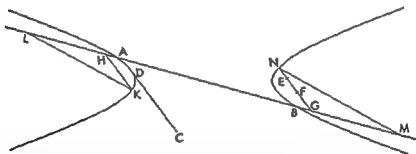
But the straight line AM is the upright side,¹ and the straight line BC is parallel to an ordinate Therefore the section passes through the point C (1 20), and the straight line CD meets the section at the point C

PROPOSITION 28

If a straight line touches one of the opposite sections, and some point is taken within the other section and through it a straight line is drawn parallel to the tangent then produced both ways it will meet the section

Let there be opposite sections whose diameter is the straight line AB , and let some straight line CD touch the section A , and let some point E be taken within the other section and through E let the straight line EF be drawn parallel to the straight line CD

I say that the straight line EF produced both ways will meet the section



Since then it has been proved that the straight line CD produced will meet the diameter AB (1 24), and EF is parallel to it therefore EF produced will meet the diameter I let it meet it at G , and let AH be made equal to GB , and through H let HK (1 18) be drawn parallel to EF , and let the straight line KL be dropped ordinately and let GM be made equal to LH , and let the straight line MN be drawn parallel to an ordinate and let GN be further produced in the same straight line And since KL is parallel to MN and KH to GN and LV is one straight line triangle KHL is similar to triangle HMN And

$$LH = GM$$

therefore

$$KL = MN$$

And so also

$$\text{sq } KL = \text{sq } MN$$

And since

$$LH = GM$$

and

$$AH = BG,$$

and AB is common therefore

$$BL = AM$$

¹The text reads $\kappa\lambda\gamma\iota\alpha$ which is impossible I have corrected to $\alpha\beta\delta\iota\alpha$

therefore

$$\text{rect } BL, LA = \text{rect } AM MB$$

Therefore

$$\text{rect } BL, LA \text{ sq } LK : \text{rect } AM MB \text{ sq } MN$$

And

$$\text{rect } BL, LA \text{ sq } LK : \text{the transverse the upright (I 21),}$$

therefore also

$$\text{rect } AM MB \text{ sq } MN \text{ the transverse the upright}$$

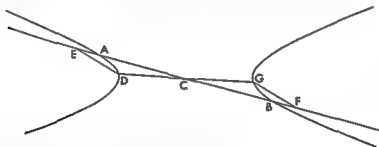
Therefore the point N is on the section. Therefore the straight line EF produced will meet the section at the point N (I 21)

Likewise then it could be shown that produced to the other side it will meet the section

PROPOSITION 29

If in opposite sections a straight line is drawn through the center to meet either of the sections, then produced it will cut the other section

Let there be opposite sections whose diameter is the straight line AB , and whose center is the point C , and let the straight line CD cut the section AD



I say that it will also cut the other section

For let the straight line ED be dropped ordinatewise and let the straight line BF be made equal to the straight line AE and let the straight line FG be drawn ordinatewise (I 19) And since

$$EA = BF$$

and AB is common therefore

$$\text{rect } BE EA = \text{rect } BF FA$$

And since

$$\text{rect } BE, EA \text{ sq } DE \text{ the transverse the upright (I 21)}$$

but also

$$\text{rect } BF FA \text{ sq } FG \text{ the transverse the upright (I 21),}$$

therefore also

$$\text{rect } BE, EA \text{ sq } DE : \text{rect } BF FA \text{ sq } FG \text{ (I 14)}$$

But

$$\text{rect } BE EA = \text{rect } BF FA$$

therefore also

$$\text{sq } DE = \text{sq } FG$$

Since then

$$EC = CF,$$

and

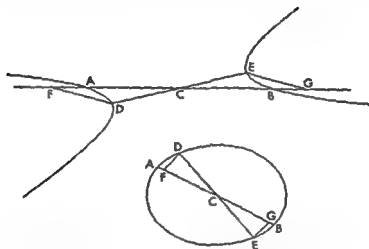
$$DF = FG,$$

and EF is a straight line, and ED is parallel to FG , therefore DG is also a straight line (Eucl vi 32) And therefore CD will also cut the other section

PROPOSITION 30

If in an ellipse or in opposite sections a straight line is drawn in both directions from the center meeting the section, it will be bisected at the center

Let there be an ellipse or opposite sections and their diameter the straight line AB , and their center C , and through C let some straight line DCE be drawn (I 29)



I say that the straight line CD is equal to the straight line CE

For let the straight lines DF and EG be drawn ordinatewise. And since
rect BF, FA sq FD the transverse the upright (I 21),
but also

rect AG, GB sq GE the transverse the upright (I 21),
therefore also

$$\text{rect } BF, FA \text{ sq } FD = \text{rect } AG, GB \text{ sq } CE \text{ (I 14)}$$

And alternately

$$\text{rect } BF, FA \text{ rect } AG, GB = \text{sq } FD \text{ sq } GE$$

But

$$\text{sq } FD \text{ sq } GE = \text{sq } FC \text{ sq } CG \text{ (Eucl vi 4),}$$

therefore alternately

$$\text{rect } BF, FA \text{ sq } FC = \text{rect } AG, GB \text{ sq } CG$$

Therefore also *componendo* in the case of the ellipse and inversely and *convertendo* in the case of the opposite sections (Eucl v Defs 14, 13, 16),

$$\text{sq } FC \text{ sq } CG = \text{sq } BC \text{ sq } CG \text{ (Eucl II 5 6),}$$

and alternately But

$$\text{sq } CB = \text{sq } AC$$

therefore also

$$\text{sq } CG = \text{sq } CF$$

Therefore

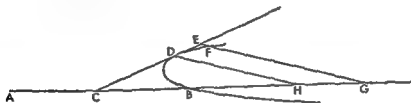
$$CG = CF$$

And the straight lines DF and GE are parallel, therefore also
 $DC = CE$

PROPOSITION 31

If on the transverse side of the figure of an hyperbola some point be taken cutting off from the vertex of the section not less than half of the transverse side of the figure and a straight line be drawn from it to meet the section, then, when further produced it will fall within the section on the near side of the section

Let there be an hyperbola whose diameter is the straight line AB and let C some point on the diameter be taken cutting off the straight line CB not less



than half of AB and let some straight line CD be drawn to meet the section

I say that the straight line CD produced will fall within the section

For if possible let it fall outside the section as the line CDF (1. 24), and from E a point at random let the straight line EG be dropped ordinatewise also DH and first let

$$AC = CB$$

And since

$$\text{sq } EG \text{ sq } DH > \text{sq } FG \text{ sq } DH \text{ (Eucl v 8),}$$

but

$$\text{sq } EG \text{ sq } DH \text{ sq } CG \text{ sq } CH$$

because of EG s being parallel to DH and

$$\text{sq } FG \text{ sq } DH \text{ rect } AG \text{ GB rect } AH \text{ HB}$$

because of the section (1. 21),

therefore

$$\text{sq } CG \text{ sq } CH > \text{rect } AG \text{ GB rect } AH \text{ HB}^1$$

Alternately therefore

$$\text{sq } CG \text{ rect } AG \text{ GB} > \text{sq } CH \text{ rect } AH, HB$$

Therefore *separando*

$$\text{sq } CB \text{ rect } AG \text{ GB} > \text{sq } CB \text{ rect } AH \text{ HB}$$

and this is impossible (Eucl v 8) Therefore the straight line CDE will not fall outside the section therefore inside And for this reason the straight line from some one of the points on the straight line AC will *a fortiori* fall inside, since it will also fall inside CD

¹The rules governing operations on inequalities in proportions are not developed by Euclid in Book V of the *Elements*. But they can be deduced on Euclid's principles.

PROPOSITION 32

If a straight line is drawn through the vertex of a section of a cone parallel to an ordinate, then it touches the section, and another straight line will not fall into the space between the conic section and this straight line

Let there be a section of a cone, first the so called parabola whose diameter is the straight line AB , and from A let the straight line AC be drawn parallel to an ordinate

Now it has been shown that it falls outside the section (1 17)

Then I say that also another straight line will not fall into the space between the straight line AC and the section

For if possible, let it fall in, as the straight line AD and let some point D be taken on it at random and let the straight line DE be dropped or dinatewise, and let the straight line AF be the parameter of the ordinates And since

$\text{sq } DE \text{ sq } EA > \text{sq } GE \text{ sq } EA$ (Eucl v 8),
and

$$\text{sq } GE = \text{rect } FA \cdot AE \text{ (1 11),}$$

therefore also

$$\text{sq } DE \text{ sq } EA > \text{rect } FA \cdot AE \text{ sq } EA$$

or

$$> FA \cdot EA$$

Let it be contrived then that

$$\text{sq } DE \text{ sq } EA = FA \cdot HA \text{ (Eucl vi 20 11),}$$

and through the point H let the straight line HLA be drawn parallel to ED
Since then

$$\text{sq } DE \text{ sq } EA = FA \cdot AH = \text{rect } FA \cdot AH \text{ sq } AH$$

and

$$\text{sq } DE \text{ sq } EA = \text{sq } KH \text{ sq } H \text{ (Eucl vi 22)}$$

and

$$\text{sq } HL = \text{rect } FA \cdot AH \text{ (1 11)}$$

therefore also

$$\text{sq } KH \text{ sq } HA = \text{sq } LH \text{ sq } HA$$

Therefore

$$KH = HL$$

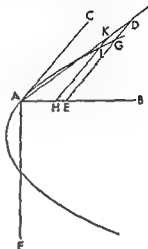
and this is absurd Therefore another straight line will not fall into the space between the straight line AC and the section

Next let the section be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line AB and whose upright side is the straight line AF and let the straight line BF be joined and produced, and from the point A let the straight line AC be drawn parallel to an ordinate

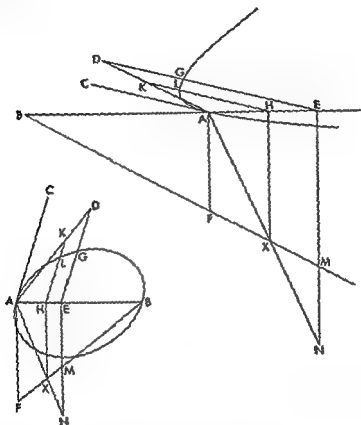
Now it has been shown that it falls outside the section (1 17)

Then I say that also another straight line will not fall into the space between the straight line AC and the section

For if possible let it fall as the straight line AD and let some point D be taken at random on it and from it let the straight line DE be dropped ordi



parallel and through E let the straight line EM be drawn parallel to the straight line AF



And since

$$\text{sq } GE = \text{rect } AE \cdot EV \quad (\text{r } 12 \text{ } 13)$$

let it be contrived that

$$\text{rect } AE \cdot EN = \text{sq } DE,$$

and let the straight line joining AN cut the straight line FM at X and through X let the straight line XH be drawn parallel to FA and through H HLA parallel to AC Since then

$$\text{sq } DE = \text{rect } AM \cdot EN,$$

hence

$$NE \cdot ED = DE \cdot EA$$

and therefore

$$NE \cdot EA = \text{sq } DE = \text{sq } EA \quad (\text{Eucl vi } 20)$$

But

$$NE \cdot EA = XH \cdot HA$$

and

$$\text{sq } DE = \text{sq } EA = \text{sq } KH = \text{sq } HA$$

Therefore

$$XH \cdot HA = \text{sq } KH = \text{sq } HA$$

therefore

$$\angle H \quad HK \quad KH \quad HA \text{ (Eucl vi 20)}$$

Therefore

$$\text{sq } \angle H = \text{rect } AH \quad HX,$$

but also

$$\text{sq } LH = \text{rect } AH, HX$$

because of the section (i 12, 13),

therefore

$$\text{sq } \angle H = \text{sq } HL$$

and this is absurd. Therefore another straight line will not fall into the space between the straight line AC and the section.

PROPOSITION 33

If in a parabola some point is taken, and from it an ordinate is dropped to the diameter and to the straight line cut off by it on the diameter from the vertex a straight line in the same straight line from its extremity is made equal then the straight line joined from the point thus resulting to the point taken will touch the section.

Let there be a parabola whose diameter is the straight line AB and let the straight line CD be dropped ordinately and let the straight line AE be made equal to the straight line ED and let the straight line AC be joined.

I say that the straight line AC produced will fall outside the section.

For if possible let it fall within as the straight line CF and let the straight line GB be dropped ordinately. And since

$$\text{sq } BG \quad \text{sq } CD > \text{sq } FB \quad \text{sq } CD$$

but

$$\text{sq } FB \quad \text{sq } CD \quad \text{sq } BA \quad \text{sq } AD$$

and

$$\text{sq } BG \quad \text{sq } CD \quad BE \quad DE \text{ (i 20),}$$

therefore

$$BE \quad DE > \text{sq } BA \quad \text{sq } AD$$

But

$$BE \quad DE \quad 4 \text{ rect } BE \quad EA \quad 4 \text{ rect } DE \quad EA$$

therefore also

$$4 \text{ rect } BE \quad EA \quad 4 \text{ rect } DE \quad EA > \text{sq } AB \quad \text{sq } AD$$

Therefore alternately

$$4 \text{ rect } BE \quad FA \quad \text{sq } AB > 4 \text{ rect } DE \quad FA \quad \text{sq } AD$$

and this is absurd for since

$$DE,$$

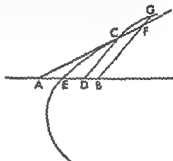
hence

But

$$\begin{array}{c} 4 \\ \text{of } A \end{array}$$

for E is not the π
 AC does not fall

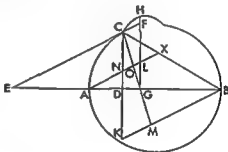
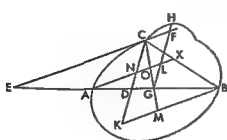
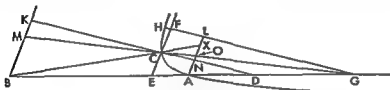
line



PROPOSITION 34

If on an hyperbola or ellipse or circumference of a circle some point is taken and from it a straight line is dropped ordinatewise to the diameter, and whatever ratio the straight lines cut off by the ordinate from the ends of the figure's transverse side have to each other, that ratio have the segments of the transverse side to each other so that the segments from the vertex are corresponding then the straight line joining the point taken on the transverse side and that taken on the section will touch the section.

Let there be an hyperbola or ellipse or circumference of a circle whose diam-



eter is the straight line AB and let some point C be taken on the section and from C let the straight line CD be drawn ordinatewise and let it be contrived that

$$BD : DA :: BE : EA^1$$

and let the straight line EC be joined

I say that the straight line CE touches the section

¹This construction is easy. In the case of the hyperbola *componendo*

$$BD + DA : DA :: BA : EA$$

and in the case of the ellipse *sepa ando*

$$BD - DA : DA :: BA : EA$$

This proportion is the same as the harmonic proportion defined by Nicomachus in his *Introduction to Arithmetic*. For if

$$BD : DA :: BE : EA$$

$$BD + DA : BD :: BA : BE$$

$$BA : BD :: BE - EA : BE$$

$$BD + DA : BA :: BA : BE - EA$$

$$DA : BD - BA :: EA : BA - BE$$

$$2BD - BA : BA :: 2BE - BA$$

then

and

Hence

But

Therefore

And so BA is the harmonic mean between BD and BE

For if possible, let it cut it, as the straight line ECF , and let some point F be taken on it, and let the straight line GFH be dropped ordinatewise and let the straight lines AL and BK be drawn through A and B parallel to the straight line EC , and let the straight lines DC , BC , and GC be joined and produced to the points M , Λ , and K . And since

$$BD \text{ DA } BE \text{ EA}$$

but

$$BD \text{ DA } BK \text{ AN},$$

and

$$BE \text{ AE } BC \text{ CY } BK \text{ YN (Eucl vi 4),}$$

therefore

$$BK \text{ AN } BK \text{ YN},$$

therefore

$$AN = NY$$

Therefore

$$\text{rect } AN \text{ NY} > \text{rect } AO \text{ OY (Eucl vi 27 or 5)}$$

Therefore

$$NY \text{ YO} > OA \text{ AN}^1$$

But

$$NY \text{ YO } AB \text{ BM (Eucl vi 4)}$$

therefore

$$AB \text{ BM} > OA \text{ AN}$$

Therefore

$$\text{rect } AB \text{ AN} > \text{rect } BM, OA$$

And so

$$\text{rect } AB \text{ AN } \text{sq } CE > \text{rect } BM \text{ OA } \text{sq } CE \text{ (Eucl v 8)}$$

But

$$\text{rect } AB \text{ AN } \text{sq } CE \text{ rect } BD \text{ DA } \text{sq } DE$$

through the similarity of the triangles BAD , ECD and NAD and

¹Eutocius commenting says For since

$$\text{rect } AN \text{ NY} > \text{rect } AO \text{ OY}$$

$$\text{let rect } AN \text{ NY} = \text{rect } AO \text{ XP}$$

where XP is some line such that

$$XP > YC$$

therefore

$$OA \text{ AN } NY \text{ YP}$$

But

$$NY \text{ YO} > NY \text{ YP (Eucl v 8)}$$

and therefore

$$NY \text{ XO} > OA \text{ AN}$$

Then the converse is also evident that if

$$NY \text{ YO} > OA \text{ AN}$$

then

$$\text{rect } NY \text{ NA} > \text{rect } AO \text{ OY}$$

For let it be that

$$OA \text{ AN } NY \text{ YP}$$

where

$$XP > YO$$

therefore

$$\text{rect } XY \text{ YA} = \text{rect } AO \text{ YP}$$

and so

$$\text{rect } YN \text{ NA} > \text{rect } AO \text{ OY}$$

¹Eutocius commenting, says Since then because AN , EC and AB are parallel

$$AN \text{ EC } AD \text{ DE}$$

and

$$EC \text{ AB } ED \text{ DB}$$

therefore ex aequali

$$AN \text{ AB } AD \text{ DB}$$

therefore also

$$\text{sq } AN \text{ rect } AN \text{ AB } \text{sq } AD \text{ rect } AD \text{ DB}$$

But

$$\text{sq } EC \text{ sq } AN$$

therefore ex aequali

$$\text{sq } EC \text{ rect } AN \text{ AB } \text{sq } ED \text{ rect } AD \text{ DB}$$

and inversely

$$\text{rect } AB \text{ YN } \text{sq } EC \text{ rect } YD \text{ DB } \text{sq } ED$$

A similar proof holds for the proportion following

rect BM, OA sq CE rect BG, GA sq GE ,

therefore

rect BD, DA sq $DE >$ rect BG, GA sq GE

Therefore alternately

rect BD, DA rect $BG, GA >$ sq DE sq GE

But

rect BD, DA rect AG, GB sq CD sq GH (r 21),

and

sq DE sq EG sq CD sq FG (Eucl vi 4),

therefore also

sq CD sq $HG >$ sq CD sq FG

Therefore

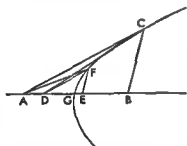
$HG < FG$ (Eucl v 10)

and this is impossible Therefore the straight line EC does not cut the section therefore it touches it

PROPOSITION 35

If a straight line touches a parabola meeting the diameter outside the section, the straight line drawn from the point of contact ordinatewise to the diameter will cut off on the diameter beginning from the vertex of the section a straight line equal to the straight line between the vertex and the tangent, and no straight line will fall into the space between the tangent and the section

Let there be a parabola whose diameter is the straight line AB , and let the straight line BC be erected ordinatewise and let the straight line AC be tangent to the section



I say that the straight line AG is equal to the straight line GB

For if possible, let it be unequal to it, and let the straight line GE be made equal to AG and let the straight line EF be erected ordinatewise and let the straight line AF be joined Therefore AF produced will meet the straight line AC (r 33) and this is impossible For two straight lines will have the same ends Therefore the straight line AG is not unequal to the straight line GB , therefore it is equal

Then I say that no straight line will fall into the space between the straight line AC and the section

For if possible let the straight line CD fall in between and let GE be made equal to GD and let the straight line EF be erected ordinatewise Therefore the straight line joined from D to F touches the section (r 33) therefore produced it will fall outside it And so it will meet DC and two straight lines will have the same ends and this is impossible Therefore a straight line will not fall into the space between the section and the straight line AC

PROPOSITION 36

If some straight line meeting the transverse side of the figure touches an hyperbola or ellipse or circumference of a circle and a straight line is dropped from the

of contact ordinatewise to the diameter then as the straight line cut off by the tangent from the end of the transverse side is to the straight line cut off by the tangent from the other end of that side so will the straight line cut off by the ordinate from the end of the side be to the straight line cut off by the ordinate from the other end of the side in such a way that the corresponding straight lines are continuous and another straight line will not fall into the space between the tangent and the section of the cone

Let there be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line AB and let the straight line CD be tangent, and let the straight line CE be dropped ordinatewise

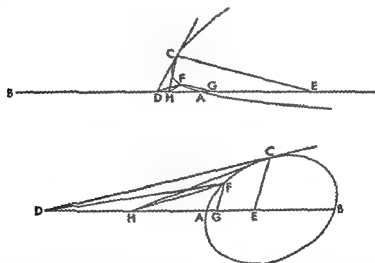
I say that

$$BE \text{ EA } BD \text{ DA}$$

For if it is not, let it be

$$BD \text{ DA } BG \text{ GA},$$

and let the straight line GF be erected ordinatewise therefore the straight line



joined from D to F will touch the section (1 31) therefore produced it will meet CD Therefore two straight lines will have the same ends, and this is impossible

I say that no straight line will fall between the section and the straight line CD

For if possible let it fall between as the straight line CH , and let it be contrived that

$$BH \text{ HA } BG \text{ GA},$$

and let the straight line CF be erected ordinatewise therefore the straight line joined from H to F when produced will meet HC (1 31) Therefore two straight lines will have the same ends and this is impossible Therefore a straight line will not fall into the space between the section and the straight line CD

PROPOSITION 37

If a straight line touching an hyperbola or ellipse or circumference of a circle meets the diameter, and from the point of contact to the diameter a straight line is dropped ordinatewise, then the straight line cut off by the ordinate from the center of the section with the straight line cut off by the tangent from the center of the section will contain an area equal to the square on the radius of the section, and with the straight line between the ordinate and the tangent will contain an area having the ratio to the square on the ordinate which the transverse has to the upright

Let there be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line AB , and let the straight line CD be drawn tangent and let the straight line CE be dropped ordinatewise, and let the point F be the center

I say that

$$\text{rect } DF, FE = \text{sq } FB,$$

and $\text{rect } DE, EF = \text{sq } EC$ the transverse the upright

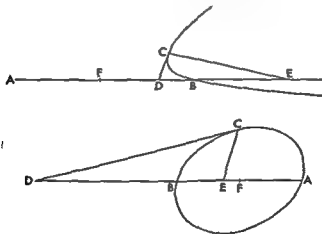
For since CD touches the section, and CE has been dropped ordinatewise hence

$$AD \cdot DB = AE \cdot EB \quad (\text{I } 36)$$

Therefore *componendo*

$$AD + DB \cdot DB = AE + EB \cdot EB$$

And let the halves of the antecedents be taken (Eucl I 15) in the case of the



hyperbola we shall say but

$$\text{half } (AE + EB) = FE,$$

and

$$\text{half } AB = FB,$$

therefore

$$FE \cdot EB = FB \cdot BD$$

Therefore *convertendo*

$$FE \cdot FB = FB \cdot FD,$$

therefore

$$\text{rect } EF, FD = \text{sq } FB$$

And since

$$FE \text{ EB } FB \text{ BD } AF \text{ BD,}$$

alternately

$$AF \text{ FE } DB \text{ BE,}$$

componendo

$$AE \text{ EF } DE \text{ EB}$$

and so

$$\text{rect } AE \text{ EB} = \text{rect } FE, ED$$

But

$\text{rect } AE \text{ EB } \text{sq } CE$ the transverse the upright (I 21),
therefore also

$$\text{rect } FE \text{ ED } \text{sq } CE \text{ the transverse the upright}$$

And in the case of the ellipse and of the circle we shall say but
half $(AD + DB) = DF$,

and

$$\text{half } AB = FB$$

therefore

$$FD \text{ DB } FB \text{ BE}$$

Therefore *convertendo*

$$DF \text{ FB } BF \text{ FE}$$

Therefore

$$\text{rect } DF \text{ FE} = \text{sq } BF$$

But

$$\text{rect } DF \text{ FE} = \text{rect } DE \text{ EF} + \text{sq } FE \text{ (Eucl II 3),}$$

and

$$\text{sq } BF = \text{rect } AE \text{ EB} + \text{sq } FE \text{ (Eucl II 5)}$$

Let the common square on CF be subtracted therefore

$$\text{rect } DE \text{ EF} = \text{rect } AE \text{ EB}$$

Therefore

$$\text{rect } DE \text{ EF } \text{sq } CE \text{ rect } AE \text{ EB } \text{sq } CE$$

But

$$\text{rect } AE \text{ EB } \text{sq } CE \text{ the transverse the upright (I 21)}$$

Therefore

$$\text{rect } DE \text{ EF } \text{sq } CE \text{ the transverse the upright}$$

PROPOSITION 38

If a straight line touching an hyperbola or ellipse or circumference of a circle meets the second diameter and from the point of contact a straight line is dropped to the same diameter parallel to the other diameter, then the straight line cut off by the dropped straight (*καταμνησ*)¹ line from the center of the section with the straight

¹When this word *καταμνησ* is used in connection with the first diameter we translate it as ordinate but we have preferred to stick more closely to the original when it is referred to the second diameter. For although it is certainly an ordinate in the case of the ellipse yet in the case of the hyperbola it is only analogically an ordinate. This analogy however becomes stronger and stronger as the treatise moves on. It is therefore no accident that *καταμνησ* is used in both cases. On the other hand in First Definitions § 5 Apollonius definitely calls both cases ordinates as if announcing the culmination of an analogy to be worked out in the course of the treatise.

line cut off by the tangent from the center of the section will contain an area equal to the square on the half of the second diameter and with the straight line between the dropped straight line and the tangent will contain an area having a ratio to the square on the dropped straight line which the upright side of the figure has to the transverse

Let there be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line AGB and whose second diameter is the straight line CGD and let the straight line ELF meeting CD at F , be a tangent to the section and let the straight line HE be parallel to AB

I say that

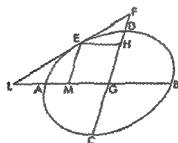
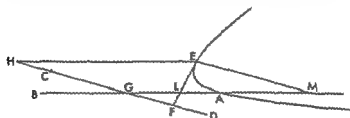
$$\text{rect } FG \cdot GH = \text{sq } GC$$

and

$$\text{rect } GH, HF \text{ sq } HE \text{ the upright the transverse}$$

Let the straight line ME be drawn ordinatewise therefore

$$\text{rect } GM \cdot ML \text{ sq } ME \text{ the transverse the upright (I 37)}$$



But

$$\text{the transverse } B1 \cdot CD \cdot CD \text{ the upright (see Def 11)}$$

and therefore

$$\text{the transverse the upright sq } BA \text{ sq } CD \text{ (Eucl vi 20)}$$

and as the quarters of them that is

$$\text{the transverse the upright sq } GA \text{ sq } GC$$

therefore also

$$\text{rect } GM \cdot ML \text{ sq } ME \text{ sq } GA \text{ sq } GC$$

But

$$\text{rect } GM \cdot ML \text{ sq } ME \text{ comp } GM \cdot ME \cdot LM \cdot MF$$

or

$$\text{rect } GM \cdot ML \text{ sq } ME \text{ comp } CM \cdot CH \cdot LM$$

Therefore inversely

$$\text{sq } CG \text{ sq } GA \text{ comp } EM \text{ MG or } HG \text{ GM } EM \text{ ML or } FG \text{ GL}$$

Therefore

$$\text{sq } GC \text{ sq } GA \text{ comp } HG \text{ GM, FG } GL,$$

which is the same as

$$\text{rect } FG \text{ GH } \text{ rect } MG \text{ GL}$$

Therefore

$$\text{rect } FG, GH \text{ rect } MG, GL \text{ sq } CG \text{ sq } GA$$

And alternately therefore

$$\text{rect } FG \text{ GH } \text{ sq } CG \text{ rect } MG \text{ GL } \text{ sq } GA$$

But

$$\text{rect } MG, GL = \text{sq } GA \text{ (1 37),}$$

therefore also

$$\text{rect } FG, GH = \text{sq } CG$$

Again since

$$\text{the upright the transverse sq } EV \text{ rect } CV \text{ VL (1 37),}$$

and

$$\text{sq } EV \text{ rect } GM \text{ ML comp } EV \text{ GM } EM \text{ ML}$$

or

$$\text{sq } EM \text{ rect } GM \text{ ML comp } HG \text{ HE } FG \text{ GL or } FH \text{ HE}$$

which is the same as

$$\text{rect } FH \text{ HG } \text{ sq } HE$$

therefore

$$\text{rect } FH \text{ HG } \text{ sq } HE \text{ the upright the transverse}$$

With the same things supposed it remains to be shown that as the straight line between the tangent and the end of the (second) diameter on the same side with the dropped straight line is to the straight line between the tangent and the second diameter so is the straight line between the other end and the dropped straight line to the straight line between the first end and the dropped straight line

For since

$$\text{rect } FG \text{ GH} = \text{sq } GC = \text{rect } CG \text{ GD (2 para above),}$$

for

$$CG = GD$$

therefore

$$\text{rect } FG \text{ GH} = \text{rect } CG \text{ GD}$$

therefore

$$FG \text{ GD } CG \text{ GH}$$

And *convertendo*

$$GF \text{ FD } GC \text{ CH}$$

And let the doubles of the antecedents be taken, but

$$2GF = CF + FD$$

because

$$CG = GD$$

and

$$2GC = CD$$

therefore

$$CF + FD \text{ FD } DC \text{ CH}$$

And *separando*

CF FD DH HC,

and this was to be shown

Then it is clear from what has been said that the straight line *EF* touches the section either if

$$\text{rect } FG, GH = \text{sq } GC,$$

or if

$$\text{rect } FH HG = \text{sq } GC$$

in the ratio we said, for it could be shown conversely

PROPOSITION 39

If a straight line touching an hyperbola or ellipse or circumference of a circle meets the diameter, and from the point of contact a straight line is dropped ordinatewise to the diameter, then whichever of the two straight lines is taken of which one is the straight line between the ordinate and the center of the section and the other is between the ordinate and the tangent then the ordinate will have to it the ratio compounded of the ratio of the other of the two straight lines to the ordinate and of the ratio of the upright side of the figure to the transverse

Let there be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line *AB* and let the center of it be the point *F*, and let the straight line *CD* be drawn tangent to the section, and the straight line *CE* be dropped ordinatewise

I say that

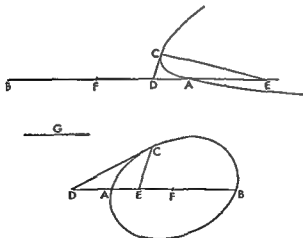
$$CE \text{ } FE \text{ comp the upright the transverse } ED \text{ } EC,$$

and

$$CE \text{ } ED \text{ comp the upright the transverse } FE \text{ } EC$$

For let

$$\text{rect } FE \text{ } ED = \text{rect } EC \text{ } G$$



And since

$$\text{rect } FE \text{ } ED = \text{sq } CE \text{ the transverse the upright (1 37),}$$

and

$$\text{rect } FE \text{ } ED = \text{rect } CE, G,$$

therefore

$$\text{rect } CE \text{ } G \text{ sq } CE \text{ } G \text{ } CE \text{ the transverse the upright}$$

And since

$$\text{rect } FE \text{ } ED = \text{rect } CE, G,$$

hence

$$FE \text{ } EC \text{ } G \text{ } ED$$

And since

$$CE \text{ } ED \text{ comp } CE \text{ } G, G \text{ } ED,$$

but

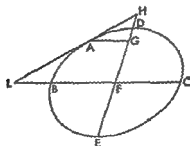
$$CE \text{ } G \text{ the upright the transverse}$$

therefore

$$CE \text{ } ED \text{ comp the upright the transverse, } FE \text{ } EC$$

PROPOSITION 40

If a straight line touching an hyperbola or ellipse or circumference of a circle meets the second diameter, and from the point of contact a straight line is dropped to the same diameter parallel to the other diameter, then whichever of the two straight lines is taken of which one is the straight line between the dropped straight line and the center of the section and the other is between the dropped straight line and the



tangent the dropped straight line will have to it the ratio compounded of the ratio of the transverse side to the upright and of the ratio of the other of the two straight lines to the dropped straight line

Let there be an hyperbola or ellipse or circumference of a circle \$AB\$ and its diameter the straight line \$BFC\$ and its second diameter the straight line \$DFE\$ and let the straight line \$HLA\$ be drawn tangent and the straight line \$AG\$ parallel to the straight line \$BC\$

I say that

AG HG comp the transverse the upright FG GA ,
and

AG FG comp the transverse the upright, HG GA
Let

$$\text{rect } GA \cdot K = \text{rect } HG \cdot GF$$

And since

the upright the transverse $\text{rect } HG, GF$ sq GA (1 38),
and

$$\text{rect } GA, K = \text{rect } HG, GF,$$

therefore also

$\text{rect } GA, K$ sq GA K AG the upright the transverse
And since

$$AG \cdot GF \text{ comp } AG \cdot K \cdot K \cdot GF,$$

but

$$AG \cdot K \cdot \text{the transverse the upright,}$$

and

$$K \cdot GF \cdot HG \cdot GA$$

because

$$\text{rect } HG \cdot GF = \text{rect } AG \cdot K$$

therefore

$$AG \cdot GF \text{ comp the transverse the upright } GH \cdot GA$$

PROPOSITION 41

If in an hyperbola or ellipse or circumference of a circle a straight line is dropped ordinatewise to the diameter and equiangular parallelogrammic figures are described both on the ordinate and on the radius and the ordinate side has to the remaining side of the figure the ratio compounded of the ratio of the radius to the remaining side of its figure, and of the ratio of the upright side of the section's figure to the transverse then the figure on the straight line between the center and the ordinate similar to the figure on the radius is in the case of the hyperbola greater than the figure on the ordinate by the figure on the radius and, in the case of the ellipse and circumference of a circle, together with the figure on the ordinate is equal to the figure on the radius

Let there be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line AB and center the point E and let the straight line CD be dropped ordinatewise and on the straight lines EA and CD let the equiangular figures AF and DG be described and let

$$CD \cdot CG \text{ comp } AE \cdot EF \text{ the upright the transverse}$$

I say that with the figure on ED similar to AF , in the base of the hyperbola
figure on $ED = AF + GD$

and in the case of the ellipse and circle

$$\text{figure on } ED + GD = AF$$

For let it be contrived that

$$\text{the upright the transverse } DC \cdot CH$$

And since

$$DC \cdot CH \text{ the upright the transverse}$$

but

$$DC \cdot CH \text{ sq } DC \text{ rect } DC \cdot CH$$

therefore

$$\text{rect } HC, CD \text{ rect } GC \text{ } CD \text{ sq } AE \text{ rect } AE \text{ } EF$$

And it has been shown that

$$\text{rect } HC, CD = \text{rect } BD \text{ } DA,$$

therefore

$$\text{rect } BD, DA \text{ rect } GC \text{ } CD \text{ sq } AE \text{ rect } AE \text{ } EF$$

Alternately

$$\text{rect } BD, DA \text{ sq } AE \text{ rect } GC \text{ } CD \text{ rect } AE \text{ } EF$$

And

$$\text{rect } GC, CD \text{ rect } AE \text{ } EF \text{ pll } g \text{ } DG \text{ pll } g \text{ } F4$$

for they are equiangular and have to one another the ratio compounded of their sides GC AE and CD EF (Eucl vi 23) and therefore

$$\text{rect } BD \text{ } DA \text{ sq } EA \text{ pll } g \text{ } DG \text{ pll } g \text{ } FA$$

Moreover in the case of the hyperbola we are to say *componendo*

$$\text{rect } BD \text{ } DA + \text{sq } AE \text{ sq } AE \text{ pll } g \text{ } GD + \text{pll } g \text{ } AF \text{ pll } g \text{ } 4F,$$

or

$$\text{sq } DE \text{ sq } EA \text{ pll } g \text{ } GD + \text{pll } g \text{ } 4F \text{ pll } g \text{ } AF \text{ (Eucl ii 6)}$$

And as the square on DE is to the square on EA so is the figure described on ED similar and similarly situated to the parallelogram AF , to the parallelogram AF (Eucl vi 20 porism) therefore, with the figure on ED similar to the parallelogram AF ,

$$\text{pll } g \text{ } GD + \text{pll } g \text{ } AF \text{ pll } g \text{ } AF \text{ figure on } ED \text{ pll } g \text{ } AF$$

Therefore

$$\text{figure on } ED = \text{pll } g \text{ } GD + \text{pll } g \text{ } AF$$

the figure on ED being similar to the parallelogram AF

And in the case of the ellipse and of the circumference of a circle we shall say since then

$$\text{whole sq } AE \text{ whole pll } g \text{ } AF$$

$$\text{rect } AD \text{ } DB \text{ subtracted pll } g \text{ } DG \text{ subtracted,}$$

also remainder is to remainder as whole to whole (Eucl v 19)

And

$$\text{sq } AE - \text{rect } BD \text{ } DA = \text{sq } DE \text{ (Eucl ii 5),}$$

therefore

$$\text{sq } DE \text{ pll } g \text{ } AF - \text{pll } g \text{ } DG \text{ sq } AE \text{ pll } g \text{ } AF$$

But

$$\text{sq } AE \text{ pll } g \text{ } 4F \text{ sq } DE \text{ figure on } DE \text{ (Eucl vi 20 porism)}$$

the figure on DE being similar to the parallelogram AF Therefore the figure on DE being similar to the parallelogram AF

$$\text{sq } DE \text{ pll } g \text{ } AF - DG \text{ sq } DE \text{ figure on } DE$$

Therefore the figure on DE being similar to the parallelogram AF ,

$$\text{figure on } DE = \text{pll } g \text{ } AF - \text{pll } g \text{ } DG$$

Therefore

$$\text{figure on } DE + \text{pll } g \text{ } DG = \text{pll } g \text{ } AF$$

PROPOSITION 42

If a straight line touching a parabola meets the diameter, and from the point of contact a straight line is dropped ordinately to the diameter and, some point being taken on the section two straight lines are dropped to the diameter, one of them parallel to the tangent and the other parallel to the straight line dropped from

the point of contact then the triangle resulting from them is equal to the parallelogram contained by the straight line dropped from the point of contact and by the straight line cut off by the parallel from the vertex of the section

Let there be a parabola, whose diameter is the straight line AB and let the straight line AC be drawn tangent to the section, and let the straight line CH be dropped ordinatewise and from some point at random let the straight line DF be dropped ordinatewise and through the point D let the straight line DE be drawn parallel to the straight line AC , and through the point C the straight line CG parallel to the straight line BF and through the point B the straight line BG parallel to the straight line HC

I say that

$$\text{trgl } DEF = \text{pllg } GF$$

For since the straight line AC touches the section and the straight line CH has been dropped ordinatewise

$$AB = BH \text{ (I 35),}$$

therefore

$$AH = 2BH$$

Therefore

$$\text{trgl } AHC = \text{pllg } BC \text{ (Eucl I 41)}$$

And since

$$\text{sq } CH \text{ sq } DF \text{ } HB \text{ } BF$$

because of the section (I 20) but

$$\text{sq } CH \text{ sq } DF \text{ } \text{trgl } ACH \text{ } \text{trgl } EDF \text{ (Eucl VI 19)}$$

and

$$HB \text{ } BF \text{ } \text{pllg } GH \text{ } \text{pllg } GF \text{ (Eucl VI 1)}$$

therefore

$$\text{trgl } ACH \text{ } \text{trgl } EDF \text{ } \text{pllg } HG \text{ } \text{pllg } FG$$

Therefore alternately

$$\text{trgl } AHC \text{ } \text{pllg } BC \text{ } \text{trgl } EDF \text{ } \text{pllg } GF$$

But

$$\text{trgl } ACH = \text{pllg } GH$$

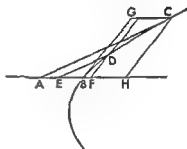
therefore

$$\text{trgl } EDF = \text{pllg } GF$$

PROPOSITION 43

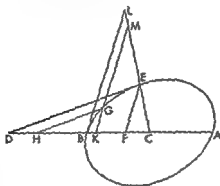
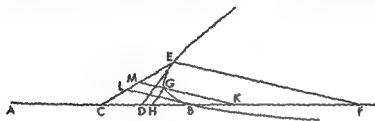
If a straight line touching an hyperbola or ellipse or circumference of a circle meets the diameter and from the point of contact a straight line is dropped ordinatewise to the diameter and is parallel to it is drawn through the vertex meeting the straight line drawn through the point of contact and the center, and some point being taken on the section two straight lines are drawn to the diameter one of which is parallel to the tangent and the other parallel to the straight line dropped from the point of contact then the triangle resulting from them in the case of the hyperbola will be less than the triangle the straight line through the center and the point of contact cuts off, by the triangle on the radius similar to the triangle cut off and in the case of the ellipse and the circumference of the circle together with the triangle cut off from the center, will be equal to the triangle on the radius similar to the triangle cut off

Let there be an hyperbola or ellipse or circumference of a circle whose diam



eter = the straight line AB , and center the point C , and let the straight line DE be drawn tangent to the section, and let the straight line CE be joined and let the straight line EF be dropped ordinatewise, and let some point G be taken on the section and let the straight line GH be drawn parallel to the tangent, and let the straight line GK be dropped ordinatewise, and through B let the straight line BL be erected ordinatewise

I say that triangle AMC differs from triangle CLB by triangle GKH



For since the straight line ED touches and the straight line EF has been dropped hence

$EF \cdot FD$ comp $CF \cdot FE$ the upright the transverse (I 39)

But

$EF \cdot FD$ $GK \cdot KH$

and

$CF \cdot FE$ $CB \cdot BL$ (Eucl VI 4)

therefore

$GK \cdot KH$ comp $BC \cdot BL$ the upright the transverse

And through those things shown in the forty first theorem (I 41), triangle CKM differs from triangle BCL by triangle GKH for the same things have also been shown in the case of the parallelograms their doubles

PROPOSITION 44

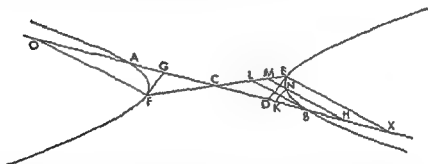
If a straight line touching one of the opposite sections meets the diameter, and from the point of contact some straight line is dropped ordinatewise to the diameter and a parallel to it is drawn through the vertex of the other section meeting the straight line drawn through the point of contact and the center and some point being taken at random on the section let straight lines be dropped to the diameter, one of which

is parallel to the tangent and the other parallel to the straight line dropped ordinatewise from the point of contact then the triangle resulting from them will be less than the triangle the dropped straight line cuts off from the center of the section, by the triangle on the radius similar to the triangle cut off

Let there be the opposite sections AF and BE , and let their diameter be the straight line AB and center the point C , and from some point F of those on the section FA let the straight line FO be drawn tangent to the section, and the straight line FO ordinatewise, and let the straight line CF be joined and produced, as CE (1 29), and through B let the straight line BL be drawn parallel to the straight line FO and let some point N be taken on the section BE , and from N let the straight line NH be dropped ordinatewise, and let the straight line NA be drawn parallel to the straight line FO

I say that

$$\text{trgl } HAN + \text{trgl } CBL = \text{trgl } CMH$$



For through E let the straight line ED be drawn tangent to the section BE , and let the straight line $E\lambda$ be drawn ordinatewise. Since then FA and BE are opposite sections whose diameter is AB and the straight line through whose center is CE and FG and ED are tangents to the section hence DE is parallel to FG ¹ And the straight line NA is parallel to FO , therefore NA is also parallel to ED , and the straight line MH to BL . Since then BE is an hyperbola, whose diameter is the straight line AB , and whose center is C and the straight line DE is tangent to the section, and $E\lambda$ drawn ordinatewise and BL is parallel to $E\lambda$ and N has been taken on the section as the point from which NH has

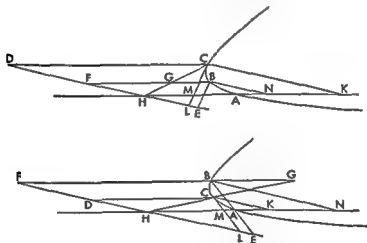
¹Futocius commenting says: For since AF is an hyperbola and DE a tangent and FO an ordinate
 rect $OC \cdot CC = \text{sq } CA$
 (1 37) likewise then also rect $\lambda C \cdot CD = \text{sq } CB$
 Therefore rect $OC \cdot CC = \text{sq } AC$ rect $\lambda C \cdot CD = \text{sq } BC$
 and alternately rect $OC \cdot CC = \text{rect } \lambda C \cdot CD = \text{sq } AC = \text{sq } CB$
 But sq $AC = \text{sq } CB$
 therefore also rect $OC \cdot CG = \text{rect } \lambda C \cdot CD$
 And $OC = C\lambda$ (1 14 30)
 and therefore $GC = CD$
 and also $FC = CE$ (1 30)
 therefore $FC = FC$ $CG = CD$
 And they contain equal angles at the point C for they are vertical. And so also $FG = FD$
 and angle $CFG = \text{angle } CFD$
 And they are alternate therefore the straight line FG is parallel to the straight line ED

been dropped ordinatewise and AN has been drawn parallel to DE , therefore
 $\text{trgl } NHA + \text{trgl } BCL = \text{trgl } HMC$,
 for this has been shown in the forty third theorem (I 43)

PROPOSITION 45

If a straight line touching an hyperbola or ellipse or circumference of a circle meets the second diameter, and from the point of contact some straight line is dropped to the same diameter parallel to the other diameter, and through the point of contact and the center a straight line is produced, and some point being taken at random on the section two straight lines are drawn to the second diameter one of which is parallel to the tangent and the other parallel to the dropped straight line, then the triangle resulting from them is greater in the case of the hyperbola, than the triangle the dropped straight line cuts off from the center, by the triangle whose base is the tangent and vertex is the center of the section, and in the case of the ellipse and circle, together with the triangle cut off will be equal to the triangle whose base is the tangent and whose vertex is the center of the section

Let there be an hyperbola or ellipse or circumference of a circle ABC , whose



diameter is the straight line AH and second diameter HD , and center H and let the straight line CML touch it at C , and let the straight line CD be drawn parallel to AH and let the straight line HC be joined and produced and let some point B be taken at random on the section, and from B let the straight lines BE and BF be drawn from B parallel to the straight lines LC and CD

I say that in the case of the hyperbola

$$\text{trgl } BEF = \text{trgl } GHF + \text{trgl } LCH,$$

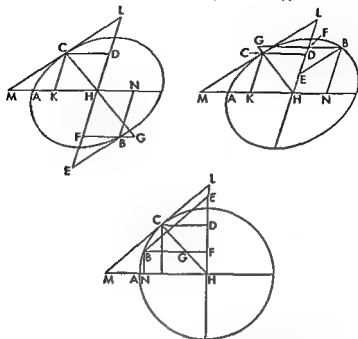
and in the case of the ellipse and circle

$$\text{trgl } BEF + \text{trgl } FGH = \text{trgl } CLH$$

For let the straight lines CK and BA be drawn parallel to DH Since then the straight line CM = tangent and the straight line CK has been dropped ordinatewise hence

CK AH comp MA AC the upright the transverse (I 39)

$MK \ AC \ CD \ DL$ (Eucl vi 4),



therefore

$CK \ KH$ comp $CD \ DL$ the upright the transverse
And triangle CDL is the figure on AH and triangle CAH , that is triangle CDH is the figure on CA that is on DH therefore, in the case of the hyperbola,

$\text{trgl } CDL = \text{trgl } CAH + \text{trgl on } AH$ similar to $\text{trgl } CDL$,
and in the case of the ellipse and the circle

$\text{trgl } CDH + \text{trgl } CDL = \text{trgl on } AH$ similar to $\text{trgl } CDL$,
for this was also shown in the case of their doubles in the forty first theorem (i 41)

Since then triangle CDL differs either from triangle CAH or from triangle CDH by the triangle on AH similar to triangle CDL , and it also differs by triangle CHL therefore

$\text{trgl } CHL = \text{trgl on } AH$ similar to $\text{trgl } CDL$

Since then triangle BFE is similar to triangle CDL , and triangle GFH to triangle CDH therefore they have the same ratio¹ And triangle BFE is described on NH between the ordinate and the center and triangle GFH on the ordinate BN , that is on FH and by things already shown (i 41) triangle BFE differs from triangle GFH by the triangle on AH similar to CDL , and so also by triangle CHL

¹That is (Eucl vi 4)

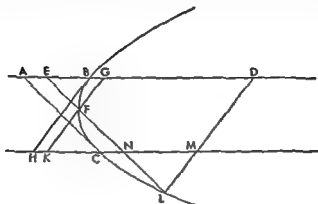
$$\frac{BF}{GF} = \frac{FE}{FH} = \frac{CD}{CD} = \frac{DL}{DH} = \frac{CK}{CK} = \frac{AH}{AH}$$

and
Therefore these first ratios can be substituted in the central proportion of the theorem
 $CK \ AH$ comp $CD \ DL$ the upright the transverse
and so satisfy i 41

PROPOSITION 46

If a straight line touching a parabola meets the diameter, the straight line drawn through the point of contact parallel to the diameter in the direction of the section bisects the straight lines drawn in the section parallel to the tangent.

Let there be a parabola whose diameter is the straight line ABD , and let the



straight line AC touch the section (1 24) and through C let the straight line HCM be drawn parallel to the straight line AD (1 26), and let some point L be taken at random on the section, and let the straight line LNF (1 18, 22) be drawn parallel to AC .

I say that

$$LN = NF$$

Let the straight lines BH , AFG , and LMD be drawn ordinatewise. Since then by the things already shown in the forty-second theorem (1 42)

$$\text{trgl } ELD = \text{pllg } BM,$$

and

$$\text{trgl } EFG = \text{pllg } BK$$

therefore the remainders

$$\text{pllg } GM = \text{quadr } LFGD$$

Let the common pentagon $MDGFN$ be subtracted therefore the remainders

$$\text{trgl } AFN = \text{trgl } LMN$$

And AF is parallel to LM therefore

$$FN = LN \text{ (Eucl vi 22 lemma)}$$

PROPOSITION 47

If a straight line touching an hyperbola or ellipse or circumference of a circle meets the diameter and through the point of contact and the center a straight line is drawn in the direction of the section it bisects the straight lines drawn in the section parallel to the tangent.

Let there be an hyperbola or ellipse or circumference of a circle whose center is the straight line AB and center C and let the straight line DE tangent to the section and let the straight line CE be joined,

let a point N be taken at random on the section, and through N let the straight line $HNOG$ be drawn parallel

I say that

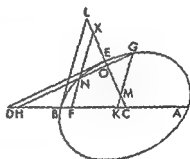
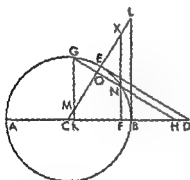
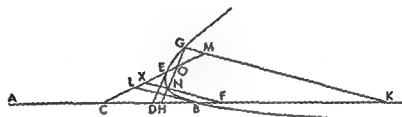
$$NO = OG$$

For let the straight lines ANF , BL and GMA be dropped ordinates to the diameter. Therefore by things already shown in the forty third theorem (1 43)

$$\text{trgl } HNF = \text{quadr } LBF\lambda,$$

and

$$\text{trgl } GHK = \text{quadr } LBK\lambda$$



Therefore the remainders

$$\text{quadr } NGKF = \text{quadr } MKF\lambda,$$

Let the common pentagon $ONFAM$ be subtracted

therefore the remainders

$$\text{trgl } OMG = \text{trgl } NXO$$

And the straight line MG is parallel to the straight line NX , therefore

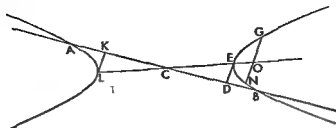
$$NO = OG \text{ (Eucl 11 22, lemma)}$$

PROPOSITION 48

If a straight line touching one of the opposite sections meets the diameter, and through the point of contact and the center a straight line produced cuts the other section then whatever line is drawn in the other section parallel to the tangent, will be bisected by the straight line produced

Let there be opposite sections whose diameter is the straight line AB and center C and let the straight line KI touch the section A and let the straight line LC be joined and produced (1 29), and let some point V be taken on the

section B , and through N let the straight line NG be drawn parallel to the straight line LA .



I say that

$$NO = OG$$

For let the straight line ED be drawn through E tangent to the section therefore ED is parallel to LA (I 44 note) And so also to NG . Since then BNG is an hyperbola whose center is C and tangent DE and since CE has been joined and a point N has been taken on the section and through it NG has been drawn parallel to DE , by a theorem already shown (I 47) for the hyperbola

$$NO = OG$$

PROPOSITION 49

If a straight line touching a parabola meets the diameter and through the point of contact is parallel to the diameter is drawn, and from the vertex a straight line is drawn parallel to an ordinate and it is contrived that as the segment of the tangent between the erected straight line and the point of contact is to the segment of the parallel between the point of contact and the erected straight line so is some straight line to the double of the tangent, then whatever straight line is drawn [parallel to the tangent] from the section to the straight line drawn through the point of contact parallel to the diameter will equal in square the rectangle contained by the straight line found and by the straight line cut off by it from the point of contact

Let there be a parabola whose diameter is the straight line MBC and CD its tangent and through D let the straight line FDN be drawn parallel to the straight line BC , and let the straight line FB be erected ordinatewise (I 17),

and let it be contrived that

$ED : DF$ some straight line

$$G : 2CD$$

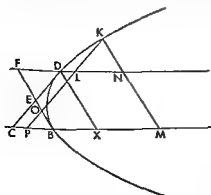
and let some point K be taken on the section and let the straight line KLP be drawn through K parallel to CD

I say that

$$\text{sq } KL = \text{rect } GDL$$

that is that with the straight line DL as diameter the straight line G is the upright side

For let the straight lines DY and KNI be dropped ordinate wise And since the straight line



CD touches the section, and the straight line $D\lambda$ has been dropped ordinate-wise, then

$$CB = B\lambda \quad (1 \ 35)$$

But

$$B\lambda = FD$$

and therefore

$$CB = FD$$

And so also

$$\text{trgl } ECB = \text{trgl } EFD$$

Let the common figure $DEBMN$ be added, therefore

$$\begin{aligned} \text{quadr } DCMN &= \text{pllg } FM \\ &= \text{trgl } APM \quad (1 \ 42) \end{aligned}$$

Let the common quadrilateral $LPMN$ be subtracted therefore the remainders

$$\text{trgl } ALN = \text{pllg } LC$$

And

$$\text{angle } DLP = \text{angle } ALN,$$

therefore

$$\text{rect } AL \ LN = 2 \text{ rect } LD, DC^1$$

And since

$$ED \ DF \ G \ 2CD$$

and

$$ED \ DF \ AL \ LN,$$

therefore also

$$G \ 2CD \ AL \ LN$$

But

$$AL \ LN \text{ sq } AL \text{ rect } AL \ LN,$$

and

$$G \ 2CD \text{ rect } G \ DL \ 2 \text{ rect } LD \ DC,$$

therefore

¹Eutocius commenting says For let the triangle AIN and the parallelogram $DLPC$ be set out And since

let the straight line NR be drawn through N parallel to LA and through A AR parallel to LN therefore LR is a parallelogram and

and so also

Then let the straight lines DC and LP be produced to S and T and let CS be made equal to DC and PT to LP and let ST be joined therefore

$$\begin{aligned} \text{pllg } DT &= 2 \text{ pllg } DP \\ \text{pllg } LR &= \text{pllg } LS \end{aligned}$$

But it is also equiangular with it because of the angles at L being vertical but in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional therefore

and

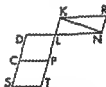
And since

hence

And if DC is parallel to LP and CP is not parallel to LD it is clear $DCPL$ is a trapezoid and so I say that

For if LP is filled out as we have said before and the straight lines DC and LP are produced and CS is made equal to LP and PT to DC and the straight line ST is joined then

and the same demonstration will fit And this will be useful in what follows (1 50)



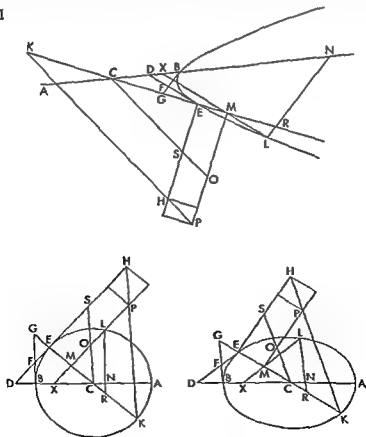
sq KL rect KL, LN rect $G DL$ 2rect $CD DL$
 And alternately but
 rect $KL LN = 2\text{rect } CD DL$,
 therefore a.l.o

$$\text{sq } KL = \text{rect } G, DL$$

PROPOSITION 50

If a straight line touching an hyperbola or ellipse or circumference of a circle meets the diameter, and a straight line is produced through the point of contact and the center and from the vertex a straight line erected parallel to an ordinate meets the straight line drawn through the point of contact and the center, and if it is con-
 trued that as the segment of the tangent between the point of contact and the straight line erected is to the segment of the straight line, drawn through the point of contact and the center, between the point of contact and the straight line erected, so some straight line is to the double of the tangent, then whatever straight line is drawn from the section to the straight line drawn through the point of contact and the center parallel to the tangent will equal in square a rectangular area applied to the straight line found having as breadth the straight line cut off by it from the

CASES I

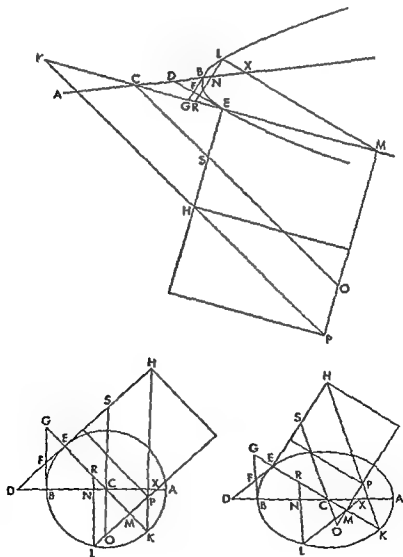


point of contact, and exceeding, in the case of the hyperbola, by a figure similar to the rectangle contained by the double of the straight line between the center and the point of contact and by the straight line found but in the case of the ellipse and circle, defectue by it

Let there be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line AB , and center C , and let the straight line DE be a tangent, and let the straight line CE be joined and produced both ways, and let the straight line CA be made equal to the straight line EC , and through B let the straight line BFG be erected ordinally and through E let the straight line EH be drawn perpendicular to FC and let it be that

$$FF' EG EH 2FD,$$

CASES II



and let the straight line HA be joined and produced and let some point L be taken on the section and through it let the straight line LMN be drawn parallel to ED , and the straight line LRN parallel to BG , and the straight line MP parallel to EH

I say that

$$\text{sq } LM = \text{rect } EM \ MP$$

For let the straight line CSO be drawn through C parallel to AP And since $EC = CA$,

and

Cases I

$$EC \ AC \ ES \ SH$$

therefore also

$$ES = SH$$

Cases II

And since

$$FE \ EG \ HE \ 2ED,$$

and

$$2ES = EH,$$

therefore also

$$FE \ EG \ SE \ ED$$

And

$$FE \ EG \ LM \ MR$$

therefore

$$LM \ MR \ SE \ ED$$

And since it was shown (I 43) that, in the case of the hyperbola,

$$\begin{aligned} \text{trgl } RNC &= \text{trgl } LNX + \text{trgl } GBC, \\ &= \text{trgl } LNY + \text{trgl } CDE^1 \end{aligned}$$

and, in the case of the ellipse and circle

$$\begin{aligned} \text{trgl } RAC + \text{trgl } LNY &= \text{trgl } GBC \\ &= \text{trgl } CDE \end{aligned}$$

therefore in the case of the hyperbola with the common triangle ECD and the common quadrilateral $NRMN$ subtracted and in the case of the ellipse and circle with the common triangle MAC subtracted ²

¹That

$$\text{trgl } GBC = \text{trgl } CDE$$

is proved by Apollonius in the course of another proof of I 43 reported by Eutocius It is also proved in III 1 without the help of intervening propositions

²The position of point L furnishes different cases which at times as in the present theorem require a change in the course of the proof The figures marked Cases I are drawn to fit the proof as set down but we have added figures marked Cases II as an example of the possible differences

For the hyperbola of Case II instead of the subtraction in the theorem above we have

$$\begin{aligned} \text{trgl } RVC &= \text{trgl } LNY + \text{trgl } CDE \\ \text{quadr } MCNL &= \text{quadr } MCNL \end{aligned}$$

Subtracting the first equals from the second identity we have

$$\text{trgl } LVR = \text{quadr } MEDY$$

The rest of the proof is the same

For the ellipse and circle of Case II we have as in the theorem above

$$\text{trgl } PAC + \text{trgl } LNY = \text{trgl } CDE$$

and subtracting the common triangle CUY

$$\text{trgl } LVP = \text{trgl } CDE - \text{trgl } CUY$$

therefore

$$\text{rect } LM \ MR = \text{rect } EM \ LD + MY$$

For let CM be made equal to CV and CA to CA Then

trgl $LMR = \text{quadr } MED\lambda$

And $M\lambda$ is parallel to DE , and

angle $LMR = \text{angle } EM\lambda$,

therefore

rect $LM, MR = \text{rect } EM ED + M\lambda$ (1.49 note, para 2)

And since

$MC CE M\lambda ED$

and

$MC CE MO FS$,

therefore

$MO ES MX ED$

And componendo

$MO + ES ES MX + ED ED$,

alternately

$MO + ES MX + ED ES ED$

But

$MO + ES MX + ED \text{ rect } MO + ES EM \text{ rect } M\lambda + ED EM$,

and

$ES ED EM MR FE EG$ (Eucl vi 4)

or

$ES ED \text{ sq } LM \text{ rect } LM, MR$

therefore

rect $MO + ES ME \text{ rect } M\lambda + ED EM \text{ sq } LM \text{ rect } LM, MR$

And alternately

rect $MO + ES ME \text{ sq } LM \text{ rect } M\lambda + ED EM \text{ rect } LM, MR$

But

rect $LM MR = \text{rect } ME M\lambda + ED$ (above),

therefore

$\text{sq } LM = \text{rect } EM MO + ES$

And

$SE = SH$

and

$SH = OP$,

therefore

$\text{sq } LM = \text{rect } EM MP$

PROPOSITION 51

If a straight line touching either of the opposite sections meets the diameter and through the point of contact and the center some straight line is produced to the other section and from the vertex a straight line is erected parallel to an ordinate and meets the straight line drawn through the point of contact and the center and if it is contrived that as the segment of the tangent between the straight line erected

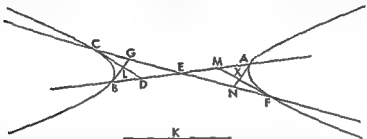
trgl $CDF - \text{trgl } CM\lambda = \text{quadr } MFDX$
and $M\lambda$ is parallel to FD and
angle $F M \lambda = \text{angle } R M I$

These cases will come up again in Book III and in general it is convenient to think of quadrilateral $MFDX$ as standing for the difference of the two triangles when one pair of its sides cross each other



and the point of contact is to the segment of the straight line, drawn through the point of contact and the center, between the point of contact and the straight line erected, so is some straight line to the double of the tangent, then whatever straight line in the other of the sections is drawn to the straight line through the point of contact and the center, parallel to the tangent, will equal in square the rectangle applied to the straight line found having as breadth the straight line cut off by it from the point of contact and exceeding by a figure similar to the rectangle contained by the straight line between the opposite sections and the straight line found

Let there be opposite sections whose diameter is the straight line AB and



center E , and let the straight line CD be drawn tangent to the section B and the straight line CE joined and produced (1 29) and let the straight line BLG be drawn ordinatewise (1 17) and let it be contrived that

$$LC \cdot CG \text{ some straight line } K \cdot 2CD$$

Now it is evident that the straight lines in the section BC , parallel to CD and drawn to EC produced are equal in square to the areas applied to K having as breadths the straight line cut off by them from the point of contact and exceeding by a figure similar to the rectangle $CF \cdot K$, for

$$GC = 2CE$$

I say then that in section FA the same thing will come about

For let the straight line MF be drawn through F tangent to the section AF , and let the straight line AN be erected ordinatewise. And since BC and AF are opposite sections and CD and MF are tangents to them therefore CD is equal and parallel to MF (1 44 note). But also

$$CE = EF$$

therefore also

$$ED = EV$$

And since

$$LC \cdot CG \text{ } K \cdot 2CD \text{ or } 2MF$$

therefore also

$$AF \cdot FN \text{ } K \cdot 2MF$$

Since then AF is an hyperbola whose diameter is AB and tangent MF , and AN has been drawn ordinatewise and

$$AF \cdot FN \text{ } K \cdot 2FM,$$

hence any lines drawn from the section to EF produced parallel to FM will equal in square the rectangle contained by the straight line K and the line cut off by them from F exceeding by a figure similar to the rectangle $CF \cdot K$ (1 50)

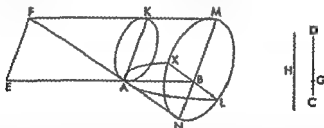
And with these things shown it is at once evident that in the parabola each of the straight lines drawn off parallel to the original diameter is a diameter

(I 46), but in the hyperbola and ellipse and opposite sections each of the straight lines drawn through the center is a diameter (I 47-48), and that in the parabola the straight lines dropped to each of the diameters parallel to the tangents will equal in square the rectangles applied to it (I 49), but in the hyperbola and opposite sections they will equal in square the areas applied to it and exceeding by the same figure (I 50-51), but in the ellipse the areas applied to it and defective by the same figure (I 50) and that all the things which have been already proved about the sections as following when the principal diameters are used,¹ will also, those very same things follow when the other diameters are taken

PROPOSITION 52 (PROBLEM)

Given a straight line in a plane bounded at one point, to find in the plane the section of a cone called parabola, whose diameter is the given straight line, and whose vertex is the end of the straight line and where whatever straight line is dropped from the section to the diameter at a given angle will equal in square the rectangle contained by the straight line cut off by it from the vertex of the section and by some other given straight line

Let there be the straight line AB given in position and bounded at the point A , and another straight line CD given in magnitude and first let the given



angle be a right angle it is required then to find a parabola in the plane of reference whose diameter is the straight line AB and whose vertex is the point A and whose upright side is the straight line CD , and where the straight lines dropped ordinatewise will be dropped at a right angle that is so that AB is the axis (First Def 1 7)

Let AB be produced to E and let CG be taken as the fourth part of CD , and let

$$EA > CG$$

and let

$$CD : H :: H : EA$$

Therefore

$$CD : EA :: sq H : sq EA$$

and

$$CD < 4EA$$

therefore also

$$sq H < 4 sq EA$$

¹The principal diameter (ὁ ἀξὺς ἀπὸς αὐτῆς) is that whose being is established in I 7 postm.

Therefore

$$H < 2EA,$$

and the two straight lines EA are greater than H . It is therefore possible for a triangle to be constructed from H and two straight lines EA . Then let the triangle EAF be constructed on EA at right angles to the plane of reference so that

$$EA = AF,$$

and

$$H = FE$$

and let the straight line AK be drawn parallel to FE and FK to EA and let a cone be conceived whose vertex is the point F and whose base is the circle about diameter KA , at right angles to the plane through AFX . Then the cone will be a right cone (First Def. 1. 3) for

$$AF = FK$$

And let the cone be cut by a plane parallel to the circle KA , and let it make as a section the circle MNX (1. 4) at right angles clearly to the plane through MFN , and let the straight line MN be the common section of the circle MNX and of the triangle MFN , therefore it is the diameter of the circle. And let the straight line XL be the common section of the plane of reference and of the circle. Since then circle MNX is at right angles to triangle MFN , and the plane of reference also at right angles to triangle MFN , therefore the straight line LX , their common section, is at right angles to triangle MFN , that is to triangle KFA (Eucl. XI. 19), and therefore it is perpendicular to all the straight lines touching it and in the triangle, and so it is perpendicular to both MN and AB .

Again since a cone, whose base is the circle MNX and whose vertex is the point F , has been cut by a plane at right angles to the triangle MFN and makes as a section circle MNX and since it has also been cut by another plane, the plane of reference, cutting the base of the cone in a straight line XL at right angles to MN which is the common section of the circle MNX and the triangle MFN , and the common section of the plane of reference and of the triangle MFN the straight line AB , is parallel to the side of the cone FKM , therefore the resulting section of the cone in the plane of reference is a parabola and its diameter AB (1. 11), and the straight lines dropped ordinately from the section to AB will be dropped at right angles for they are parallel to XL which is perpendicular to AB . And since

$$CD \cdot H = H \cdot EA,$$

and

$$EA = AF = FK$$

and

$$H = EF = AK$$

therefore

$$CD \cdot AK = AK \cdot AF$$

And therefore

$$CD \cdot AF = \text{sq } AK = \text{sq } AF \text{ or rect } AF, FK$$

Therefore CD is the upright side of the section, for this has been shown in the eleventh theorem (1. 11)

PROPOSITION 53 (PROBLEM)

With the same things supposed let the given angle not be right, and let the angle HAE be made equal to it and let

$$AH = \text{half } CD,$$

and from H let the straight line HE be drawn perpendicular to AE , and through E let the straight line EL be drawn parallel to BH , and from A let the straight line AL be drawn perpendicular to EL , and let EL be bisected at K , and from A let the straight line AM be drawn perpendicular to EL and produced to F and G and let rect $LA \cdot AM = \text{sq } AL$. And given the two straight lines LA and AM in position and bounded at A and AM in magnitude and let a parabola be described with a right angle whose diameter is the straight line AL , and whose vertex is the point A and whose upright side is the straight line AM , as has been shown before (I 52), and it will pass through the point A because

$$\text{sq } AL = \text{rect } LA \cdot AM \text{ (I 11),}$$

and the straight line EA will touch the section because

$$EA = AL \text{ (I 33)}$$

And HA is parallel to EKL , therefore HAB is the diameter of the section and the straight lines dropped to it parallel to AE will be bisected by AB (I 46) And they will be dropped at angle HAE . And since

$$\text{angle } AEH = \text{angle } AGF,$$

and angle at A is common therefore triangle AHE is similar to triangle AGF . Therefore

$$HA : EA :: FA : AG,$$

therefore

$$2AH : 2AE :: FA : AG$$

But

$$CD = 2AH,$$

therefore

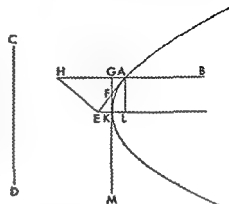
$$FA : AG :: CD : 2AE$$

Then by things already shown in the forty ninth theorem (I 49) the straight line CD is the upright side

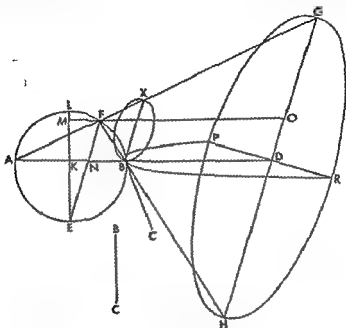
PROPOSITION 54

Given two bounded straight lines perpendicular to each other one of them being produced on the side of the right angle to find on the straight line produced the section of a cone called hyperbola in the same plane with the straight lines, so that the straight line produced is a diameter of the section and the point at the angle is the vertex, and where whatever straight line is dropped from the section to the diameter making an angle equal to the given angle will equal in square the rectangle applied to the other straight line having as breadth the straight line cut off by the dropped straight line beginning with the vertex and exceeding by a figure similar and similarly situated to that contained by the original straight lines

Let there be the two bounded straight lines AB and BC perpendicular to



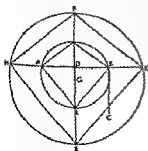
each other and let AB be produced to D it is required then to find in the plane through the lines AB, BC an hyperbola whose diameter will be the straight



line ABD and vertex B and upright side the straight line BC and where the straight lines dropped from the section to BD at the given angle will equal in square the rectangles applied to BC having as breadths the straight lines cut off by them from B and exceeding by a figure similar and similarly situated to the rectangle $AB \cdot BC$

First let the given angle be a right angle and on AB let a plane be erected at right angles to the plane of reference and let the circle $AEBF$ be described in it about AB so that the segment of the circle's diameter within the sector AEB has to the segment of the diameter within the sector AFB a ratio not greater than that of AB to BC ¹ and let AEB be bisected at E and let the straight line

¹Eutocius commenting, adds Let there be two straight lines AB and BC and let it be required to describe a circle on AB so that its diameter is cut by AB in such a way that the part of it on the side of C has to the remainder a ratio not greater than that of AB to BC



Now let it be supposed that they have the same ratio and let AB be bisected at D and through it let the straight line EDF be drawn perpendicular to AB and let it be contrived that $AB \cdot BC = ED \cdot DF$

and let EF be bisected then it is clear that if

$$AB = BC$$

$$ED = DF$$

and the point D will be the midpoint of EF and if

$$AB > BC$$

$$ED > DF$$

and the midpoint will be below D and if

EA be drawn perpendicular from E to the straight line AB and let it be produced to L , therefore the straight line EL is a diameter (Eucl III 1) If then

$$AB \ BC \ EA \ AL$$

we use point L , but if not, let it be contrived that

$$AB \ BC \ EA \ AM$$

with

$$AM < AL \text{ (Eucl V 8),}$$

and through M let MF be drawn parallel to AB , and let AF , EF , and FB be joined, and through B let $B\lambda$ be drawn parallel to FE Since then

$$\text{angle } AFE = \text{angle } EFB,$$

but

$$\text{angle } AFE = \text{angle } A\lambda B,$$

and

$$\text{angle } EFB = \text{angle } \lambda BF,$$

therefore also

$$\text{angle } \lambda BF = \text{angle } F\lambda B,$$

therefore also

$$FB = F\lambda$$

Let a cone be conceived whose vertex is the point F and whose base is the circle about diameter $B\lambda$ at right angles to triangle $BF\lambda$ Then the cone will be a right cone for

$$FB = F\lambda$$

Then let the straight lines BF , $F\lambda$ and MF be produced, and let the cone be cut by a plane parallel to the circle $B\lambda$ then the section will be a circle (I 4) Let it be the circle GPR and so GH will be the diameter of the circle (I 4 end) And let the straight line PDR be the common section of circle GH and of the plane of reference then PDR will be perpendicular to both of the straight lines GH and DB for both of the circles λB and HG are perpendicular to triangle FGH and the plane of reference is perpendicular to triangle FGH , and therefore their common section the straight line PDR is perpendicular to triangle FGH therefore it makes right angles also with all the straight lines touching it and in the same plane

$$AB < BC$$

it will be above D

And now let it be below as G and with center G and radius GF let a circle be described, then it will have to pass either within or without the points A and B And if it should pass through the points A and B what was enjoined would be done but let it fall beyond the points A and B and let the straight line AB produced both ways meet the circumference at H and K and let FH , HF , EA and AF be joined and let MB be drawn through B parallel to FA and BL parallel to AF and let MA and AL be joined then these will also be parallel to FH and HE because

$$AD = DB$$

$$DH = DA$$

and FDE is perpendicular to HA And since the angle at K is a right angle and MB and BL are parallel to FA and AF therefore the angle at B is a right angle then for the same reasons also the angle at A And so the circle described on ML will pass through the points A and B (Eucl III 31) Let the circle $MALB$ be described And since MB is parallel to FA ,

$$FD \ DM \ KD \ DB$$

$$\text{Then likewise also} \quad KD \ DB \ FD \ DL$$

$$\text{And therefore} \quad FD \ DM \ ED \ DL$$

$$\text{And I alternately} \quad ED \ DF \ AB \ BC \ LD \ DM$$

And likewise if the circle described on FE cuts AB the same thing could be shown.

And since a cone whose base is circle GH and vertex F , has been cut by a plane perpendicular to triangle FGH , and has also been cut by another plane the plane of reference, in the straight line PDR perpendicular to the straight line GDH and the common section of the plane of reference and of triangle GFH , that is the straight line DB , produced in the direction of B , meets the straight line GF at A , therefore by things already shown before (1 12) the section PBR will be an hyperbola whose vertex is the point B , and where the straight lines dropped ordinatewise to BD will be dropped at a right angle, for they are parallel to straight line PDR . And since

$$AB \cdot BC = EA \cdot KM,$$

and

$$EA \cdot KM = EN \cdot NF \text{ rect } EN \cdot NF \text{ sq } NF,$$

therefore

$$AB \cdot BC = \text{rect } EN \cdot NF \text{ sq } NF$$

And

$$\text{rect } EN \cdot NF = \text{rect } AN \cdot NB,$$

therefore

$$AB \cdot BC = \text{rect } AN \cdot NB \text{ sq } NF$$

But

$$\text{rect } AN \cdot NB \text{ sq } NF \text{ comp } AN \cdot NF \text{ BN } NF,$$

but

$$AN \cdot NF = AD \cdot DG = FO \cdot OG,$$

and

$$BN \cdot NF = FO \cdot OH,$$

therefore

$$AB \cdot BC \text{ comp } FO \cdot OG, FO \cdot OH,$$

that is

$$\text{sq } FO = \text{rect } OG \cdot OH$$

Therefore

$$AB \cdot BC = \text{sq } FO = \text{rect } OG \cdot OH$$

And the straight line FO is parallel to the straight line AD , therefore the straight line AB is the transverse side and BC the upright side, for these things have been shown in the twelfth theorem (1 12)

PROPOSITION 55 (PROBLEM)

Then let the given angle not be a right angle and let there be the two given straight lines AB and AC , and let the given angle be equal to angle BAH then it is required to describe an hyperbola whose diameter will be the straight line AB , and upright side AC and where the ordinates will be dropped at angle HAB

Let the straight line AB be bisected at D and let the semicircle AFD be described on AD and let some straight line FG parallel to AH , be drawn to the semicircle making

$$\text{sq } FG = \text{rect } DG \cdot GA = AC \cdot AB^1$$

¹Eutocius, commenting gives this construction. Let there be the semicircle ABC on the diameter AC and the given ratio EF to FG and let it be required to do what is proposed.

Let FH be made equal to EF and let HG be bisected at K and let the straight line CB be drawn in the semicircle at angle ACB (the required angle) and from the center L let the straight line LS be drawn perpendicular to it and produced let it meet the circumference at

of which the greater is the straight line AB , then it is required to describe in the plane of reference an ellipse whose diameter will be the straight line AB and vertex A , and upright side AC , and where the ordinates will be dropped from the section to the diameter at a given angle and will equal in square the rectangles applied to AC having as breadths the straight lines cut off by them from A and defective by a figure similar and similarly situated to rectangle BAC .

And first let the given angle be a right angle, and let a plane be erected from AB at right angles to the plane of reference and in it on AB , let the sector of a circle ADB be described and its midpoint be D and let the straight lines DA and DB be joined and let the straight line AX be made equal to AC , and through X let the straight line XO be drawn parallel to DB , and through O let OF be drawn parallel to AB , and let DF be joined and let it meet AB produced at E , then we will have

$$AB \cdot AC = AB \cdot AX = DA \cdot AO = DE \cdot EF$$

And let the straight lines AF and FB be joined and produced, and let some point G be taken at random on FA , and through it let the straight line GL be drawn parallel to DE and let it meet AB produced at K , then let FO be produced and let it meet GK at L . Since then

$$\begin{aligned} \text{arc } AD &= \text{arc } DB, \\ \text{angle } ABD &= \text{angle } DFB \quad (\text{Eucl. iii. 27}) \end{aligned}$$

And since

$$\text{angle } EFA = \text{angle } FDA + \text{angle } FAD,$$

but

$$\text{angle } FAD = \text{angle } FBD,$$

and

$$\text{angle } FDA = \text{angle } FBA,$$

therefore also

$$\text{angle } EFA = \text{angle } DBA = \text{angle } DFB$$

And also DE is parallel to LG therefore

$$\text{angle } EFA = \text{angle } FGH,$$

and

$$\text{angle } DFB = \text{angle } FHG$$

And so also

$$\text{angle } FGH = \text{angle } FHG,$$

and

$$FG = FH$$

Then let circle $GHIN$ be described about HG at right angles to triangle HGF , let a cone be conceived whose base is the circle $GHIN$, and whose vertex is the point F , then the cone will be a right cone because

$$FG = FH$$

And since the circle $GHIN$ is at right angles to plane HGF and the plane of reference is also at right angles to the plane through GH and HF therefore their common section will be at right angles to the plane through GH and HF . Then let their common section be the straight line AM therefore the straight line AM is perpendicular to both of the straight lines AK and AG .

And since a cone whose base is the circle $GHIN$ and whose vertex is the point F has been cut by a plane through the axis and makes as a section triangle GHF , and has been cut also by another plane through AK and AM ,

which is the plane of reference in the straight line AM which is perpendicular to GA and the plane meets the sides of the cone FG and FH , therefore the resulting section is an ellipse whose diameter is AB and where the ordinates will be dropped at a right angle (I 13), for they are parallel to AM And since

$$DE \cdot EF = \text{rect } DE \cdot EF \text{ or } \text{rect } BE, EA = \text{sq } EF,$$

and

$$\text{rect } BE, EA = \text{sq } EF \text{ comp } BE \cdot EF, AE \cdot EF,$$

but

$$BE \cdot EF = BK \cdot KH,$$

and

$$AE \cdot EF = AL \cdot KG = FL \cdot LG,$$

therefore

$$BA \cdot AC \text{ comp } FL \cdot LG = FL \cdot LH \text{ (see above),}$$

which is the same as

$$\text{sq } FL = \text{rect } GL, LH$$

therefore

$$BA \cdot AC = \text{sq } FL = \text{rect } GL, LH$$

And whenever this is so, the straight line AC is the upright side of the figure as has been shown in the thirteenth theorem (I 13)

PROPOSITION 57 (PROBLEM)

With the same things supposed let the straight line AB be less than AC , and let it be required to describe an ellipse about diameter AB so that AC is the upright

Let AB be bisected at D and from D let the straight line EDF be drawn perpendicular to AB , and let

$$\text{sq } FE = \text{rect } BA \cdot AC$$

so that

$$FD = DE,$$

and let FG be drawn parallel to AB , and let it be contrived that

$$AC \cdot AB = EF \cdot FG,$$

therefore also

$$EF > FG$$

And since

$$\text{rect } CA, AB = \text{sq } EF,$$

hence

$$CA \cdot AB = \text{sq } FE = \text{sq } AB + \text{sq } DF + \text{sq } DA$$

But

$$CA \cdot AB = EF \cdot FG$$

therefore

$$EF \cdot FG = \text{sq } FD + \text{sq } DA$$

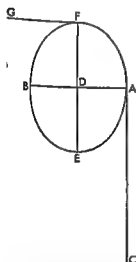
But

$$\text{sq } FD = \text{rect } FD, DE,$$

therefore

$$EF \cdot FG = \text{rect } ED, DF + \text{sq } AD$$

Then with two bounded straight lines situated at right angles to each other and with EF greater let an ellipse be described whose diameter is EF and upright side FG (I 56) then the section will pass through A because



rect FD, DE sq DA EF FG (I 21)

And

$$AD = DB$$

then it will also pass through B Then an ellipse has been described about AB
And since

$$CA \ AB \ \text{sq} \ FD \ \text{sq} \ DA,$$

and

$$\text{sq} \ DA = \text{rect} \ AD, DB,$$

therefore

$$CA \ AB \ \text{sq} \ DF \ \text{rect} \ AD, DB$$

And so the straight line AC is an upright side (I 21)

PROPOSITION 58 (PROBLEM)

But then let the given angle not be a right angle and let the angle BAD be equal to it and let the straight line AB be bisected at E , and let the semicircle AFE be described on AE , and in it let the straight line FG be drawn parallel to AD making

$$\text{sq} \ FG \ \text{rect} \ AG \ GE \ CA \ AB^1$$

and let the straight lines AF and EF be joined and produced, and let

$$DE \ EH \ EH \ EF,$$

¹Eutocius commenting gives this construction Let there be the semicircle ABC and within it some straight line AB (at the required angle to AC) and let two unequal straight lines DE and EF be laid down, and let EF be produced to G and let FG be made equal to DE and let the whole line EG be bisected at H and let the center of the circle K be taken and from it let a perpendicular be drawn to AB and let it meet the circumference at L and through L let LM be drawn parallel to AB and let KA produced meet LM at M and let it be contrived that

$$HF \ FG \ LM \ MN$$

and let

$$LK = LN$$

and let the straight lines NA and LA be joined and produced and let the circle finished out cut them at P and O and let the straight line

ORP be joined

Since then $FH \ FG \ LM \ MN$

componendo $HG \ GF \ LV \ NV$

inversely $FG \ GH \ LV \ NL$

and $FG \ GE \ VN \ NY$

separando $FG \ FE \ NV \ MY$

And since $NL = LY$

and the straight line LA is common and at right angles therefore also

$$LA = LA$$

And also $AO = AP$

therefore NL is parallel to OP Therefore triangle LMN is similar to triangle OKR and triangle LMN to triangle PKA

Therefore $LM \ KP \ NV \ RO$

But also $LM \ KR \ NV \ PR$

and therefore $NV \ RO \ NV \ PR$

and alternately $NV \ NV \ PO \ RP$

But $NV \ NV \ GF \ FE \ DE \ EF$

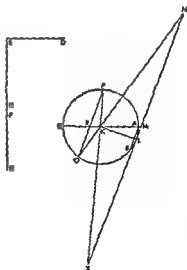
and $OR \ RP \ \text{sq} \ OL \ \text{rect} \ OR \ RP$

and therefore

$$DF \ EF \ \text{sq} \ OR \ \text{rect} \ OR \ RP$$

And $\text{rect} \ OR \ RP = \text{rect} \ AR \ RC$ (Eucl III 35)

Therefore $DE \ EF \ \text{sq} \ OR \ \text{rect} \ AR \ RC$



and let

$$EA = EH$$

and let it be contrived that

rect HF $FL = sq$ AF .

and let the straight line AL be joined and from H let the straight line HM be drawn perpendicular to HF and so parallel to the straight line AFL for the angle at F is right. And with the two given bounded straight lines KH , and HM perpendicular to each other, let an ellipse be described whose transverse diameter is KL , and the upright side of whose figure is HM , and where the ordinates to HK will be dropped at right angles (156-57) then the section will pass through A because $sq\ FA = rect\ HF, FL$ (113). And since

HE-EK.

end

$AE - EB,$

the section will also pass through B , and E will be the center, and the straight line AEB the diameter. And the straight line DA will touch the section because $\text{rect } DE \cdot EF = \text{sq } EH$

And since

$CA \cdot AB = sq \ FG \text{ rect } AG, GE$

but

CA AB comp CA 2AD 2AD AB or DA AE,

and

$$\text{sq } FG = \text{rect } AG, GE \text{ comp } FG \quad GE \quad FG \quad GA,$$

therefore

$$\text{ratio comp } CA \quad 2AD \quad DA \quad AE = \text{ratio comp } FG \quad GE \quad FG \quad GA$$

But

DA AE FG GE

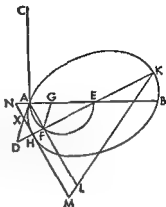
and the common ratio being taken away we will have

CA 2AD FG GA

or

CA 2AD YA 4N

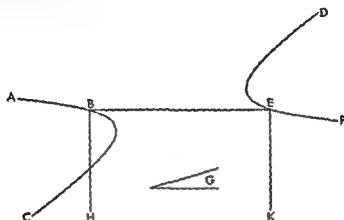
And whenever this is so the straight line AC is the upright side of the figure (r. 50)

**PROPOSITION 59 (PROBLEM)**

Given two bounded straight lines perpendicular to each other to find opposite sections whose diameter is one of the given straight lines and whose vertex is the ends of the straight line and where the straight lines dropped in each of the sections at a given angle will equal in square the rectangles applied to the other of the straight lines and exceeding by a figure similar to the rectangle contained by the given straight lines

Let there be the two given bounded straight lines BE and BH perpendicular to each other and let the given angle be G then it is required to describe

opposite sections about one of the straight lines BE and BH , so that the ordinates are dropped at an angle G



And given the two straight lines BE and BH let an hyperbola be described whose transverse diameter will be the straight line BE and the upright side of whose figure will be BH and where the ordinates to BE produced will be at an angle G , and let it be the line ABC for we have already described how this must be done (1 55) Then let the straight line EA be drawn through E perpendicular to BE and equal to BH , and let another hyperbola DEF be likewise described whose diameter is BE and the upright side of whose figure is EA and where the ordinates from the section will be dropped at a same angle G Then it is evident that B and E are opposite sections, and there is one diameter for them, and their uprights are equal

PROPOSITIO 60 (PROBLEM)

Given two straight lines bisecting each other to describe about each of them opposite sections so that the straight lines are their conjugate diameters and the diameter of one pair of opposite sections is equal in square to the figure of the other pair and likewise the diameter of the second pair of opposite sections is equal in square to the figure of the first pair

Let there be the two given straight lines AC and DE bisecting each other then it is required to describe opposite sections about each of them as a diameter so that the straight lines AC and DE are conjugates in them and DE is equal in square to the figure about AC and AC is equal in square to the figure about DE

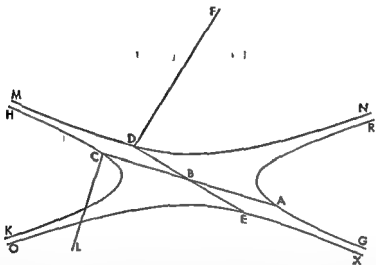
Let

$$\text{rect } AC \cdot CL = \text{sq } DE$$

and let LC be perpendicular to CA And given two straight lines AC and CL perpendicular to each other let the opposite sections RAG and HCA be described whose transverse diameter will be CA and whose upright side will be CL and where the ordinates from the sections to C will be dropped at the given angle (1 59) Then the straight line DE will be a second diameter of the opposite sections (See Def 1 11), for it is the mean proportion between the sides of the figure, and parallel to an ordinate it has been bisected at B

Then again let

$$\text{rect } DE, DF = \text{sq } AC,$$



and let DF be perpendicular to DE . And given two straight lines ED and DF lying perpendicular to each other, let the opposite sections MDN and OEX be described whose transverse diameter will be DE and the upright side of whose figure will be DF , and where the ordinates from the sections will be dropped to DE at the given angle (I 59) then the straight line AC will also be a second diameter of the sections MDN and XEO . And so AC bisects the parallels to DE between the sections RAG and HCK and DE the parallels to AC , and this it was required to do

And let such sections be called conjugate

BOOK TWO

APOLLONIUS to EUDÆMUS

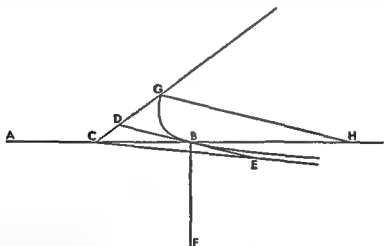
If you are well well and good and I too fare pretty well

I have sent you my son Apollonius bringing you the second book of the conics as arranged by us Go through it then carefully and acquaint those with it worthy of sharing in such things And Philonides the geometer, I introduced to you in Ephesus, if ever he happen about Pergamum, acquaint him with it too And take care of yourself, to be well Good bye

PROPOSITION I

If a straight line touch an hyperbola at its vertex and from it on both sides of the diameter a straight line is cut off equal to the straight line equal in square to the fourth of the figure, then the straight lines drawn from the center of the section to the ends thus taken on the tangent will not meet the section

Let there be an hyperbola whose diameter is the straight line AB and center C and upright the straight line BF , and let the straight line DE touch the



section at B and let the squares on BD and BE each be equal to the fourth of the figure $AB \cdot BF$ and the straight lines CD and CE be joined and produced

I say that they will not meet the section

For if possible, let CD meet the section at G and from G let the straight line GH be dropped ordinately therefore it is parallel to DB (I. 17) Since then

$$AB \cdot BF = \text{sq } AB = \text{rect } AB \cdot BF,$$

therefore also

$$GL = LM$$

And since

$$GH = DB,$$

therefore

$$GK > DB$$

And also

$$KV > BE,$$

since also

$$LM > BE$$

therefore

$$\text{rect } MA, AG > \text{rect } DB \text{ } BE \\ > \text{sq } DB$$

Since then

$$AB \text{ } BF \text{ } \text{sq } CB \text{ } \text{sq } BD \text{ (ii 1),}$$

but

$$AB \text{ } BF \text{ } \text{rect } AL \text{ } LB \text{ } \text{sq } LA \text{ (i 21),}$$

and

$$\text{sq } CB \text{ } \text{sq } BD \text{ } \text{sq } CL \text{ } \text{sq } LG,$$

therefore also

$$\text{sq } CL \text{ } \text{sq } LG \text{ } \text{rect } AL \text{ } LB \text{ } \text{sq } LK$$

Since then

$$\begin{array}{l} \text{whole sq } LC \text{ } \text{whole sq } LG \\ \text{part subtr rect } AL, LB \text{ } \text{part subtr sq } LA, \end{array}$$

therefore also

$$\text{sq } LC \text{ } \text{sq } LG \text{ } \text{remainder sq } CB \text{ } \text{remainder rect } MA \text{ } AG$$

that is

$$\text{sq } CB \text{ } \text{rect } MA \text{ } AG \text{ } \text{sq } CB \text{ } \text{sq } DB$$

Therefore

$$\text{sq } DB = \text{rect } MA \text{ } AG,$$

and this is absurd for it has been shown to be greater than it. Therefore the straight line *CH* is not an asymptote to the section

PROPOSITION 3

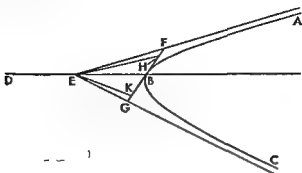
If a straight line touches an hyperbola it will meet both of the asymptotes and it will be bisected at the point of contact, and the square on each of its segments will be equal to the fourth of the figure resulting on the diameter drawn through the point of contact

Let there be the hyperbola *ABC* and its center *E*, and asymptotes *FE* and *EG* and let some straight line *HA* touch it at *B*

I say that the straight line *HA* produced will meet the straight lines *FE* and *EG*

For if possible, let it not meet them and let *EB* be joined and produced and let *ED* be made equal to *EB* therefore the straight line *BD* is a diameter. Then let the squares on *HB* and *BA* each be made equal to the fourth of the figure on *BD*, and let *HI* and *FA* be joined. Therefore they are asymptotes (ii 1) and this is absurd (ii 2) for *FE* and *EG* are supposed asymptotes. Therefore *HA* produced will meet the asymptotes *EF* and *EC* at *F* and *C*

I say then also that the squares on BF and BG will each be equal to the fourth of the figure on BD

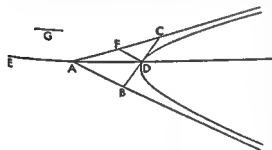


For let it not be, but if possible let the squares on BH and BK each be equal to the fourth of the figure. Therefore HE and EK are asymptotes (II 1), and this is absurd (II 2). Therefore the squares on BF and BG will each be equal to the fourth of the figure on BD .

PROPOSITION 4 (PROBLEM)

Given two straight lines containing an angle and a point within the angle to describe through the point the section of a cone called hyperbola so that the given straight lines are its asymptotes

Let there be the two straight lines AC and AB containing a chance angle at A and let some point D be given and let it be required to describe through D an hyperbola to the asymptotes CA and AB



Let the straight line AD be joined and produced to E and let AE be made equal to DA and let the straight line DF be drawn through D parallel to AB , and let FC be made equal to AF and let CD be joined and produced to B and let it be contrived that
 $\text{rect } DE, G = \text{sq } CB$
 and with AD extended let an hyperbola be described about it through D so that the

ordinates equal in square the areas applied to G and exceeding by a figure similar to rectangle DE, G . Since then DF is parallel to BA , and
 $CF = FA$

therefore

$$CD = DB$$

and so

$$\text{sq } CB = 4 \text{ sq } CD$$

And

$$\text{sq } CB = \text{rect } DE, G$$

therefore the squares on CD and DB are each equal to the fourth part of the figure DE, G . Therefore the straight lines AB and AC are asymptotes to the hyperbola described.

PROPOSITION 5

If the diameter of a parabola or hyperbola bisects some straight line, the tangent to the section at the end of the diameter will be parallel to the bisected straight line.

Let there be the parabola or hyperbola ABC whose diameter is the straight line DBE and let the straight line FBG touch the section and let some straight line AEC be drawn in the section making AE equal to EC .

I say that AC is parallel to FG .

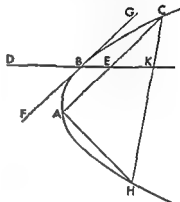
For if not let the straight line CH be drawn through C parallel to FG and let HA be joined. Since then ABC is a parabola or hyperbola whose diameter is DE , and tangent FG , and CH is parallel to it therefore

$$CA = AH \quad (\text{I } 46, 47)$$

But also

$$CE = EA$$

Therefore AH is parallel to KE , and this is absurd, for produced it meets BD (I 22).



PROPOSITION II

If the diameter of an ellipse or circumference of a circle bisects some straight line not through the center, the tangent to the section at the end of the diameter will be parallel to the bisected straight line.

Let there be an ellipse or circumference of a circle whose diameter is the straight line AB and let AB bisect CD a straight line not through the center at the point E .

I say that the tangent to the section at A is parallel to CD .

For let it not be but if possible, let DF be parallel to the tangent at A therefore

$$DG = FG$$

But also

$$DE = EC$$

therefore CF is parallel to GE and this is absurd.

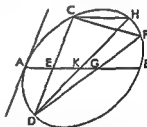
For if G is the center of the section AB the straight line CF will meet the straight line AB (I 23), and if it is not suppose it to be K and let DK be joined and produced to H and let CH be joined. Since then

$$DK = KH,$$

and also

$$DE = EC$$

therefore CH is parallel to AB . But also CF and this is absurd. Therefore the tangent at A is parallel to CD .



PROPOSITION 7

If a straight line touches a section of a cone or circumference of a circle and a parallel to it is drawn in the section and bisected the straight line joined from the point of contact to the midpoint will be a diameter of the section

Let there be a section of a cone or circumference of a circle ABC , and FG tangent to it, and AC parallel to FG and bisected at E , and let BE be joined

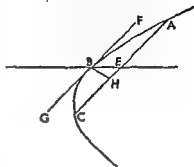
I say that BE is a diameter of the section

For let it not be, but if possible, let BH be a diameter of the section. Therefore

$AH = HC$ (First Def 1 4),
and thus is absurd, for

$$AE = FC$$

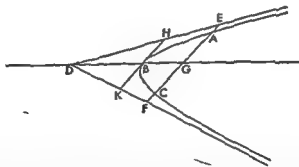
Therefore BH will not be a diameter of the section. Then likewise we could show that there is no other than BE



PROPOSITION 8

If a straight line meets an hyperbola in two points produced both ways it will meet the asymptotes and the straight lines cut off on it by the section from the asymptotes will be equal

Let there be the hyperbola ABC and the asymptotes ED and DF , and let some straight line AC meet ABC



I say that produced both ways it will meet the asymptotes

Let AC be bisected at G and let DG be joined. Therefore it is a diameter of the section (II 7) therefore the tangent at B is parallel to AC (II 5 6). Then let HBA be the tangent (I 32), then it will meet ED and DF (II 3). Since then AC is parallel to HA and HA meets DK and DH therefore also AC will meet DE and DF

Let it meet them at E and F and

$$HB = BA \text{ (II 3),}$$

therefore also

$$FG = GE$$

And so also

$$CF = AE$$

PROPOSITION 9

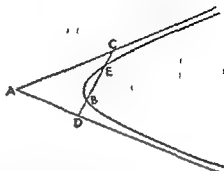
If a straight line meeting the asymptotes is bisected by the hyperbola, it will touch the section in one point only

For let the straight line CD meeting the asymptotes CA, AD be bisected by the hyperbola at the point E

I say that it touches the hyperbola at no other point

For if possible, let it touch it at B

Therefore $CE = BD$ (II 8), and this is absurd for CE is supposed equal to ED . Therefore it will not touch the section at another point



PROPOSITION 10

If some straight line cutting the section meet both of the asymptotes the rectangle contained by the straight lines cut off between the asymptotes and the section is equal to the fourth of the figure resulting on the diameter bisecting the straight lines drawn parallel to the drawn straight line

Let there be the hyperbola ABC and let DE, LF be its asymptotes, and let some straight line DF be drawn cutting the section and the asymptotes and let AC be bisected at G , and let GE be joined, and let EH be made equal to BE , and let BM be drawn from B perpendicular to HEB , therefore BH is a diameter (II 7), and BM the upright side

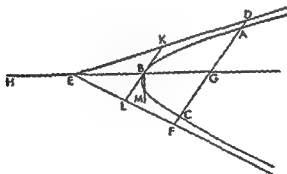
I say that

rect $DA AF$ = fourth rect HB, BM ,

then likewise also

rect $DC CF$ = fourth rect HB, BM

For let AL be drawn through B tangent to the section therefore it is



parallel to DF (II 6) And since it has been shown

$HB \cdot BM$ sq EB sq BA sq FG sq GD (II 1 3)

and

$HB \cdot BM$ rect $HG \cdot GB$ sq GA (I 21),

therefore

$$\text{sq } EG \text{ sq } GD \text{ rect } HG, GB \text{ sq } GA$$

Since then

$$\begin{array}{l} \text{whole sq } EG \text{ whole sq } GD \\ \text{part subtr rect } HG, GB \text{ part subtr sq } AG, \end{array}$$

therefore also

$$\text{remainder sq } EB \text{ remainder rect } DA, AF \text{ sq } EG \text{ sq } GD,$$

or

$$\text{remainder sq } EB \text{ remainder rect } DA, AF \text{ sq } EB \text{ sq } BA$$

Therefore

$$\text{rect } FA, AD = \text{sq } BK$$

Then likewise it could be shown also that

$$\text{rect } DC, CF = \text{sq } BL,$$

therefore also

$$\text{rect } FA, AD = \text{rect } DC, CF$$

PROPOSITION 11

If some straight line cut each of the straight lines containing the angle adjacent to the angle containing the hyperbola, it will meet the section in one point only and the rectangle contained by the straight lines cut off between the containing straight lines and the section will be equal to the fourth part of the square on the diameter drawn parallel to the cutting straight line

Let there be an hyperbola whose asymptotes are CA, AD and let DA be produced to E and through some point E let EF be drawn cutting EA and AC

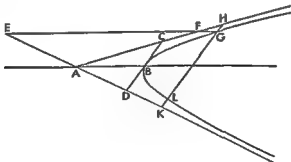
Now it is evident that it meets the section in one point only for the straight line drawn through A parallel to EF as AB will cut angle CAD and will meet the section (Π 2) and be its diameter (Γ 50), therefore EF will meet the section in one point only (Γ 26)

Let it meet it at G

I may then also that

$$\text{rect } EG, GF = \text{sq } AB$$

For let the straight line $HGLK$ be drawn ordinatewise through G , therefore



the tangent through B is parallel to GH (Π 5) Let it be CD Since then $CB = BD$ (Π 3),

therefore

$$\text{sq } CB \text{ or rect } CB, BD \text{ sq } BA \text{ comp } CB \text{ } BA \text{ } DB \text{ } BA$$

But

$$CB \ BA \ HG \ GF,$$

and

$$DB \ BA \ GK \ GE$$

therefore

$$\text{sq } CB \ \text{sq } B1 \ \text{comp } HG \ GF, \ \Lambda G \ GE$$

But also

$$\text{rect } \Lambda G, GH \ \text{rect } EG \ GF \ \text{comp } HG \ GF, \ KG \ GE,$$

therefore

$$\text{rect } \Lambda G \ GH \ \text{rect } EG \ GF \ \text{sq } CB \ \text{sq } BA$$

Alternately

$$\text{rect } \Lambda G, GH \ \text{sq } CB \ \text{rect } EG, GF \ \text{sq } BA$$

But it was shown

$$\text{rect } \Lambda G, GH = \text{sq } CB \ (\text{ii } 10),$$

therefore also

$$\text{rect } EG \ GF = \text{sq } AB$$

PROPOSITION 12

If two straight lines at chance angles are drawn to the asymptotes from some point of those on the section and parallels are drawn to them from some point of those on the section then the rectangle contained by the parallels will be equal to that contained by those straight lines to which they were drawn parallel

Let there be an hyperbola whose asymptotes are AB and BC and let some point D be taken on the section, and from it let DE and DF be dropped to AB and BC and let some other point on the section G be taken and through G let GH and GA be drawn parallel to ED and DF

I say that

$$\text{rect } ED \ DF = \text{rect } HG, GA$$

For let DG be joined and produced to A and C Since then

$$\text{rect } AD \ DC = \text{rect } AG, GC \ (\text{ii } 8)$$

therefore

$$AG \ AD \ DC \ CG$$

But

$$AG \ AD \ GH \ ED,$$

and

$$DC \ CG \ DF \ GA$$

therefore

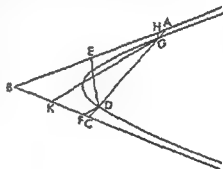
$$GH \ DE \ DF \ GA$$

Therefore

$$\text{rect } ED \ DF = \text{rect } HG \ GA$$

PROPOSITION 13

If in the place bounded by the asymptotes and the section some straight line is drawn parallel to one of the asymptotes, it will meet the section in one point only



Let there be an hyperbola whose asymptotes are CA and AB , and let some point E be taken and through it let EF be drawn parallel to AB

I say that it will meet the section

For if possible, let it not meet it and let some point G on the section be taken and through G let GC and GH be drawn parallel to CA and AB and let

$$\text{rect } CG, GH = \text{rect } AE, EF$$

and let AF be joined and produced, then it will meet the section (1 2) Let it meet it at K , and through K parallel to CA and AB let KL and KD be drawn, therefore

$$\text{rect } CG, GH = \text{rect } LK, KD \text{ (II 12)}$$

And it is supposed that also

$$\text{rect } CG, GH = \text{rect } AE, EF$$

therefore

$$\text{rect } LK, KD \text{ or } \text{rect } KL, LA = \text{rect } AE, EF$$

and thus is impossible for both

$$KL > EF$$

and

$$LA > AE$$

Therefore EF will meet the section

Let it meet it at M

I say then that it will not meet it at any other point

For if possible let it also meet it at N and through M and N let MX and NB be drawn parallel to CA Therefore

$$\text{rect } EM, MX = \text{rect } EN, NB \text{ (II 12),}$$

and this is impossible Therefore it will not meet the section in another point

PROPOSITION 14

The asymptotes and the section if produced indefinitely draw nearer to each other and they reach a distance less than any given distance

Let there be an hyperbola whose asymptotes are AB and AC and a given distance K

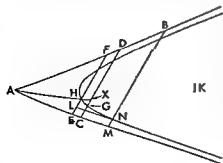
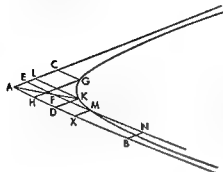
I say that AB and AC and the section if produced, draw nearer to each other and will reach a distance less than K

For let EHF and CGD be drawn parallel to the tangent and let AH be joined and produced to Y Since then $\text{rect } CG, GD = \text{rect } FH, HE$ (II 10), therefore

$$DG \cdot FH = HE \cdot CG$$

But

$$DG > FH \text{ (I 8, 26),}$$



therefore also

$$HE > CG$$

Then likewise we could show that the succeeding straight lines are less

Then let the distance EL be taken less than K and through L let LN be drawn parallel to AC therefore it will meet the section (II 13) Let it meet it at N , and through N let $MA B$ be drawn parallel to EF Therefore

$$MN = EL$$

and so

$$MN < K$$

PORISM

Then from this it is evident that the straight lines AB and AC are nearer than all the asymptotes to the section and the angle contained by BA, AC is clearly less than that contained by other asymptotes to the section

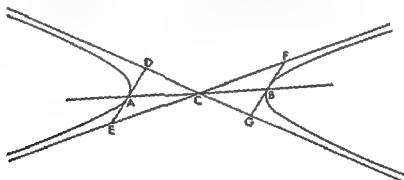
PROPOSITION 15

The asymptotes of opposite sections are common

Let there be opposite sections whose diameter is AB and center C

I say that the asymptotes of the sections A and B are common

Let the straight lines DAE and FBG be drawn tangent to the sections through the points A and B they are therefore parallel (I 44, note) Then let



each of the straight lines DA, AE, FB and BG be cut off equal in square to the fourth of the figure applied to AB therefore

$$DA = AE = FB = BG$$

Then let CD, CE, CF and CG be joined Then it is evident that DC is in a straight line with CG and CE with CF because of the parallels Since then it is an hyperbola whose diameter is AB and tangent DE and DA and AE are each equal in square to the fourth of the figure applied to AB therefore DC and CE are asymptotes (II 1) For the same reasons then FC and CG are also asymptotes to section B Therefore the asymptotes of opposite sections are common

PROPOSITION 16

If in opposite sections some straight line is drawn cutting each of the straight lines containing the angle adjacent to the angles containing the sections, it will meet

each of the opposite sections in one point only, and the straight lines cut off on it by the sections from the asymptotes will be equal

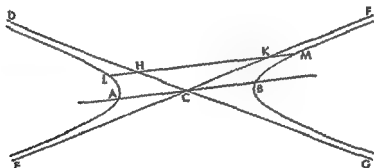
For let there be the opposite sections A and B whose center is C and asymptotes DCG and ECF , and let some straight line HA be drawn through cutting each of the straight lines DC and CF

I say that produced it will meet each of the sections in one point only

For since DC and CE are asymptotes of section A , and some straight line HK has been drawn across cutting both of the straight lines containing the adjacent angle DCF , therefore HA produced will meet the section (II 11) Then likewise also B

Let it meet them at L and M

Let the straight line ACB be drawn through C parallel to LM , therefore



$$\text{rect } AL LH = \text{sq } AC \text{ (II 11)}$$

and

$$\text{rect } HM MK = \text{sq } CB \text{ (II 11)}$$

And so also

$$\text{rect } AL LH = \text{rect } HM, MK,$$

and

$$LH = KM$$

PROPOSITION 17

The asymptotes of conjugate opposite sections are common

Let there be conjugate opposite sections whose conjugate diameters are AB and CD , and whose center is E



I say that their asymptotes are common

For let the straight lines FAG , GDH , HBA , and KCF be drawn through the points A , B , C , and D touching the sections, therefore $FGHK$ is a parallelogram (I 44, note) Then let FEH and KEG be joined therefore they are straight lines (II 15) and diagonals of the parallelogram, and they are all bisected at the point E And since the figure on AB is equal to the square on CD (I 60), and

$$CE = ED,$$

therefore each of the squares on FA , AG , KB , and BH is equal to a fourth of the figure on AB Therefore the straight lines FEH and KEG are asymptotes of the sections A and B (II 1) Then likewise we could show that the same straight lines are also asymptotes of the sections C and D Therefore the asymptotes of conjugate opposite sections are common

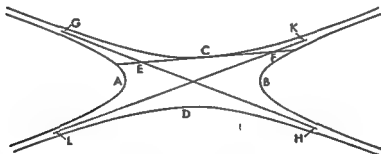
PROPOSITION 18

If a straight line meeting one of the conjugate opposite sections when produced both ways, falls outside the section, it will meet both of the adjacent sections in one point only

Let there be the conjugate opposite sections A , B , C , and D , and let some straight line EF meet the section C and produced both ways fall outside the section

I say that it will meet both of the sections A and B in one point only

For let GH and KL be asymptotes of the sections Therefore EF meets both



GH and KL (II 3) Then it is evident that it will also meet the sections A and B in one point only (II 16)

PROPOSITION 19

If some straight line is drawn touching some one of the conjugate opposite sections at random, it will meet the adjacent sections and will be bisected at the point of contact

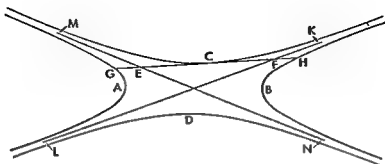
Let there be the conjugate opposite sections A , B , C and D , and let some straight line ECF touch it at C

I say that produced it will meet sections A and B and will be bisected at C

It is evident now that it will meet sections A and B (II 18), let it meet them at G and H

I say that

$$CG=CH$$



For let the asymptotes of the sections KL and MN be drawn. Therefore $EG=FH$ (II 16),

and

$$CE=CF \text{ (II 3),}$$

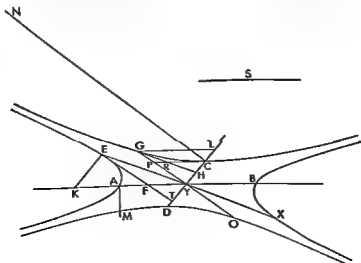
and

$$CG=CH$$

PROPOSITION 20

If a straight line touches one of the conjugate opposite sections, and two straight lines are drawn through their center, one through the point of contact and one parallel to the tangent until it meet one of the adjacent sections then the straight line touching the section at the point of meeting will be parallel to the straight line drawn through the point of contact and the center, and those through the points of contact and the center will be conjugate diameters of the opposite sections

Let there be conjugate opposite sections whose conjugate diameters are the



straight lines AB and CD and center Y , and let EF be drawn touching the section A , and produced let it meet CY at T , and let EY be joined and produced to λ and through Y let YG be drawn parallel to EF , and through G let HG be drawn touching the section

I say that HG is parallel to YE and GO and $E\lambda$ are conjugate diameters

For let the straight lines KE , GL and CRP be drawn ordinatewise, and let AM and CN be the parameters Since then

$$BA \cdot AM = AC \cdot CD \text{ (I 60),}$$

but

$$BA \cdot AM = \text{rect } Y\lambda \cdot \lambda F = \text{sq } KE \text{ (I 37),}$$

and

$$NC \cdot CD = \text{sq } GL = \text{rect } YL \cdot LH \text{ (I 37),}$$

therefore also

$$\text{rect } Y\lambda \cdot \lambda F = \text{sq } EK = \text{sq } GL = \text{rect } YL \cdot LH$$

But

$$\text{rect } Y\lambda \cdot \lambda F = \text{sq } EK \text{ comp } YK = \lambda E \cdot \lambda F = \lambda E,$$

and

$$\text{sq } GL = \text{rect } YL \cdot LH \text{ comp } GL = LY \cdot GL = LH,$$

therefore

$$\text{ratio comp } Y\lambda = \lambda E \cdot \lambda F : \lambda E = \text{ratio comp } GL : LY, GL : LH$$

and of these

$$FA : AE = GL : LY$$

for each of the straight lines EK , λF and FE is parallel to each of the straight lines YL , LG and GL respectively Therefore as remainder

$$YA : AE = GL : LH$$

Also the sides about the equal angles at K and L are proportional therefore triangle $E\lambda Y$ is similar to triangle GHL and will have equal the angles the corresponding sides subtend Therefore

$$\text{angle } EY\lambda = \text{angle } LGH$$

But also

$$\text{angle } \lambda YG = \text{angle } LGY$$

and therefore

$$\text{angle } EY\lambda = \text{angle } HGY$$

Therefore $E\lambda$ is parallel to GH

Then let it be contrived that

$$PG : GR = HG : S$$

therefore S is a half of the parameter of the ordinates to the diameter GO in sections C and D (I 51) And since CD is the second diameter of the sections A and B and CT meets it therefore

$$\text{rect } TY \cdot EA = \text{sq } CY$$

for if we draw from E a parallel to λY the rectangle contained by TY and the straight line cut off by the parallel will be equal to the square on CY (I 38) And therefore

$$TY \cdot EA = \text{sq } TY = \text{sq } TC \text{ (Eucl VI 20)}$$

But

$$TY \cdot EA = TF \cdot FE$$

or

$$TY \cdot FA = \text{trgl } TYF = \text{trgl } EYF \text{ (Eucl VI 1),}$$

and

sq TY sq CY trgl YTF trgl YCP (Eucl vi 19)

or

sq TY sq CY trgl YTF trgl GHY (iii 1)

Therefore

trgl TYF trgl EFY trgl TFY trgl YGH

Therefore

trgl $GHY = \text{trgl } YEF$

But they also have

angle $HGY = \text{angle } YEF$

for EY is parallel to GH , and EF to GY . Therefore the sides about the equal angles are reciprocally proportional (Eucl vi 15) Therefore

$GH : EY = EF : GY$

therefore

rect $HG : GY = \text{rect } YE : EF$

And since

$S : HG = RG : GP$,

and

$RG : GP = YE : EF$,

for they are parallel therefore also

$S : HG = YE : EF$

But with YG taken as common height,

$S : HG = \text{rect } SYG : \text{rect } HGY$,

and

$YE : EF = \text{rect } YEF : \text{rect } YEF$

And therefore

rect $S, YG : \text{rect } HGY = \text{rect } YEF : \text{rect } YEF$

Alternately

rect $S, GY : \text{rect } EY = \text{rect } HGY : \text{rect } YEF$

But

rect $HG, GY = \text{rect } YE : EF$ (above),

therefore also

rect $S, GY = \text{rect } EY$

And rectangle S, GY is a fourth of the figure on GO for

$GY = \text{half } GO$

and S is the parameter and

sq $EY = \text{fourth sq } EY$

for

$EY = YX$

Therefore the square on EY is equal to the figure on GO . Then likewise we could show also that GO is equal in square to the figure on EX . Therefore EY and GO are conjugate diameters of the opposite sections A, B, C , and D .

PROPOSITION 21

The same things being supposed it is to be shown that the point of meeting of the tangents is on one of the asymptotes

Let there be conjugate opposite sections whose diameters are the straight lines AB and CD and let the straight lines AE and EC be drawn tangent

I say that the point E is on the asymptote

PROPOSITION 23

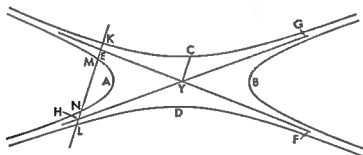
If in conjugate opposite sections some radius is drawn to any one of the sections, and a parallel is drawn to it meeting the three adjacent sections, then the rectangle contained by the segments produced between the three sections on the straight line drawn is twice the square on the radius

Let there be the conjugate opposite sections A, B, C , and D and let the center of the section be Y , and from the point Y let some straight line CY be drawn to meet any one of the sections, and let KL be drawn parallel to CY cutting the three adjacent sections

I say that

$$\text{rect } KM, ML = 2 \text{ sq } CY$$

Let the asymptotes to the sections, EF and GH , be drawn therefore



$$\text{sq } CY = \text{rect } HM, ME \text{ (II 22)} = \text{rect } HA, AE \text{ (II 11)}$$

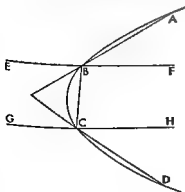
And

$$\text{rect } HM, ME + \text{rect } HA, KE = \text{rect } LM, MK$$

because of the straight lines on the ends being equal (II 8 16) Therefore also
 $\text{rect } LM, MK = 2 \text{ sq } CY$

PROPOSITION 24

If two straight lines meet a parabola each at two points and if a point of meeting of neither one of them is contained by the points of meeting of the other, then the straight lines will meet each other outside the section



Let there be the parabola $ABCD$ and let the two straight lines AB and CD meet $ABCD$ and let a point of meeting of neither of them be contained by the points of meeting of the other

I say that the straight lines produced will meet each other

Let the diameters of the section EBF and GCH be drawn through the points B and C therefore they are parallel (I 51, end) and each one cuts the section in one point only (I 26) Then let BC be joined therefore

$$\text{angle } EBC + \text{angle } BCG = 2 \text{ rt angles}$$

and DC and BA produced make angles less than two right angles. Therefore they will meet each other outside the section (I 10, Eucl Post 5)

PROPOSITION 25

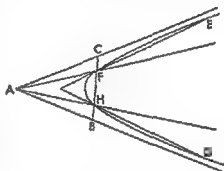
If two straight lines meet an hyperbola each at two points and if a point of meeting of neither of them is contained by the points of meeting of the other, then the straight lines will meet each other outside the section, but within the angle containing the section

Let there be an hyperbola, whose asymptotes are AB and AC , and let the two straight lines EF and GH cut the section, and let a point of meeting of neither of them be contained by the points of meeting of the other

I say that the straight lines EF and GH produced will meet outside the section, but within the angle CAB

For let the straight lines AF and AH be joined and produced and let FH be joined. And since the straight lines EF and GH produced cut the angles AFH and AHF , and the said angles are less than two right angles (Eucl I 17), the straight lines EF and GH produced will meet each other outside the section but within the angle BAC

Then we could likewise show it even if the straight lines EF and GH are tangents to the sections

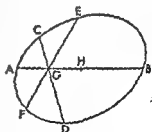


PROPOSITION 26

If in an ellipse or circumference of a circle two straight lines not through the center cut each other then they do not bisect each other

For if possible, in the ellipse or circumference of a circle let the two straight lines CD and EF not through the center bisect each other at G and let the point H be the center of the section and let GH be joined and produced to A and B

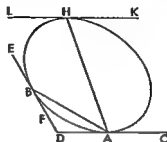
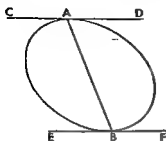
Since then the straight line AB is a diameter bisecting EF , therefore the tangent at A is parallel to EF (II 6). We could then likewise show that it is also parallel to CD . And so also EF is parallel to CD . And this is impossible. Therefore CD and EF do not bisect each other



PROPOSITION 27

If two straight lines touch an ellipse or circumference of a circle, and if the straight line joining the points of contact is through the center of the section the tangents will be parallel but if not they will meet on the same side of the center

Let there be the ellipse or circumference of a circle AB , and let the straight lines CD and EBF touch it, and AB be joined, and first let it be through the center



I say that CD is parallel to EF

For since AB is a diameter of the section, and CD touches it at A , therefore CD is parallel to the ordinates to AB (I 17). Then for the same reasons BF is also parallel to the same ordinates. Therefore CD is also parallel to EF .

Then let AB not be through the center, as in the second drawing, and let the diameter AH be drawn, and let KHL be drawn tangent through H therefore KL is parallel to CD . Therefore EF produced will meet CD on the same side of the center as AB .

PROPOSITION 28

If in a section of a cone or circumference of a circle some straight line bisects two parallel straight lines, then it will be a diameter of the section.

For let AB and CD two parallel straight lines in a conic section, be bisected at E and F , and let EF be joined and produced.

I say that it is a diameter of the section.

For if not let the straight line GFH be so if possible. Therefore the tangent at G is parallel to AB (I 6). And so the same straight line is parallel to CD . And GH is a diameter therefore

$$CH = HD \text{ (First Def 1 4)}$$

and this is impossible for it is supposed

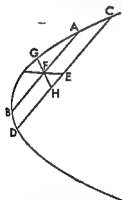
$$CE = ED$$

Therefore GH is not a diameter. Then likewise we could show that there is no other except EF . Therefore EF will be a diameter of the section.

PROPOSITION 29

If in a section of a cone or circumference of a circle two tangents meet the straight line drawn from their point of meeting to the midpoint of the straight line joining the points of contact is a diameter of the section.

Let there be a section of a cone or circumference of a circle to which let the straight lines AB and AC meeting at A be drawn tangent and let BC be joined and bisected at D , and let AD be joined.



I say that it is a diameter of the section

For if possible let DE be a diameter, and let EC be joined, then it will cut the section (1 35 36) Let it cut it at F , and through F let PKG be drawn parallel to CDB Since then

$$CD = DB$$

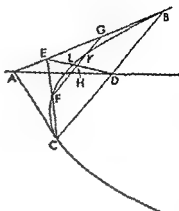
also

$$FH = HG$$

And since the tangent at L is parallel to BC (11 5, 6), and FG is also parallel to BC , therefore also FG is parallel to the tangent at L Therefore

$$FH = HK \text{ (1 46, 47)}$$

and this is impossible Therefore DF is not a diameter Then likewise we could show that there is no other except AD



PROPOSITION 30

If two straight lines tangent to a section of a cone or to a circumference of a circle meet the diameter drawn from the point of meeting will bisect the straight line joining the points of contact

Let there be the section of a cone or circumference of a circle BC , and let two tangents BA and AC be drawn to it meeting at A and let BC be joined and let AD be drawn through A as a diameter of the section

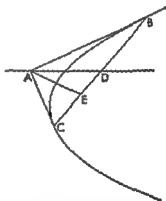
I say that

$$DB = DC$$

For let it not be but if possible let

$$BE = EC$$

and let BE be joined therefore AF is a diameter of the section (11 29) But AD is also a diameter and this is absurd For if the section is an ellipse the point A at which the diameters meet each other, will be a center outside the section and this is impossible and if the section is a parabola, the diameters meet each other (1 51 end) and if it is an hyperbola and the straight lines BA and AC meet the section without containing one another's points of meeting then the center is within the angle containing the hyperbola (11 25), but it is also on it, for it has been supposed a center since DA and AE are diameters (1 51 end) and this is absurd Therefore BE is not equal to EC



PROPOSITION 31

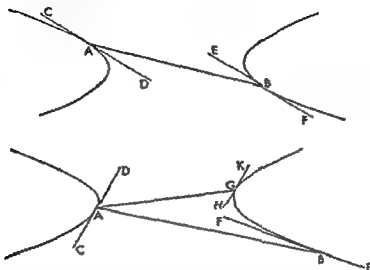
If two straight lines touch each of the opposite sections then if the straight line joining the points of contact falls through the center the tangents will be parallel, but if not, they will meet on the same side as the center

Let there be the opposite sections 1 and B, and let the straight lines CAD and EBF be tangent to them at A and B and let the straight line joined from

A to B fall first through the center of the sections

I say that CD is parallel to EF

For since they are opposite sections of which AB is a diameter and CD touches one of them at A , therefore the straight line drawn through B parallel

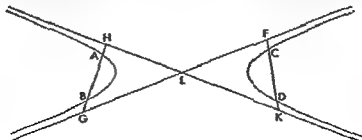


to CD touches the section (1 44 note) But EF also touches it therefore CD is parallel to EF

Then let the straight line from A to B not be through the center of the sections and let AG be drawn as a diameter of the sections and let HA be drawn tangent to the section therefore HK is parallel to CD and since the straight lines EF and HK touch an hyperbola therefore they will meet (11 25 end) And HK is parallel to CD therefore also the straight lines CD and EF produced will meet And it is evident they are on the same side as the center

PROPOSITION 32

If straight lines meet each of the opposite sections in one point when touching or in two points when cutting, and when produced the straight lines meet then their point of meeting will be in the angle adjacent to the angle containing the section



Let there be opposite sections and the straight lines AB and CD

touching the opposite sections in one point or cutting them in two points, and let them meet when produced

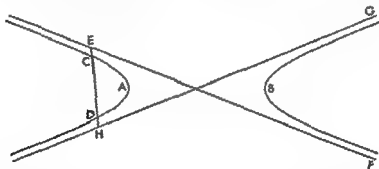
I say that their point of meeting will be in the angle adjacent to the angle containing the section

Let FG and HA be asymptotes to the sections, therefore AB produced will meet the asymptotes (II 8). Let it meet them at H and G . And since FA and HG are supposed as meeting, it is evident that either they will meet in the place under angle HLF or in that under angle ALG . And likewise also, if they touch (II 3).

PROPOSITION 33

If a straight line meeting one of the opposite sections, when produced both ways, falls outside the section, it will not meet the other section, but will fall through the three places of which one is that contained by the angle containing the section, and two are those contained by the angle adjacent to the angle containing the section.

Let there be the opposite sections A and B , and let some straight line CD



cut A , and when produced both ways, let it fall outside the section

I say that the straight line CD does not meet the section B .

For let EF and GH be drawn as asymptotes to the sections, therefore CD produced will meet the asymptotes (II 8). And it only meets them in the points E and H . And so it will not meet the section B .

And it is evident that it will fall through the three places. For if some straight line meets both of the opposite sections, it will meet neither of the opposite sections in two points. For if it meets it in two points, by what has just been proved it will not meet the other section.

PROPOSITION 34

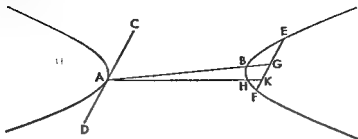
If some straight line touch one of the opposite sections and a parallel to it be drawn in the other section, then the straight line drawn from the point of contact to the midpoint of the parallel will be a diameter of the opposite sections.

Let there be the opposite sections A and B , and let some straight line CD touch one of them A at A , and let EF be drawn parallel to CD in the other section, and let it be bisected at G , and let AG be joined.

I say that AG is a diameter of the opposite sections.

For if possible, let AHA be. Therefore the tangent at H is parallel to CD .

(π.31) But CD is also parallel to EF , and therefore the tangent at H is parallel to EF . Therefore



$$E\Lambda = \Lambda\Gamma \quad (1.47),$$

and this is impossible, for

$EG = GF$

Therefore AH is not a diameter of the opposite sections

Therefore $AB \parallel$

PROPOSITION 35

If a diameter in one of the opposite sections bisects some straight line the straight line touching the other section at the end of the diameter will be parallel to the bisected straight line

Let there be the opposite sections A and B and let their diameter AB bisect the straight line CD in section B at E



I say that the tangent to the section at A is parallel to CD

For if possible let DF be parallel to the tangent to the section at A , therefore $DG = GF$ (I. 48)

But also

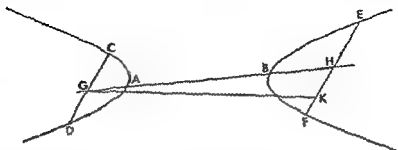
$DE = EC$

Therefore CF is parallel to EG , and this is impossible, for produced it meets it (1.22) Therefore DF is not parallel to the tangent to the section at A nor to any other straight line except CD

PROPOSITION 36

If parallel straight lines are drawn one in each of the opposite sections then the straight line joining their midpoints will be a diameter of the opposite sections

Let there be the opposite sections A and B , and let the straight lines CD and EF be drawn one in each of them, and let them be parallel, and let them both



be bisected at points G and H and let GH be joined

I say that GH is a diameter of the opposite sections

For if not, let GA be one. Therefore the tangent to A is parallel to CD (II 5), and so also to EF . Therefore

$$EA = AF \text{ (I 48)}$$

and thus is impossible since also

$$EH = HF$$

Therefore GA is not a diameter of the opposite sections. Therefore GH ■

PROPOSITION 37

If a straight line not through the center cuts the opposite sections then the straight line joined from its midpoint to the center is a so-called upright diameter of the opposite sections and the straight line drawn from the center parallel to the bisected straight line is a transverse diameter conjugate to it

Let there be the opposite sections A and B , and let some straight line CD not through the center cut the sections A and B and let it be bisected at E , and let Y be the center of the sections, and let YE be joined, and through Y let AB be drawn parallel to CD

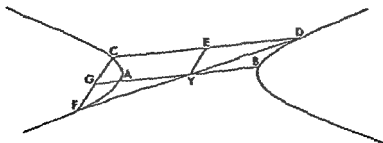
I say that the straight lines AB and YE are conjugate diameters of the sections

For let DY be joined and produced to F , and let CF be joined. Therefore

$$DY = YF \text{ (I 30)}$$

But also

$$DE = EC,$$



therefore EY is parallel to FC . Let BA be produced to G . And since

$$DY = YF,$$

therefore also

$$EY = FG,$$

and so also

$$CG = FG$$

Therefore the tangent at A is parallel to CF (II 5), and so also to EY . Therefore EY and AB are conjugate diameters (I 16)

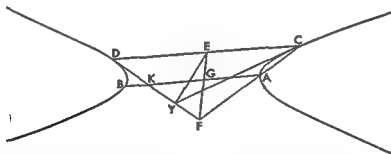
PROPOSITION 38

If two straight lines meeting touch opposite sections the straight line joined from the point of meeting to the midpoint of the straight line joining the points of contact will be a so-called upright diameter of the opposite sections, and the straight line drawn through the center parallel to the straight line joining the points of contact is a transverse diameter conjugate to it

Let there be the opposite sections A and B , and CY and YD touching the sections and let CD be joined and bisected at E , and let EY be joined

I say that the diameter EY is a so-called upright and the straight line drawn through the center parallel to CD is a transverse diameter conjugate to it

For if possible, let EF be a diameter and let F be a point taken at random, therefore DY will meet EF . Let it meet it at F , and let CF be joined, therefore



CF will hit the section (I 32). Let it hit it at A , and through A let AB be drawn parallel to CD . Since then EF is a diameter, and bisects CD , it also bisects the parallels to it (First Def 1 4). Therefore

$$AG = GB$$

And since

$$CE = ED$$

and is on triangle CFD therefore also

$$AG = GK$$

And so also

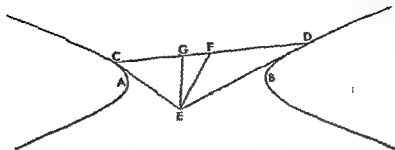
$$GK = GB$$

and thus is impossible. Therefore EF will not be a diameter

PROPOSITION 39

If two straight lines meeting touch opposite sections the straight line drawn through the center and the point of meeting of the tangents bisects the straight line joining the points of contact

Let there be the opposite sections A and B , and let two straight lines CE and



ED be drawn touching A and B , and let CD be joined, and let EF be drawn as a diameter

I say that

$$CF = FD$$

For if not, let CD be bisected at G , and let GE be joined, therefore GE is a diameter (II 38). But EF is also, therefore E is the center (I 31, end). Therefore the point of meeting of the tangents is at the center of the sections and this is absurd (II 32). Therefore CF is not unequal to FD . Therefore equal.

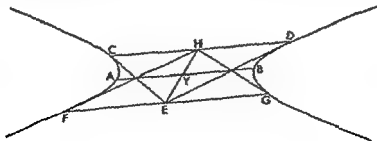
PROPOSITION 40

If two straight lines touching opposite sections meet and through the point of meeting a straight line is drawn, parallel to the straight line joining the points of contact and meeting the sections then the straight lines drawn from the points of meeting to the midpoint of the straight line joining the points of contact touch the sections.

Let there be the opposite sections A and B , and let two straight lines CE and ED be drawn touching A and B , and let CD be joined and through E let FEG be drawn parallel to CD , and let CD be bisected at H , and let FH and HG be joined.

I say that FH and HG touch the sections.

Let EH be joined, therefore EH is an upright diameter and the straight line drawn through the center parallel to CD a transverse diameter conjugate to it (II 38). And let the center Y be taken and let AYB be drawn parallel to CD therefore HE and AB are conjugate diameters. And CH has been drawn



ordinatewise to the second diameter and CE has been drawn touching the section and meeting the second diameter. Therefore the rectangle EY, YH is

equal to the square on the half of the second diameter (I 38); that is to the fourth part of the figure on AB (Second Def 1 10) And since FE has been drawn ordinatewise and FH joined, therefore FH touches the section A (I 38) Likewise then also GH touches section B Therefore FH and HG touch sections A and B

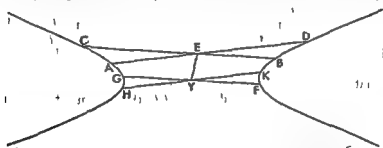
PROPOSITION 41

If in opposite sections two straight lines not through the center cut each other, then they do not bisect each other

Let there be the opposite sections A and B , and in A and B let the two straight lines CB and AD not through the center cut each other at E

I say that they do not bisect each other

For if possible, let them bisect each other, and let Y be the center of the sections and let EY be joined, therefore EY is a diameter (II 37) Let YF be drawn through Y parallel to BC , therefore EF is a diameter and conjugate to

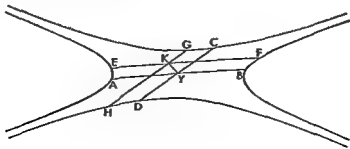


EY (II 37) Therefore the tangent at F is parallel to EY (First Def 1 6) Then for the same reasons with HK drawn parallel to AD the tangent at H is parallel to EY and so also the tangent at F is parallel to the tangent at H , and this is absurd for it has been shown it also meets it (II 31) Therefore the straight lines CB and AD not being through the center do not bisect each other

PROPOSITION 42

If in conjugate opposite sections two straight lines not through the center cut each other, they do not bisect each other

Let there be the conjugate opposite sections A , B , C and D , and in A , B , C



and D let the two straight lines not through the center EF and GH , cut each other at K

I say that they do not bisect each other

For if possible, let them bisect each other, and let the center of the sections be Y , and let AB be drawn parallel to EF and CD to HG , and let AY be joined, therefore AY and AB are conjugate diameters (II 37) Likewise YK and CD are also conjugate diameters. And so also the tangent at A is parallel to the tangent at C , and this is impossible, for it meets it, since the tangent at A cuts the sections A and B (II 19), and the tangent at A sections C and D , it is evident also that their point of meeting is in the place under angle AYC (II 21). Therefore the straight lines EF and GH not being through the center do not bisect each other

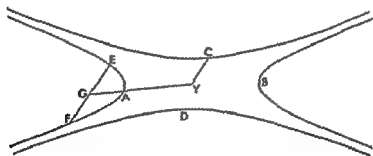
PROPOSITION 43

If a straight line cuts one of the conjugate opposite sections in two points, and through the center one straight line is drawn to the midpoint of the cutting straight line and another straight line is drawn parallel to the cutting straight line, they will be conjugate diameters of the opposite sections

Let there be the conjugate opposite sections A , B , C and D , and let some straight line cut section A at the two points E and F , and let EF be bisected at G , and let Y be center, and let YG be joined, and let CY be drawn parallel to EF

I say that AY and YC are conjugate diameters

For since AY is a diameter and bisects EF , the tangent at A is parallel to



EF (II 5) and so also to CY . Since then they are opposite sections and a tangent has been drawn to one of them A at A and from the center Y one straight line YI is joined to the point of contact and another CY has been drawn parallel to the tangent therefore YA and CY are conjugate diameters for this has been shown before (II 20)

PROPOSITION 44 (PROBLEM)

Given a section of a cone to find a diameter

Let there be the given conic section on which are the points A , B , C , D and E . Then it is required to find a diameter

Let it have been done and let it be CH . Then with DF and EH drawn or drawnwise and produced

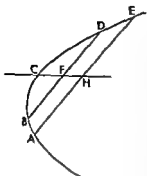
$$DF = FB,$$

and

$$EH = HA \text{ (First Def 1 4)}$$

If then we fix the straight lines BD and EA in position to be parallel, the points H and F will be given. And so HFC will be given in position.

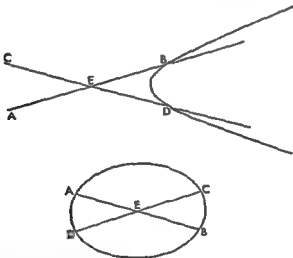
Then it will be constructed (*συμβεησεται*) thus let there be the given conic section on which are the points A, B, C, D and E , and let the straight lines BD and AE be drawn parallel and be bisected at F and H . And the straight line FH joined will be a diameter of the section (First Def 1 4). And in the same way we could also find an indefinite number of diameters.



PROPOSITION 45 (PROBLEM)

Given an ellipse or hyperbola to find the center

And this is evident, for if two diameters of the section, AB and CD are



drawn through (11 44) the point at which they cut each other will be the center of the section, as indicated below.

PROPOSITION 46 (PROBLEM)

Given a section of a cone to find the axis

Let the given section of a cone first be a parabola on which are the points F, C and E . Then it is required to find its axis.

For let AB be drawn as a diameter of it (1 44). If then AB is an axis what was enjoined would have been done but if not let it have been done and let CD be the axis therefore the axis CD is parallel to AB (1 51, end) and bisects the straight lines drawn perpendicular to it (First Def 1 7). And the perpendiculars to CD are also perpendiculars to AB and so CD bisects the perpendic-

ulars to AB . If then I fix EF the perpendicular to AB , it will be given in position, and therefore

$$ED = DF,$$

therefore the point D is given. Therefore through the given point D , CD has been drawn parallel in position to AB , therefore CD is given in position.

Then it will be constructed thus: let there be the given parabola on which are the points F , E and A , and let AB , a diameter of it, be drawn (I 44), and let BE be drawn perpendicular to it and let it be produced to F . If then

$$EB = BF$$

it is evident that AB is the axis (First Def 1 7), but if not, let EF be bisected by D , and let CD be drawn parallel to AB . Then it is evident that CD is the axis of the section, for being parallel to a diameter that is being a diameter (I 51, end), it bisects EF at right angles. Therefore CD has been found as the axis of the given parabola (First Def 1 7).

And it is evident that the parabola has only one axis. For if there is another, as AB , it will be parallel to CD (I 51, end) and it cuts EF , and so it also bisects it (First Def 1 4).

Therefore

$$BE = BF$$

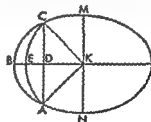
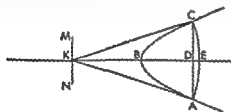
and this is absurd.

PROPOSITION 47 (PROBLEM)

Given an hyperbola or ellipse to find the axis

Let there be the hyperbola or ellipse ABC then it is required to find its axis.

Let it have been found and let it be AD , and A the center of the section, therefore AD bisects the ordinates to itself and at right angles (First Def 1 7).



Let the perpendicular CDA be drawn and let KA and KC be joined. Since then

$$CD = DA,$$

therefore

$$CA = KA$$

If then we fix the given point C , CA will be given. And so the circle described with center A and radius AC will also pass through A and will be given in position. And the section ABC is also given in position, therefore the point A

is given. But the point C is also given therefore CA is given in position. Also $CD = DA$ therefore the point D is given. But also A is given therefore DA is given in position.

Then it will be constructed thus let there be the given hyperbola or ellipse ABC , and let K be taken as its center and let a point C be taken at random on the section and let the circle CEA with center A and radius AC be described, and let CA be joined and bisected at D , and let AC , AD and AK be joined and let DK be drawn through to B

Since then

$$AD = DC$$

and DK is common, therefore the two straight lines CD and DA are equal to the two straight lines AD and DA and

$$\text{base } AK = \text{base } KC$$

Therefore KBD bisects ADC at right angles. Therefore KD is an axis (First Def 1 7)

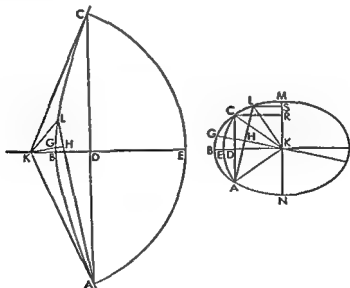
Let MKN be drawn through K parallel to CA therefore MN is the axis of the section conjugate to BK (First Def 1 8)

PROPOSITION 48 (PROBLEM)

Then with these things shown, let it be next in order to show that there are no other axes of the same sections

For if possible let there also be another axis KH . Then in the same way as before with AL drawn perpendicular

$$AH = HL \text{ (First Def 1 4),}$$



and so also

$$AK = KL$$

But also

$$AK = KC$$

therefore

$$AL = KC,$$

and this is absurd

Now that the circle AEC does not hit the section also in another point between the points A , B and C , is evident in the case of the hyperbola, and in the case of the ellipse let the perpendiculars CR and LS be drawn Since then

$$KC = KL,$$

for they are radii, also

$$\text{sq } KC = \text{sq } AL$$

But

$$\text{sq } CR + \text{sq } RK = \text{sq } CK,$$

and

$$\text{sq } AS + \text{sq } SL = \text{sq } LA,$$

therefore

$$\text{sq } CR + \text{sq } RA = \text{sq } AS + \text{sq } SL$$

Therefore

$$\begin{aligned} &\text{difference between sq } CR \text{ and sq } SL = \\ &\text{difference between sq } AS \text{ and sq } RA \end{aligned}$$

Again since

$$\text{rect } MR, RN + \text{sq } RA = \text{sq } AM,$$

and also

$$\text{rect } MS, SN + \text{sq } SA = \text{sq } AM \text{ (Eucl II 5)}$$

therefore

$$\text{rect } MR, RN + \text{sq } RK = \text{rect } MS, SN + \text{sq } SA$$

Therefore

$$\begin{aligned} &\text{difference between sq } SA \text{ and sq } AR = \\ &\text{difference between rect } MR, RN \text{ and rect } MS, SN \end{aligned}$$

And it was shown that

$$\begin{aligned} &\text{difference between sq } SA \text{ and sq } AR = \\ &\text{difference between sq } CR \text{ and sq } SL \end{aligned}$$

therefore

$$\begin{aligned} &\text{difference between sq } CR \text{ and sq } SL = \\ &\text{difference between rect } MR, RN \text{ and rect } MS, SN \end{aligned}$$

And since CR and LS are ordinates

$$\text{sq } CR \text{ rect } MR, RN \text{ sq } SL \text{ rect } MS, SN \text{ (I 21)}$$

But the same difference was also shown for both therefore

$$\text{sq } CR = \text{rect } MR, RN$$

and

$$\text{sq } SL = \text{rect } MS, SN \text{ (Eucl I 16, 17 9)}$$

Therefore the line ICW is a circle, and this is absurd, for it is supposed an ellipse

PROPOSITION 49 (PROBLEM)

Given a section of a cone and a point not within the section to draw from the point a straight line touching the section in one point

Let the given section of a cone first be a parabola whose axis is BD Then it is

required to draw a straight line as prescribed from the given point which is not within the section

Then the given point is either on the line or on the axis or somewhere else outside

Now let it be on the line, and let it be A , and let it have been done, and let it be AE , and let AD be drawn perpendicular, then it will be given in position And

$$BE = BD \text{ (I 35)}$$

and BD is given, therefore BE is also given And the point B is given, therefore E is also given But A also therefore AE is given in position

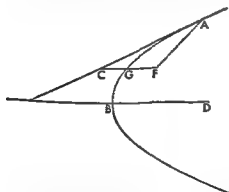
Then it will be constructed thus let AD be drawn perpendicular from A , and let BE be made equal to BD , and let AE be joined Then it is evident that it touches the section (I 33)

Again let the given point E be on the axis and let it have been done and let AE be drawn tangent, and let AD be drawn perpendicular, therefore

$$BE = BD \text{ (I 35)}$$

And BE is given therefore also BD is given And the point B is given, therefore D is also given And DA is perpendicular, therefore DA is given in position Therefore the point A is given But also E , therefore AE is given in position

Then it will be constructed thus let BD be made equal to BE , and from D let DA be drawn perpendicular to ED , and let AE be joined Then it is evident that AE touches (I 33)



And it is evident also that even if the given point is the same as B the straight line drawn from B perpendicular touches the section (I 17)

Then let C be the given point and let it have been done and let CA be it and through C let CF be drawn parallel to the axis that is to BD therefore CF is given in position And from A let AF be drawn ordinatewise to CF then

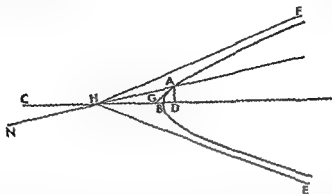
$$CG = FG \text{ (I 35)}$$

And the point G is given therefore F is also given And FA has been

erected ordinatewise that is parallel to the tangent at G (I 32) therefore FA is given in position Therefore A is also given but also C Therefore CA is given in position

It will be constructed thus let CF be drawn through C parallel to BD , and let FG be made equal to CG , and let FA be drawn parallel to the tangent at G (above) and let AC be joined It is evident then that this will do the problem (I 33)

Again let it be an hyperbola whose axis is DBC and center H , and asymptotes HE and HF . Then the given point will be given either on the section or



on the axis or within angle EHF or in the adjacent place or on one of the asymptotes containing the section or in the place between the straight lines containing the angle vertical to angle EHF .

Let the point A first be on the section and let it have been done, and let AG be tangent and let AD be drawn perpendicular and let BC be the transverse side of the figure, then

$$CD : DB :: CG : GB \quad (\text{I } 36)$$

And the ratio of CD to DB is given, for both the straight lines are given therefore also the ratio of CG to GB is given. And BC is given, therefore point G is given. But also A therefore AG is given in position.

It will be constructed thus: let AD be drawn perpendicular from A , and let

$$CG : GB :: CD : DB$$

and let AG be joined. Then it is evident that AG touches the section (I 34).

Then again let the given point G be on the axis and let it have been done, and let AG be drawn tangent, and let AD be drawn perpendicular. Then for the same reasons

$$CD : DB :: CG : GB \quad (\text{I } 36)$$

And BC is given therefore the point D is given. And DA is perpendicular, therefore DA is given in position. And also the section is given in position therefore the point A is given. But also G therefore AG is given in position.

Then it will be constructed thus: let the other things be supposed the same, and let it be contrived that

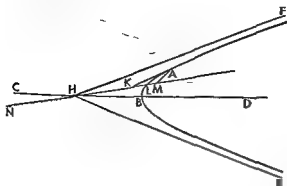
$$CG : GB :: CD : DB$$

and let DA be drawn perpendicular and let AC be joined. Then it is evident that AG does the problem (I 34) and that from G another tangent to the section could be drawn on the other side.

With the same things supposed let the given point K be in the place inside angle EHF , and let it be required to draw a tangent to the section from K . Let it have been done and let it be KI and let KH be joined and produced, and let HN be made equal to LH therefore they are all given. Then also LN will be given. Then let AM be drawn ordinatewise to MN then also

$$AK : KI :: AN : NI$$

And the ratio of NA to AL is given, therefore also the ratio of NM to ML is given. And the point L is given, therefore also M is given. And MA has been erected parallel to the tangent at L therefore MA is given in position. And also the section ALB is given in position, therefore the point A is given. But K is also given, therefore AK is given.



Then it will be constructed thus: let the other things be supposed the same and the given point K , and AK be joined and produced, and

let HN be made equal to HL and let it be contrived that

$$NK : KL :: NM : ML$$

and let MA be drawn parallel to the tangent at L (above), and let KA be joined, therefore KA touches the section (I 34).

And it is evident that a tangent to the section could also be drawn to the other side.

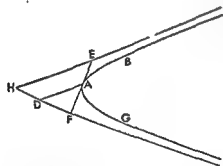
With the same things supposed let the given point F be on one of the asymptotes containing the section, and let it be required to draw from F a tangent to the section. And let it have been done, and let it be FAE , and through A let AD be drawn parallel to EH then

$$DH = DF$$

since also

$$FA = AE \text{ (II 3)}$$

And FH is given, therefore also point D is given. And through the given point D DA has been drawn parallel in position to EH , therefore DA is given in position. And the section is also given in position, therefore the point A is given. But F is also given, therefore the straight line FAE is given in position.



Then it will be constructed thus: let there be the section AB and asymptotes EH and HF , and the given point F on one of the asymptotes containing the section and let FH be bisected at D and through D

let DA be drawn parallel to HE and let FA be joined. And since

$$FD = DH,$$

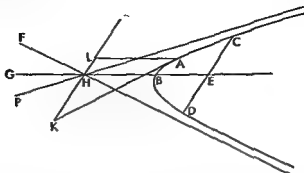
therefore also

$$FA = AE$$

And so by things shown before, the straight line FAE touches the section (II 9).

With the same things supposed let the given point be in the place under the

angle adjacent to the straight lines containing the section, and let it be Λ , it is required then to draw a tangent to the section from Λ . And let it have been



done and let it be ΛK and let KH be joined and produced, then it will be given in position. If then a given point C is taken on the section, and through C , CD is drawn parallel to KH , it will be given in position. And if CD is bisected at E and HE is joined and produced, it will be in position, a diameter conjugate to KH (First Def 1 6). Then let HG be made equal to BH and through G let AL be drawn parallel to BH , then because KL and BG are conjugate diameters and AK a tangent and AL a straight line drawn parallel to BG therefore rectangle $KH \cdot HL$ is equal to the fourth part of the figure on BG (1 38). Therefore rectangle $KH \cdot HL$ is given. And KH is given therefore HL is also given. But it is also given in position, and the point H is given, therefore L is also given. And through L LA has been drawn parallel in position to BG , therefore LA is given in position. And the section is also given in position, therefore the point A is given. But also Λ therefore AK is given in position.

Then it will be constructed thus: let the other things be supposed the same and let the given point Λ be in the aforesaid place and let KH be joined and produced, and let some point C be taken and let CD be drawn parallel to KH , and let CD be bisected by E and let EH be joined and produced and let HG be made equal to BH therefore GB is a transverse diameter conjugate to KHL (First Def 1 6). Then let rectangle $KH \cdot HL$ be made equal to the fourth of the figure on BG and through L let LA be drawn parallel to BG and let AK be joined then it is clear that AK touches the section by the converse of the theorem (1 38).

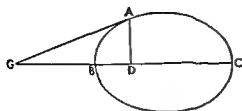
And if it is given in the place between the straight lines FHI and HIP the problem is impossible. For the tangent will cut CHI . And so it will meet both FHI and HIP and this is impossible by the things shown in the thirty first theorem of the first book (1 31) and in the third of this book (11 3).

With the same things supposed let the section be an ellipse and the given point A on the section and let it be required to draw from A a tangent to the section. Let it have been done and let it be AG and let AD be drawn from A ordinatewise to the axis BC then the point D will be given, and

$$CD : DB :: CG : CB \quad (1 \text{ 36})$$

And the ratio of CD to DB is given therefore the ratio of CG to GB is also given. Therefore the point G is given. But also A therefore AG is given in position.

Then it will be constructed thus let AD be drawn perpendicular, and let CG GB CD DE ,

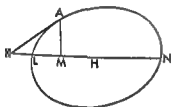


and let AG be joined Then it is evident that AG touches, as also in the case of the hyperbola (1 34)

Then again let the given point be K , and let it be required to draw a tangent Let it have been done, and let it be KA , and let the straight line KLH be joined to the center H and produced to N , then

it will be given in position And if AM is drawn ordinatewise, then

NK KI NM ML (1 36)



And the ratio of NK KL is given, therefore the ratio of MN to LM is also given Therefore the point M is given And MA has been erected ordinatewise, for it is parallel to the tangent at L , therefore MA is given in position Therefore the point A is given But also K , therefore KA is given in position

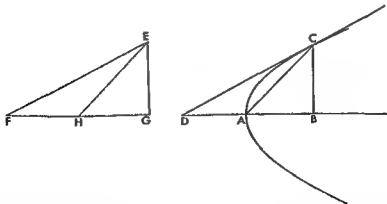
And the construction (*συνθεσις*) is the same as for the preceding

PROPOSITION 50 (PROBLEM)

Given the section of a cone to draw a tangent which will make with the axis, on the same side as the section, an angle equal to a given acute angle

Let the section of a cone first be a parabola whose axis is AB then it is required to draw a tangent to the section which will make with the axis AB , on the same side as the section an angle equal to the given acute angle

Let it have been done and let it be CD therefore angle BDC is given Let BC



be drawn perpendicular then the angle at B is also given Therefore the ratio of DB to BC is given But the ratio of BD to BA is given therefore also the ratio of AB to BC is given And the angle at B is given therefore angle BAC

is also given. And it is in position with respect to BA and the given point A , therefore CA is given in position. And the section is also given in position, therefore the point C is given. And CD touches, therefore CD is given in position.

Then the problem will be constructed thus: let the given section of a cone first be a parabola whose axis is AB , and the given acute angle, angle EFG , and let some point E be taken on EF , and let EG be drawn perpendicular, and let FG be bisected by H , and let HE be joined, and let angle BAC be constructed equal to angle GHE , and let BC be drawn perpendicular, and let AD be made equal to BA , and let CD be joined. Therefore CD is tangent to the section (I 33).

I say then that

$$\text{angle } CDB = \text{angle } EFG$$

For since

$$FG \quad GH \quad DB \quad BA$$

and

$$HG \quad GE \quad AB \quad BC,$$

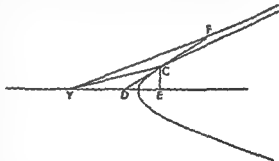
therefore ex æquali

$$FG \quad GE \quad DB \quad BC,$$

And the angles at G and B are right angles, therefore

$$\text{angle at } F = \text{angle at } D$$

Let the section be an hyperbola and let it have been done and let CD be tangent and let the center of the section I be taken, and let CI be joined and let CE be perpendicular: therefore the ratio of rectangle $YE \cdot ED$ to the square on CE is given for it is the same as the transverse to the upright (I 37). And the ratio of the square on CE to the square on ID is given, for each of the rectangles $CD \cdot DE$ and $DE \cdot EC$ is given. Therefore the ratio of rectangle $YE \cdot ED$ to the square on ED is given and so also the ratio of YE to ED is given. And the angle at E is given, therefore the angle at I is also given. Then some straight line CY has been drawn across in position with respect to the straight line YE and to the given point I at a given angle: therefore CI is given in position. And the section



is also given in position: therefore the point C is given. And CD has been drawn across as tangent: therefore CD is given in position.

Let the asymptote to the section YF be drawn: therefore CD produced will meet the asymptote (II 3). Let it meet it at F . Therefore

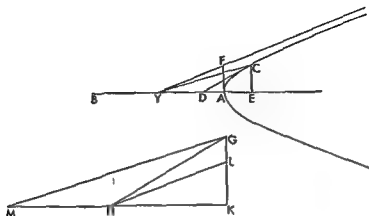
$$\text{angle } FDE > \text{angle } FID$$

Therefore, for the construction the given acute angle will have to be greater than half the angle contained by the asymptotes.

Then the problem will be constructed thus: let there be the given hyperbola whose axis is AB , and asymptote YF , and the given acute angle KJG greater than angle AYF , and let

angle $\angle HLA = \angle A\gamma F$,

and let AF be drawn from A perpendicular to AB , and let some point G be



taken on GH , and let GA be drawn from it perpendicular to HK . Since then

angle $\angle F\gamma A = \angle LHA$,

and also the angles at A and A are right, therefore

$$\frac{\gamma A}{HK} = \frac{AF}{AL} = \frac{HK}{AG}$$

therefore also

$$\gamma A \cdot AF > HA \cdot KG$$

And so also

$$\text{sq } \gamma A + \text{sq } AF > \text{sq } HA + \text{sq } KG$$

But

$$\text{sq } \gamma A + \text{sq } AF = \text{transverse upright (II 1)}$$

therefore also

$$\text{transverse upright} > \text{sq } HA + \text{sq } AG$$

If then we shall contrive that

$$\text{sq } \gamma A + \text{sq } AF \text{ some other sq } AG,$$

it will be greater than the square on HA . Let it be the rectangle MK, KH and let GM be joined. Since then

$$\text{sq } MA > \text{rect } MK, KH,$$

therefore

$$\begin{aligned} \text{sq } MA + \text{sq } AG &> \text{rect } MA, KH + \text{sq } AG \\ &> \text{sq } \gamma A + \text{sq } AF \end{aligned}$$

And if we shall contrive that

$$\text{sq } MA + \text{sq } AG \text{ sq } \gamma A \text{ some other}$$

it will be to \equiv magnitude less than the square on AF and the straight line joined from γ to the point taken will make similar triangles and therefore

$$\angle FYA > \angle CMK$$

Let angle $\angle AYC$ be made equal to angle $\angle GMK$ therefore γC will cut the section (II 2) Let it cut it at C , and from C let CD be drawn tangent to the section (II 49) and CE drawn perpendicular, therefore triangle $CYE \equiv$ similar to triangle GMA . Therefore

$$\text{sq } \gamma E + \text{sq } EC = \text{sq } MA + \text{sq } AG$$

But also

transverse upright rect IF, ED sq EC (r 37),

and

transverse upright rect MA, AH sq AG

And inversely

sq CE rect IE, ED sq GA rect MA, AH ,

therefore *ex aequali*

sq IF rect IE, ED sq MA rect MA, AH

And therefore

IE, ED MA, AH

But also we had

CE, EI GA, AM ,

therefore *ex aequali*

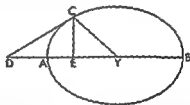
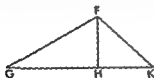
CE, ED GA, AH

And the angles at E and A are right angles, therefore

angle at D = angle GHA

Let the section be an ellipse whose axis is AB . Then it is required to draw a tangent to the section which with the axis will contain, on the same side as the section, an angle equal to the given acute angle

Let it have been done and let it be CD . Therefore angle CDA is given. Let CE be drawn perpendicular, therefore the ratio of the square on DE to the square on EC is given. Let Y be the center of the section, and let OY be joined. Then the ratio of the square on CE to the rect angle DE, EY is given for it is the same as the ratio of the upright to the transverse (r 37) and therefore the ratio of the square on DE to rectangle DE, EY is given, and therefore the ratio of DE to EY is given. And of DE to EC therefore also the ratio of CE to EY is given. And the angle at E is right, therefore the angle at Y is given. And it is given with respect to a straight line given in position and to a given point, therefore the point C is given. And from the given point C let CD be drawn tangent, therefore CD is given in position.



Then the problem will be constructed thus: let there be the given acute angle FGH and let some point I be taken on FG , and let FH be drawn perpendicular and let it be contrived that

upright transverse sq FH rect GH, HA ,

and let AF be joined and let I be the center of the section and let angle AIC be constructed equal to angle GAF and let CD be drawn tangent to the section (II 49)

I say that CD does the problem that is

angle CDE = angle FGH

For since

$$YE \cdot EC = KH \cdot FH,$$

therefore also

$$\text{sq } YE \text{ sq } EC = \text{sq } KH \text{ sq } FH$$

But also

$$\text{sq } EC \text{ rect } DE, EY = \text{sq } FH \text{ rect } KH, HG,$$

for each is the same ratio as that of the upright to the transverse (I 37, and above) And *ex aequali*, therefore

$$\text{sq } YE \text{ rect } DE EY = \text{sq } KH \text{ rect } KH, HG$$

And therefore

$$YE \cdot ED = KH \cdot HG$$

But also

$$YE \cdot EC = KH \cdot FH,$$

ex aequali, therefore

$$DE \cdot EC = HG \cdot FH$$

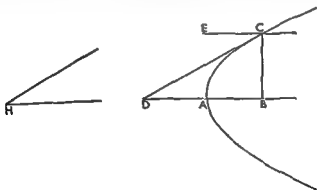
And the sides about the right angles are proportional therefore
angle $CDF = \text{angle } FGH$

Therefore CD does the problem

PROPOSITION 51 (PROBLEM)

Given a section of a cone to draw a tangent which with the diameter drawn through the point of contact will contain an angle equal to a given acute angle

Let the given section of a cone first be a parabola whose axis is AB and the given angle H then it is required to draw a tangent to the parabola which with



the diameter from the point of contact will contain an angle equal to the angle at H

Let it have been done and let CD be drawn a tangent making with the diameter EC drawn through the point of contact angle ECD equal to angle H and let CD meet the axis at D (I 24) Since then $AD \parallel EC$ (I 51, end)

$$\text{angle } ADC = \text{angle } ECD$$

But angle ECD is given for it is equal to angle H , therefore angle ADC is also given

Then it will be constructed thus let there be a parabola whose axis is AB ,

and the given angle H . Let CD be drawn a tangent to the section making with the axis the angle ADC equal to angle H (II 50), and through C let EC be drawn parallel to AB . Since then

$$\text{angle } H = \text{angle } ADC,$$

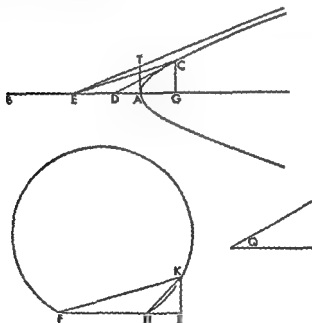
and

$$\text{angle } ADC = \text{angle } ECD,$$

therefore also

$$\text{angle } H = \text{angle } ECD$$

Let the section be an hyperbola whose axis is AB and center E , and asymptote ET , and the given acute angle Q , and let CD be tangent and let CF be joined doing the problem, and let CG be drawn perpendicular. Therefore the ratio of the transverse to the upright is given, and so also the ratio of rectangle EG, GD to the square on CG (I 37). Then let some given straight line FI be laid out, and on it let there be described a segment of a circle admitting an angle equal to angle Q (Fuel III 33) therefore it will be greater than a semi circle (Fuel III 31). And from some point K of those on the circumference let KL be drawn perpendicular making



$$\text{rect } FL \cdot IH \text{ sq } LK \quad \text{transverse upright}$$

and let FA and KH be joined. Since then

$$\text{angle } FKH = \text{angle } LCD,$$

but also

$$\text{rect } EG \cdot GD \text{ sq } GC \quad \text{transverse upright,}$$

and

$$\text{rect } FL \cdot LH \text{ sq } LK \quad \text{transverse upright,}$$

therefore triangle FL is similar to triangle ECG , and triangle FHA to triangle ECD ¹ And so

$$\text{angle } HFA = \text{angle } CED$$

Then it will be constructed thus let there be the given hyperbola AC , and axis AB , and center E , and given acute angle Q and let the given ratio of the transverse to the upright be the same as YZ to YW , and let WZ be bisected at

U and let a given straight line FH be laid out, and on it let there be described a segment of a circle, greater than semicircle and admitting an angle equal to angle Q (Eucl III 31, 33), and let it be FkH , and let the center of the circle N be taken, and from N let NO be drawn perpendicular to FH and let NO be cut at P in the ratio of UW to WY and through P let PK be drawn parallel to FH , and from K let KL be drawn perpendicular to FH produced and let FK and KH be joined and let LK be produced to M , and from N let NA be drawn perpendicular to it, therefore it is parallel to FH And therefore

$$NP \text{ } PO \text{ or } UW \text{ } WY \text{ } XA \text{ } AL$$

And doubling the antecedents

$$ZW \text{ } WY \text{ } MA \text{ } KL,$$

componendo

$$ZY \text{ } YW \text{ } ML \text{ } LK$$

But

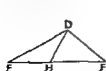
$$ML \text{ } LA \text{ } \text{rect } ML \text{ } LA \text{ } \text{sq } LK$$

therefore

$$ZY \text{ } YW \text{ } \text{rect } ML, LA \text{ } \text{sq } LK \text{ } \text{rect } PL \text{ } LH \text{ } \text{sq } LA \text{ (Eucl III 36)}$$

¹Pappus in lemma IX to this book Let triangle ABC be similar to triangle DEF and triangle AGB to DEH the result is

$$\text{rect } BC \text{ } CG \text{ } \text{sq } CA \text{ } \text{rect } EF \text{ } FH \text{ } \text{sq } DF$$



For since because of similarity

$$\text{whole angle } A = \text{whole angle } D$$

$$\text{angle } BAG = \text{angle } EDH$$

$$\text{remaining angle } GAC = \text{remaining angle } HDF$$

$$\text{angle } C = \text{angle } F$$

$$GC \text{ } CA \text{ } HF \text{ } FD$$

$$BC \text{ } CA \text{ } EF \text{ } FD$$

and
therefore

But also

therefore

But also

therefore also compounded ratio is the same with compounded. Therefore

$$\text{rect } BC \text{ } CG \text{ } \text{sq } CA \text{ } \text{rect } EF \text{ } FH \text{ } \text{sq } FD$$

But

ZI II transverse upright,

therefore also

rect FL, LH sq LK transverse upright

Then let AT be drawn from A perpendicular to AB Since then
sq EA sq AT transverse upright (ii 1),

and also

transverse upright rect FL, LH sq LK ,

and

sq FL sq LK > rect FI LH sq LK ,

therefore also

sq FL sq IA > sq EA sq AT

And the angles at A and L are right angles, therefore

angle F < angle E

Then let angle AEC be constructed equal to angle LFA , therefore EC will meet
the section (ii 2) I let it meet it at C Then let CD be drawn tangent from C
(ii 49), and let CG be drawn perpendicular, then

transverse upright rect FG, GD sq CG (i 37)

Therefore also

rect FL, LH sq LK rect EG, GD sq CG

therefore triangle KFL is similar to triangle ECG and triangle KHL to tri-
angle CGD , and triangle KFH to triangle CED And so

angle ECD = angle FHL = angle Q

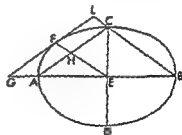
And if the ratio of the transverse to the upright is equal to equal, AI
touches the circle FAH (Eucl iii 37), and the straight line joined from the
center to A will be parallel to FH and itself will do the problem

PROPOSITION 52

If a straight line touches an ellipse making an angle with the diameter drawn
through the point of contact it is not less than the angle adjacent to the one con-
tained by the straight lines deflected at the middle of the section

Let there be an ellipse whose axes are AB and CD and center E and let AB
be the major axis and let the straight line
 GFL touch the section and let AC CB and
 FE be joined and let BC be produced to L

I say that angle LFE is not less than angle
 LCA



For FE is either parallel to LB or not

Let it first be parallel and

$AF = EB$,

therefore also

$AH = HC$

And FE is a diameter therefore the tangent at F is parallel to AC (ii 6) But
also FF is parallel to LB therefore $FHCL$ is a parallelogram and therefore
angle IFH = angle ICH

And since AE and FB are each greater than FC angle ACB is obtuse there-
fore angle ICA is acute And so also angle LFE And therefore angle GPE is
obtuse

Then let FE not be parallel to LB , and let FA be drawn perpendicular,

therefore LBE is not equal to angle FEA But
 $\text{rt angle at } E = \text{rt angle at } A,$

therefore it is not true that

$$\text{sq } BE \text{ sq } EC \text{ sq } EA \text{ sq } AF$$

But

$$\text{sq } BE \text{ sq } EC \text{ rect } AE, EB \text{ sq } EC \text{ transverse upright (I 21)}$$

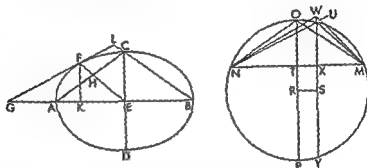
and

$$\text{transverse upright rect } GA, AE \text{ sq } AF \text{ (I 37)}$$

Therefore it is not true that

$$\text{rect } GA, AE \text{ sq } AF \text{ sq } AE \text{ sq } AF$$

Therefore GK is not equal to AE Let there be laid out a segment of a circle



MUN admitting an angle equal to angle ACB (Eucl III 33) and angle ACB is obtuse therefore MUN is a segment less than a semicircle (Eucl III 31) Then let it be contrived that

$$GA \cdot AE = NX \cdot XV$$

and from X let UXY be drawn at right angles and let NU and UM be joined, and let MN be bisected at T and let OTP be drawn at right angles, therefore it is a diameter. Let the center be R and from it let RS be drawn perpendicular, and ON and OM be joined. Since then

$$\text{angle } MON = \text{angle } ACB$$

and AB and MN have been bisected the one at E and the other at T and the angles at E and T are right angles therefore triangles OTN and BEC are similar. Therefore

$$\text{sq } TN \text{ sq } TO \text{ sq } BE \text{ sq } EC$$

And since

$$TR < TX$$

and

$$RO > SU$$

therefore

$$RO \cdot TR > SU \cdot SX$$

and *conterendo*

$$RO \cdot OT < SU \cdot UX$$

And doubling the antecedents therefore

$$PO \cdot TO < YU \cdot UX$$

And *separando*

$$PT \cdot TO < YX \cdot UX$$

But

$PT \perp O$ sq TN sq TO sq BE sq EC transverse upright (i 21),
and

transverse upright rect GA, AE sq AF (i 37),

therefore

$$\begin{aligned} \text{rect } GA, AE \text{ sq } AF &< YX \text{ XU} \\ &< \text{rect } YX, XU \text{ sq } XU \\ &< \text{rect } NY, YW \text{ sq } XU \end{aligned}$$

If then we contrive it that

rect GA, AE sq AF rect MA, XN some other,
it will be greater than the square on AU Let it be to the square on AW Since
then

$$GK \cdot KE = NX \cdot XM,$$

and AF and YW are perpendicular and

$$\text{rect } GA, AF \text{ sq } AF = \text{rect } MY, YN \text{ sq } YW,$$

therefore

$$\text{angle } GFE = \text{angle } MYN$$

Therefore

$$\text{angle } MUN \text{ or angle } ACB > \text{angle } GFE,$$

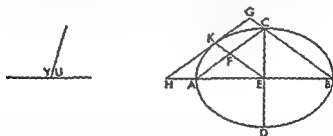
and the adjacent angle LFH is greater than angle LCH

Therefore angle LFH is not less than angle LCH

PROPOSITION 53 (PROBLEM)

Given an ellipse to draw a tangent which will make with the diameter drawn through the point of contact an angle equal to a given acute angle then it is required that the given acute angle be not less than the angle adjacent to the angle contained by the straight lines deflected at the middle of the section.

Let there be the given ellipse whose major axis is AB and minor axis CD , and center E , and let AC and CB be joined and let angle U be the given angle



not less than angle ACG , and so also angle ACB is not less than angle Y

Therefore angle U is either greater than or equal to angle ACG

Let it first be equal and through E let EA be drawn parallel to BC and through A let AH be drawn tangent to the section (ii 49) Since then

$$AE = EB$$

and

$$AE \parallel EB \quad AF \parallel FC,$$

therefore

$$AF = FC$$

And AE is a diameter therefore the tangent to the section at A , that is HAG , is parallel to CA (ii 6) And also EA is parallel to GB , therefore $AFCG$ is a parallelogram, and therefore

$$\text{angle } GKF = \text{angle } GCF$$

And angle GCF is equal to the given angle that is U , therefore also
 $\text{angle } GKE = \text{angle } U$

Then let

$$\text{angle } U > \text{angle } ACB,$$

then inversely

$$\text{angle } Y < \text{angle } ACB$$

Let a circle be laid out and let a segment be taken from it and let it be MNP admitting an angle equal to angle Y , and let MP be bisected at O , and from O let NOR be drawn at right angles to MP , and let NM and NP be joined therefore

$$\text{angle } MNP < \text{angle } ACB$$

But

$$\text{angle } MNO = \text{half angle } MNP,$$

and

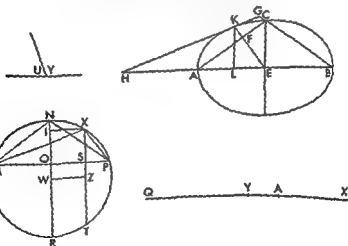
$$\text{angle } ACE = \text{half angle } ACB,$$

therefore

$$\text{angle } MNO < \text{angle } ACE$$

And the angles at E and O are right angles, therefore

$$AE \cdot EC > OM \cdot ON$$



And so also

$$\text{sq } AE \cdot EC > \text{sq } MO \cdot NO$$

But

$$\text{sq } AE = \text{rect } AE \cdot EB,$$

and

$$\text{sq } MO = \text{rect } MO \cdot OP = \text{rect } NO \cdot OR \text{ (Eucl. iii 3.)}$$

therefore

$$\text{rect } AE \cdot EB \cdot \text{sq } EC \text{ or transverse upright (i 21)} > PO \cdot OR$$

Then let it be that

transverse upright $QA' \quad I'Y'$,
and let QY' be bisected at Y' Since then
transverse upright $> RO \quad ON$,
also

$$QA' \quad A'Y' > RO \quad ON$$

And *componendo*

$$QY' \quad Y'A' > RV \quad NO$$

Let the center of the circle be W , and so also

$$Y'Y' \quad Y'A' > WN \quad NO$$

And *separando*

$$A'Y' \quad A'Y' > WO \quad ON$$

Then let it be contrived that

$$A'Y' \quad A'X' \quad WO \text{ less than } ON$$

such as IO , and let YI' and YT and WZ be drawn parallel
Therefore

$$A'Y' \quad A'Y' \quad WO \quad OI \quad ZS \quad SY,$$

and *componendo*

$$Y'Y' \quad Y'A' \quad ZY \quad YS$$

And doubling the antecedents

$$QY' \quad Y'A' \quad TX \quad YS$$

And *separando*

$$QA' \quad A'X' \text{ or transverse upright } TS \quad SX$$

Then let MY and XP be joined, and let angle AEK be constructed on straight line AE at point E equal to angle MPY and through K let KN be drawn touching the section (in 49), and let KL be dropped ordinatewise Since then
angle MPY = angle AEK ,

and

$$\text{rt angle at } S = \text{rt angle at } L$$

therefore triangle YSP is equiangular with triangle KEL

And

$$\text{transverse upright } TS \quad SY \quad \text{rect } TS \quad SY$$

$$\text{sq } SX \quad \text{rect } MY \quad SP \quad \text{sq } SY$$

therefore triangle KLE is similar to triangle $SYSP$ and triangle MXP to triangle KHE and therefore

$$\text{angle } MXP = \text{angle } HKE$$

But

$$\text{angle } MYP = \text{angle } MNP = \text{angle } Y$$

therefore also

$$\text{angle } HKE = \text{angle } Y$$

And therefore

$$\text{adjacent angle } GKE = \text{adjacent angle } U$$

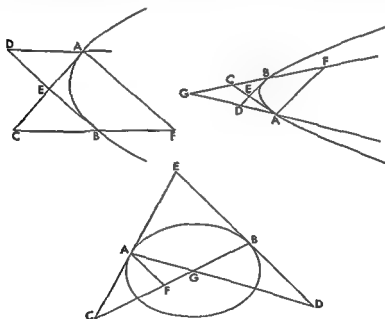
Therefore GH has been drawn across tangent to the section and making with the diameter KE drawn through the point of contact angle GKE equal to the given angle U ; and this it was required to do

BOOK THREE

PROPOSITION 1

If straight lines touching a section of a cone or circumference of a circle, meet and diameters are drawn through the points of contact meeting the tangents, the resulting vertically related triangles will be equal

Let there be the section of a cone or circumference of a circle AB and let AC and BD , meeting at E , touch AB , and let the diameters of the section CB and



DA be drawn through A and B meeting the tangents at C and D

I say that

$$\text{trgl } ADE = \text{trgl } EBC$$

For let AF be drawn from A parallel to BD therefore it has been dropped ordinatewise (1 32) Then in the case of the parabola

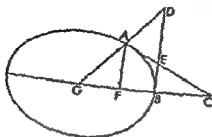
$$\text{pllg } ADBF = \text{trgl } ACF \text{ (1 42)}$$

and with the common area $AEBF$ subtracted

$$\text{trgl } ADE = \text{trgl } CBE$$

And in the case of the others let the diameters meet at center G Since then

AF has been dropped on latitude and longitude
rect FG, GC = sq EG (17)



Therefore

therefore also

FG GB BG GC,

But

$$FG \cong GC \quad \text{sq } FG \quad \text{sq } GC \text{ (Euc II 9)}$$

and

sq FG sq GB trgl AGF trgl DOB (Ex)

therefore also

FG GC trgl AGF trgl AGC,

Therefore

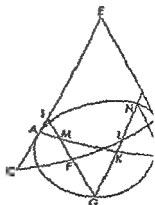
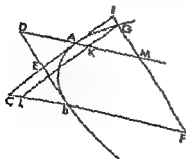
trgl AGF trgl AGC trgl AGF trgl AGC

Let the common area $DOBE$ be subtracted, therefore as re

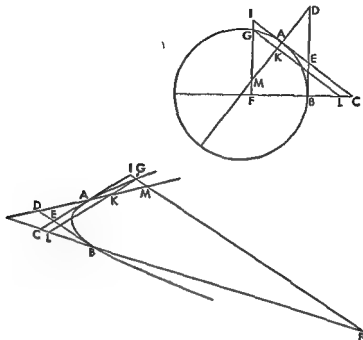
Proposition 2

PROPOSITION 2

For let there be a section of a cone or circumference of a circle



AEC and BED be tangents, and AD and BC diameters, and let some point G



be taken on the section and GKL and GMF be drawn parallel to the tangents
I say that

| | | |
| --- | --- | --- |
| trgl | AIM=quadr | CLGI |

For triangle GKM has been shown equal to quadrilateral AL (1 42, 43), let the common quadrilateral IK be added or subtracted, and

| | |
| --- | --- |
| trgl AIM = quadr CG¹ | |

PROPOSITION 3

With the same things supposed, if two points are taken on the section or circumference of a circle and through them parallels to the tangents are drawn as far as the diameters, the quadrilaterals produced by the straight lines drawn and standing on the diameters as bases are equal to each other

For let there be the section and tangents and diameters as said before and let two points at random F and G be taken on the section, and through F let the straight lines FHL and $NFIM$ be drawn parallel to the tangents and through G the straight lines GYO and HPR

I say that

$$\text{quadr } LG \approx \text{quadr } MH$$

¹Eutocius commenting gives the proof for another and important case. It must be remarked that if the point G is taken between A and B so that the parallels are for instance

and

$$\text{quadr } LN = \text{quadr } RN$$

For since it has already been shown that

$$\text{trgl } RPA = \text{quadr } CG \text{ (iii 2),}$$

and

$$\text{trgl } AMI = \text{quadr } CF \text{ (iii 2),}$$

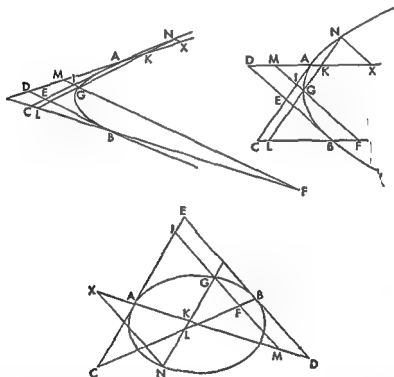
and

$$\text{trgl } RPA = \text{trgl } AMI + \text{quadr } PM,$$

therefore also

$$\text{quadr } CG = \text{quadr } CF + \text{quadr } PM,$$

MI and *LK* one must draw *IA* to the section at *A* for instance and through *N* draw



infer
the
the

AN parallel to *BD* for by what was said in the forty ninth and fiftieth theorems of the first book (i 49 50) and in the notes to them

$$\text{trgl } ANA = \text{quadr } AC$$

But triangle *ANA* is similar to triangle *AMC* because *MG* is parallel to *AN* but it is also equal to it because *AC* is a tangent and *CV* is parallel to it and *MY* is a diameter and

$$GA = AN$$

Since then

$$\text{trgl } ANA = \text{quadr } AC = \text{trgl } AMG$$

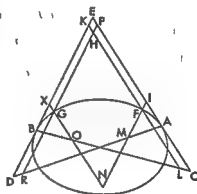
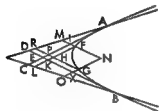
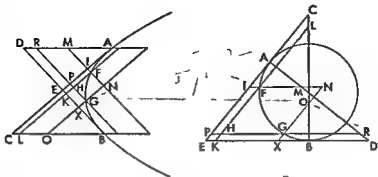
with the common quadrilateral *AC* subtracted as remainders

$$\text{trgl } ANM = \text{quadr } CF$$

It will be noticed that, just as in the second note to i 50 the quadrilateral *CG* is to be considered as the difference between the triangles *CIF* and *CFI*

and so

$$\text{quadr } CG = \text{quadr } CH + \text{quadr } RF$$



be taken

I say

For the common quadrilateral CH be subtracted therefore as remainders

$$\text{quadr } LG = \text{quadr } HM$$

$$\text{quadr } LN = \text{quadr } RN$$

PROPOSITION 4

If two straight lines touching opposite sections meet each other, and diameters are drawn through the points of contact meeting the tangents then the triangles at the tangents will be equal

Let there be the opposite sections A and B and let the tangents to them AC and BC meet at C and let D be the center of the sections and let AB and CD be joined and CD produced to E and let DA and BD also be joined and produced to F and G

I say that

$$\text{trgl } AGD = \text{trgl } BDF,$$

and

$$\text{trgl } ACF = \text{trgl } BCC$$

For let HL be drawn through H tangent to the section, therefore it is parallel to AG (I 44 note) And since

$$AD = DH \text{ (I 30),}$$

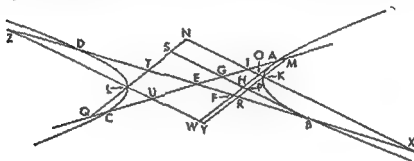
trgl $AIN = \text{quadr } AF$ (iii 2)¹

PROPOSITION 7

With the same things supposed, if points are taken on each of the sections, and from them parallels to the tangents are drawn meeting the tangents and the diameters, then the quadrilaterals produced by the straight lines drawn and standing on the diameters as bases will be equal to each other

For let the aforesaid things be supposed and let points K and L be taken on both sections and through them let $MAPI$ and $NSTLQ$ be drawn parallel to AF and $NIOKX$ and $WULZ$ parallel to BG

I say that what was said in the enunciation will be so



For since

$$\text{trgl } AOI = \text{quadr } RO \text{ (iii 2),}$$

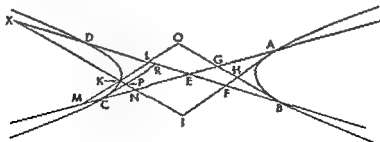
¹ Another and important case where the point A falls between C and D is given by Eutocius in his commentary to this proposition. It is as follows: and let CPR be drawn tangent to the section then it is evident that it is parallel to AF and VL (i 44 note) since it has been shown in the second theorem (iii 2) in the figure of the hyperbola

$$\text{trgl } PNC = \text{quadr } LP \text{ (iii 2 note)}$$

let the common quadrilateral WLP be added therefore

$$\text{trgl } WLN = \text{quadr } MLRC$$

Let there be added the common triangle CRE which is equal to triangle AEF by i 44 (and i 30) therefore



$$\text{whole trgl } MEL = \text{trgl } WLN + \text{trgl } AEF$$

With common triangle AEF subtracted as remainders

$$\text{trgl } LEF = \text{quadr } AIFN$$

Let the common quadrilateral $FFNI$ be added therefore in whole

$$\text{trgl } AIN = \text{quadr } AIFN$$

And likewise also

$$\text{trgl } BOL = \text{quadr } AIFN$$

let the quadrilateral EO be added to both therefore
whole $\text{trgl } AEF = \text{quadr } KE$

But also

$$\text{trgl } BGE = \text{quadr } LE \text{ (III 5, note),}$$

and

$$\text{trgl } AEF = \text{trgl } BGE \text{ (III 1),}$$

therefore

$$\text{quadr } LE = \text{quadr } IARE$$

Let the common quadrilateral NE be added, therefore as wholes
whole $\text{quadr } TH = \text{quadr } IL$,

and also

$$\text{quadr } KU = \text{quadr } RL$$

PROPOSITION 8

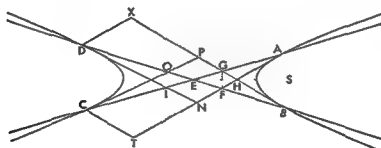
With the same things supposed, instead of K and L let there be taken the points C and D at which the diameters hit the sections and through them let the parallels to the tangents be drawn

I say that

$$\text{quadr } DG = \text{quadr } FC$$

and

$$\text{quadr } XI = \text{quadr } OT$$



For since it was shown

$$\text{trgl } AGH = \text{trgl } HBF \text{ (III 1)}$$

and the straight line from A to B is parallel to the straight line from G to F ¹
therefore

$$AE \cdot EG = BE \cdot EF$$

and *contertendo*

$$EA \cdot AG = EB \cdot BF$$

And also

$$CA \cdot AE = DB \cdot BE,$$

¹For the point H falls within the angle AEB (II 25) and the straight line drawn from H to the midpoint of AB that is S is a diameter (II 29) and must therefore pass through E (I 51 end) An analogous series of propositions is found for the opposite sections II. 32 38 39

Then, since
therefore

$$\text{trgl } GHA = \text{trgl } FHB$$

$$\text{trgl } GFB = \text{trgl } GPA$$

Their bases are the same therefore their heights are equal (Eucl. VI 1)

for each is double the other therefore *ex aequali*

$$CA \cdot AG = DB \cdot BF$$

And the triangles are similar because of the parallels, therefore

$$\text{trgl } CTA = \text{trgl } AHG = \text{trgl } ABD = \text{trgl } HBF \text{ (Eucl VI 19)}$$

And alternately, but

$$\text{trgl } AHG = \text{trgl } HBF \text{ (III 1),}$$

therefore

$$\text{trgl } CTA = \text{trgl } ABD$$

As parts of these it was shown

$$\text{trgl } AHG = \text{trgl } HBF,$$

therefore also as remainders

$$\text{quadr } DH = \text{quadr } CH$$

And so also

$$\text{quadr } DG = \text{quadr } CF$$

And since $CO \parallel AF$

$$\text{trgl } COE = \text{trgl } AEF$$

And likewise also

$$\text{trgl } DEI = \text{trgl } BEG$$

But

$$\text{trgl } BEG = \text{trgl } AEF \text{ (III 1),}$$

therefore also

$$\text{trgl } COE = \text{trgl } DEI$$

And also

$$\text{quadr } DG = \text{quadr } CF \text{ (above)}$$

Therefore as wholes,

$$\text{quadr } XI = \text{quadr } OT$$

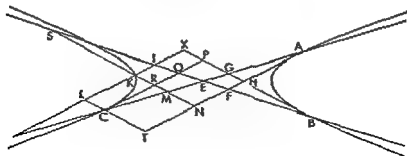
PROPOSITION 9

With the same things supposed if one of the points is between the diameters as K , and the other is the same with one of the points C and D for instance C , and the parallels are drawn I say that

$$\text{trgl } CEO = \text{quadr } AEF$$

and

$$\text{quadr } LO = \text{quadr } LM$$



And this is evident For since it was shown

$$\text{trgl } CEO = \text{trgl } AEF,$$

and

$$\text{trgl } AEF = \text{quadr } \Delta E \text{ (III 5, note),}$$

therefore also

$$\text{trgl } CEO = \text{quadr } \Delta E$$

And so also

$$\text{trgl } CRM = \text{quadr } \Delta O,$$

and

$$\text{quadr } \Delta C = \text{quadr } \Delta O$$

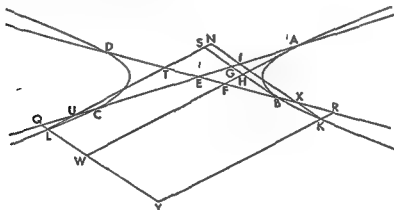
PROPOSITION 10

With the same things supposed, let A and L be taken not as points at which the diameters hit the sections

Then it is to be shown that

$$\text{quadr } LTRY = \text{quadr } QYAI$$

For since the straight lines AF and BG touch, and AE and BE are diameters



through the points of contact and LT and KI are parallel to the tangents,

$$\text{trgl } TUE = \text{trgl } UQL + \text{trgl } EFA \text{ (I 44)}$$

And likewise also

$$\text{trgl } \Delta EI = \text{trgl } XRK + \text{trgl } BEG$$

But

$$\text{trgl } EFA = \text{trgl } BEG \text{ (III 1)}$$

therefore

$$\text{trgl } TUE - \text{trgl } UQL = \text{trgl } XEI - \text{trgl } XRK$$

Therefore

$$\text{trgl } TUE + \text{trgl } \Delta RK = \text{trgl } \Delta EI + \text{trgl } UQL$$

Let the common area $\Delta YEULY$ be added therefore

$$\text{quadr } LTRY = \text{quadr } QYAI$$

PROPOSITION 11

With the same things supposed if some point is taken on either of the sections and from it parallels are drawn one parallel to the tangent and the other parallel to the straight line joining the points of contact then the triangle produced by them on the diameter drawn through the point of meeting of the tangents differs from the tri-

for each is double the other, therefore *ex aequali*

$$CA \cdot AG = DB \cdot BF$$

And the triangles are similar because of the parallels, therefore

$$\text{trgl } CTA \sim \text{trgl } AIG \sim \text{trgl } XBD \sim \text{trgl } HBF \text{ (Eucl vi 19)}$$

And alternately, but

$$\text{trgl } AIG = \text{trgl } HBF \text{ (iii 1),}$$

therefore

$$\text{trgl } CTA = \text{trgl } XBD$$

As parts of these it was shown

$$\text{trgl } AIG = \text{trgl } HBF$$

therefore also as remainders

$$\text{quadr } DH = \text{quadr } CH$$

And so also

$$\text{quadr } DG = \text{quadr } CF$$

And since CO is parallel to AF

$$\text{trgl } COE = \text{trgl } AEF$$

And likewise also

$$\text{trgl } DEI = \text{trgl } BEG$$

But

$$\text{trgl } BEG = \text{trgl } AEF \text{ (iii 1),}$$

therefore also

$$\text{trgl } COE = \text{trgl } DEI$$

And also

$$\text{quadr } DG = \text{quadr } CF \text{ (above)}$$

Therefore, as wholes

$$\text{quadr } MI = \text{quadr } OT$$

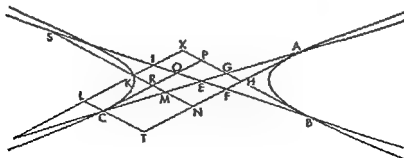
PROPOSITION 9

With the same things supposed, if one of the points is between the diameters, as K , and the other is the same with one of the points C and D for instance C , and the parallels are drawn, I say that

$$\text{trgl } CEO = \text{quadr } KE$$

and

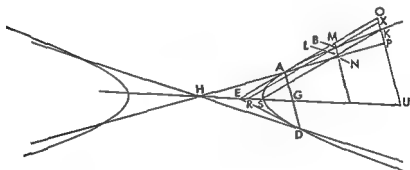
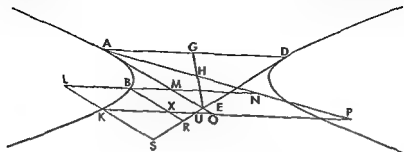
$$\text{quadr } LO = \text{quadr } LM$$



And this is evident For since it was shown

$$\text{trgl } CEO = \text{trgl } AEF,$$

For let there be the same things as before and let the points B and A be taken at random on section AB and through them let $LBMN$ and $A\lambda OUP$



be drawn parallel to AD and $B\lambda R$ and LKS parallel to AE

I say that

$$\text{quadr } BP = \text{quadr } KR$$

For since it has been shown

$$\text{trgl } AOP = \text{quadr } AOE\lambda \text{ (III 11 end)}$$

and

$$\text{trgl } AMN = \text{quadr } BMER \text{ (III 11 end),}$$

therefore, as remainders either

$$\text{quadr } KR - \text{quadr } BO = \text{quadr } MP$$

or

$$\text{quadr } AR + \text{quadr } BO = \text{quadr } MP$$

And with the common quadrilateral BO added or subtracted

$$\text{quadr } BP = \text{quadr } XS$$

PROPOSITION 13

If in conjugate opposite sections straight lines tangent to the adjacent sections meet and diameters are drawn through the points of contact then the triangles whose common vertex is the center of the opposite sections will be equal

Let there be conjugate opposite sections on which there are the points A , B , C and D and let BE and AE meeting at E touch the sections A and B and let H be the center and let AH and BH be joined and produced to D and C

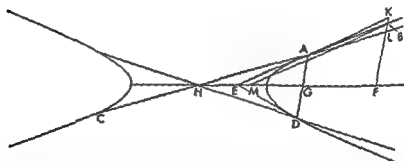
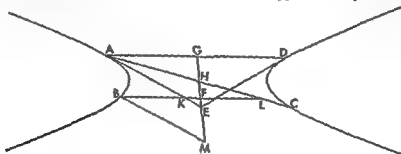
angle cut off on the tangent and the diameter drawn through the point of contact by the triangle cut off at the point of meeting of the tangents

Let there be the opposite sections AB and CD , and let the tangents AE and DE meet at E and let the center be H , and let AD and EHG be joined, and let some point B be taken at random on the section AB , and through it let BFL be drawn parallel to AG , and BM parallel to AE

I say that triangle BFM differs from triangle AKL by triangle AEF

For it is evident that AD is bisected by EH (II 39 and II 29), and that EH is a diameter conjugate to the diameter drawn through H parallel to AD (II 38), and so AG is an ordinate to EG (First Def 1 6)

Since then GE is a diameter and AE touches and AG is an ordinate and with point B taken on the section, BF has been dropped to EG parallel to AG ,



and BM parallel to AE , therefore it is clear that triangle BMF differs from triangle LHF by triangle HAE (I 45, I 43)¹ And so also triangle BMF differs from triangle AKL by triangle KFE

And it has been proved at the same time that
 $\text{quadr } BAEV = \text{trgl } LKA$

PROPOSITION 12

With the same things being so if on one of the sections two points are taken and parallels are drawn from each of them, likewise the quadrilaterals produced by them will be equal

¹That is in the first case

$\text{trgl } BMF = \text{trgl } LHF + \text{trgl } HAF$ (I 45)

In the second case only the more general statement differs, but is true (I 43). It will be noticed these are different cases I 43 and I 45 from those given in the text itself

B and through it let λRS be drawn parallel to AG and λTO parallel to BE

I say that triangle OHT differs from triangle λST by triangle $HB F$

For let AU be drawn from A parallel to BF Since then because of the same things as before, LHM is a diameter of the section AL , and DHB is a second diameter and conjugate to it (II 20), and AG is a tangent at A , and AU has been dropped parallel to LM , therefore

AU UG comp HU UA ,
transverse side of figure on LM upright (I 40)

But

AU UG λT TS ,

and

HU UA HT TO HB BF ,

and

transverse side of figure on LM upright
upright side of figure on BD transverse (I 60)

Therefore

λT TS comp HB BF , upright side of figure on BD transverse

or

λT TS comp HT TO upright side of figure on BD transverse

And by things shown in the forty first theorem of the first book (I 41), triangle THO differs from triangle λTS by triangle BFH

And so also by triangle AGH (III 13)

PROPOSITION 15

*If straight lines touching one of the conjugate opposite sections meet and diameters are drawn through the points of contact and some point is taken on any one of the conjugate sections and from it parallels to the tangents are drawn as far as the diameters then the triangle produced by them at the section is greater than the triangle produced at the center by the triangle having the tangent as base and the center of the opposite sections as vertex*¹

Let there be conjugate opposite sections AB GS T and λ whose center is H and let ADE and BDC touch the section AB and let the diameters $AHFW$ and BHT be drawn through the points of contact A and B and let some point S be taken on the section GS and through it let SFL be drawn parallel to BC and SU parallel to AE

I say that

trgl SLU = trgl HLF + trgl HCB

For let λHG be drawn through H parallel to BC and λIG through G parallel to AE and SO parallel to BT then it is evident that λG is a diameter conjugate to BT (II 20) and that SO being parallel to BT has been dropped ordinatewise to HGO (First Def I 6) and that $SLHO$ is a parallelogram

Since then BC touches and BH is through the point of contact and AE is another tangent let it be contrived that

DB BE λN $2BC$,

therefore λN is the so-called upright side of the figure on BT (I 50) Let MN be bisected at P therefore

¹This proposition comes as a climax to a long series and shows that the conjugate opposite sections taken as a unit have the same property as the other conic sections. The conjugate opposite sections seem to be a sort of fifth section.

Again since

$$HB \ BC \text{ comp } HB \ MP, MP \ BC,$$

but

$$HB \ MP \ TB \ MN \ R \ \lambda G \text{ (above } \alpha \text{ and } \gamma),$$

and

$$MP \ BC \ DB \ BE \text{ (above, } \beta)$$

therefore

$$HB \ BC \text{ comp } DB \ BE \ R \ \lambda G$$

And since BC is parallel to SL , and triangle HCB is similar to triangle HLF ,
and

$$HB \ BC \ HL \ LF,$$

therefore

$$HL \ LF \text{ comp } R \ \lambda G \ DB \ BE$$

or

$$HL \ IF \text{ comp } R \ \lambda G \ HG \ HI$$

Since then GS is an hyperbola having λG as a diameter and R as an upright side and from some point S SO has been dropped ordinatewise and figure HIG has been described on radius HG and figure HLF has been described on the ordinate SO or its equal HL , and on HO the straight line between the center and the ordinate or on SL its equal the figure SLU has been described similar to the figure HIG described on the radius, and there are the compounded ratios as already given therefore

$$\text{trgl } SLU = \text{trgl } HLF + \text{trgl } HCB \text{ (I 41)}$$

PROPOSITION 16

If two straight lines touching a section of a cone or circumference of a circle meet and from some point of those on the section a straight line is drawn parallel to one tangent and cutting the section and the other tangent then as the squares on the tangents are to each other so the area contained by the straight lines between the section and the tangent will be to the square cut off at the point of contact

Let there be the section of a cone or circumference of a circle AB and let the straight lines AC and CB meeting at C touch it and let some point D be taken on the section AB and through it let EDF be drawn parallel to CB

I say that

$$\text{sq } BC \ \text{sq } AC \ \text{rect } FE, ED \ \text{sq } EA$$

For let the diameters AGH and ABL be drawn through A and B , and DMN through D parallel to AL , it is at once evident that

$$DA = AF \text{ (I 46 47)}$$

and

$$\text{trgl } AEG = \text{quadr } LD \text{ (III 2),}$$

and

$$\text{trgl } BLC = \text{trgl } ACH \text{ (III 1)}$$

Since then

$$DA = AF$$

and DE is added

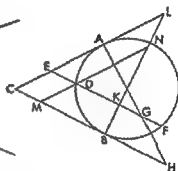
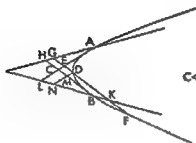
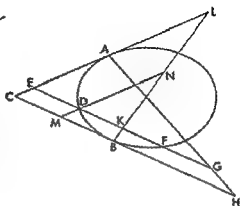
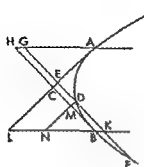
$$\text{rect } FE \ ED + \text{sq } DK = \text{sq } KE$$

And since triangle ELK is similar to triangle DNK

$$\text{sq } EK \ \text{sq } KD \ \text{trgl } EKL \ \text{trgl } DNK$$

And alternately

whole sq EK whole trgl ELK
 part subtracted sq DK part subtracted trgl DNK



Therefore also

remainder rect $FE ED$ remainder quadr DL sq EA trgl FLA

But

sq EA trgl ELA sq CB trgl BLC

therefore also

rect $FE ED$ quadr LD sq CB trgl LCB

But

quadr $LD = \text{trgl } AFG$

and

trgl $BLC = \text{trgl } ACH$,

therefore also

rect $FE ED$ trgl AEG sq CB trgl ACH

Alternately

rect $FE ED$ sq CB trgl AFG trgl ACH

But

trgl AEC trgl ACH sq EA sq AC

therefore also

rect $FE ED$ sq CB sq FA sq AC

And alternately

PROPOSITION 17

If two straight lines touching a section of a cone or circumference of a circle meet, and two points are taken at random on the section, and from them in the section are drawn parallel to the tangents straight lines cutting each other and the line of the section, then as the squares on the tangents are to each other, so will the rectangles contained by the straight lines taken similarly

Let there be the section of a cone or circumference of a circle AB , and tangents to AB AC and CB meeting at C , and let points D and E be taken at random on the section and through them at $EFIA$ and $DFGH$ be drawn parallel to AC and CB

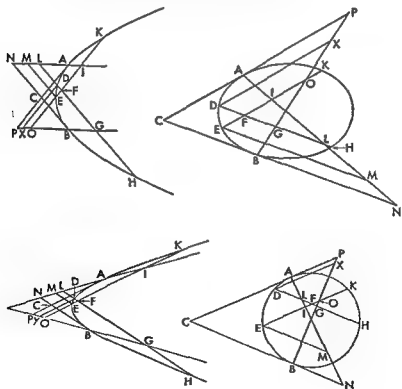
I say that

$$\text{sq } CA \text{ sq } CB \text{ rect } KF \text{ FE rect } HF \text{ FD}$$

For let the diameters $ALMN$ and $BOXP$ be drawn through A and B , and let the tangents and parallels be produced to the diameters and let DX and EM be drawn from D and E parallel to the tangents, then it is evident that

$$KI = IE, HG = GD \text{ (I 46 47)}$$

Since then KE has been cut equally at I and unequally at F ,
rect $KF, FE + \text{sq } FI = \text{sq } EI$ (Eucl II 5)



And since the triangles are similar because of the parallels
whole sq EI whole trgl IME

part subtracted sq IF part subtracted trgl FIL

Therefore also

remainder rect KF, FE remainder quadr FM whole
sq EI whole trgl IME

But

sq EI trgl IME sq CA trgl CAN

Therefore

rect KF, FE quadr FM sq CA trgl CAN

But

trgl $CAN = \text{trgl } CPB$ (in 1),

and

quadr $FM = \text{quadr } F\lambda$ (in 3)

therefore

rect KF, FE quadr $F\lambda$ sq CA trgl CPB

Then likewise it could be shown that

rect HF, FD quadr FY sq CB trgl CPB

Since then

rect KF, FE quadr FY sq CA trgl CPB

and inversely

quadr $F\lambda$ rect HF, FD trgl CPB sq CB

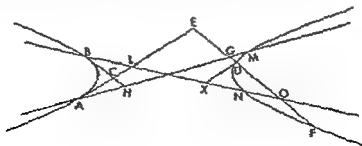
therefore ex aequalis

sq CA sq CB rect KF, FE rect HF, FD

PROPOSITION 18

If two straight lines touching opposite sections meet and some point is taken on either one of the sections and from it some straight line is drawn parallel to one of the tangents cutting the section and the other tangent then as the squares on the tangents are to each other so will the rectangle contained by the straight lines between the section and the tangent be to the square on the straight line cut off at the point of contact

Let there be the opposite sections AB and MN and tangents ACL and BCH and through the points of contact the diameters AM and BN and let some point D be taken at random on the section MN and through it let EDF be drawn parallel to BH



I say that

sq BC sq CA rect FE, FD sq AF

For let $D\lambda$ be drawn through D parallel to AL Since then AB is an hyper

bola and BN its diameter and BH a tangent and DF parallel to BH , therefore
 $FO = OD$ (I 48)

And ED is added therefore

$$\text{rect } FE, ED + \text{sq } DO = \text{sq } EO \text{ (Eucl II 6)}$$

And since EL is parallel to DY , triangle EOL is similar to triangle $D\lambda O$
 Therefore

$$\begin{array}{ll} \text{whole sq } EO & \text{whole trgl } EOL \\ \text{part subtracted sq } DO & \text{part subtracted trgl } D\lambda O, \end{array}$$

therefore also

$$\text{remainder rect } DE, EF \quad \text{remainder quadr } DL \quad \text{sq } EO \quad \text{trgl } EOL$$

But

$$\text{sq } OE \quad \text{trgl } EOL \quad \text{sq } BC \quad \text{trgl } BCL$$

therefore also

$$\text{rect } FE, ED \quad \text{quadr } DL \quad \text{sq } BC \quad \text{trgl } BCL$$

And

$$\text{quadr } DL = \text{trgl } AEG \text{ (III 6 note),}$$

and

$$\text{trgl } BCL = \text{trgl } ACH \text{ (III 1),}$$

therefore

$$\text{rect } FE, ED \quad \text{trgl } AEG \quad \text{sq } BC \quad \text{trgl } ACH$$

But also

$$\text{trgl } AEG \quad \text{sq } EA \quad \text{trgl } ACH \quad \text{sq } AC,$$

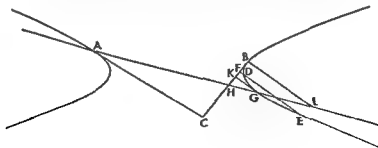
therefore *ex aequali*

$$\text{sq } BC \quad \text{sq } AC \quad \text{rect } FE, ED \quad \text{sq } EA^1$$

¹Eutocius gives an alternative proof of Apollonius demonstrating another and important case For let there be the opposite sections A and B and tangents to them AC and CB meeting at C and let D be taken on section B and through it let EDF be drawn parallel to AC I say that

$$\text{sq } AC \quad \text{sq } CB \quad \text{rect } EF, FD \quad \text{sq } FB$$

For let AHG be drawn as a diameter through A and through B and G GK and BL parallel to EF Since then BH touches the hyperbola at B and BL has been drawn ordinate-wise
 $AL \quad LG \quad AH \quad HG$ (I 36)



But

$$AL \quad LG \quad CB \quad BK$$

and

$$AH \quad HG \quad AC \quad KG$$

therefore also

$$CB \quad BA \quad AC \quad AG$$

And alternately

$$AC \quad CB \quad KG \quad KB$$

and

$$\text{sq } AC \quad \text{sq } CB \quad \text{sq } GK \quad \text{sq } AB$$

But it was shown

$$\text{sq } GK \quad \text{sq } KB \quad \text{rect } EF, FD \quad \text{sq } FB$$

therefore also

$$\text{sq } AC \quad \text{sq } CB \quad \text{rect } EF, FD \quad \text{sq } FB$$

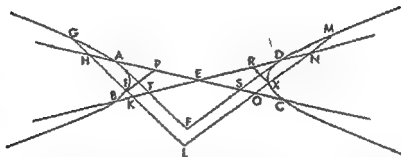
PROPOSITION 19

If two straight lines touching opposite sections meet and parallels to the tangents are drawn cutting each other and the section then, as the squares on the tangents are to each other so will the rectangle contained by the straight lines between the section and the point of meeting of the straight lines be to the rectangle contained by the straight lines taken similarly

Let there be opposite sections whose diameters are AC and BD and center in E , and let the tangents AF and FD meet at F , and let $GHIKL$ and $MNAL$ be drawn from any points parallel to AF and FD

I say that

$$\text{sq } AF : \text{sq } FD :: \text{rect } GL : LI :: \text{rect } ML : LX$$



Let IP and XR be drawn through I and L parallel to AF and FD
 BH touches the hyperbola at B and BL has been drawn ordinatewise,
 $AL : LG :: AH : HG$ (I, 36)

And since

$$\text{sq } AF : \text{trgl } AFS :: \text{sq } HL : \text{trgl } HLO :: \text{sq } HI : \text{trgl } HIP$$

therefore

$$\text{remainder rect } GL : LI :: \text{remainder quadr } IPOL : \text{sq } AF : \text{trgl } AFS$$

But

$$\text{trgl } AFS = \text{trgl } DTF \text{ (III 4)}$$

and

$$\text{quadr } IPOL = \text{quadr } ARXL \text{ (III 7)}$$

therefore also

$$\text{sq } AF : \text{trgl } DTF :: \text{rect } GL : LI :: \text{quadr } ARXL$$

But

$$\text{trgl } DTF : \text{sq } FD :: \text{quadr } RXYL : \text{rect } ML : LX \text{ (likewise),}$$

and therefore ex aequali

$$\text{sq } AF : \text{sq } FD :: \text{rect } GL : LI :: \text{rect } ML : LX$$

PROPOSITION 20

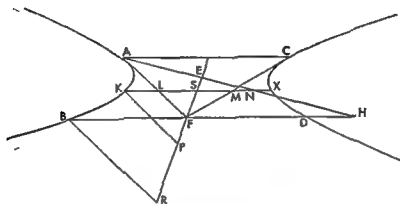
If two straight lines touching opposite sections meet and through the point of meeting some straight line is drawn parallel to the straight line joining the points of contact and meeting each of the sections and some other straight line is drawn parallel to the same straight line and cutting the sections and the tangents then as the rectangle contained by the straight lines drawn from the point of meeting to cut the sections is to the square on the tangent so is the rectangle contained by the

straight lines between the sections and the tangent to the square on the straight line cut off at the point of contact

Let there be the opposite sections AB and CD whose center is E and tangents AF and CF and let AC be joined, and let EF and AE be joined and produced and let BFH be drawn through F parallel to AC , and let the point K be taken at random and through it let $KLSMNX$ be drawn parallel to AC

I say that

$$\text{rect } BF, FD \text{ sq } FA \quad \text{rect } KL, LX \text{ sq } AL$$



For let KP and BR be drawn from K and B parallel to AF Since then
 $\text{sq } BF \text{ trgl } BFR \quad \text{sq } AS \text{ trgl } KSP \quad \text{sq } LS \text{ trgl } LSF$
 and

$$\begin{aligned} & \text{sq } AS \text{ trgl } ASP \\ & \text{remainder rect } KL, LX \text{ (Eucl II 5)} \\ & \text{remainder quadr } KLFP \text{ (Eucl V 19)} \end{aligned}$$

and
 $\text{sq } BF = \text{rect } BF, FD$ (II 39, 38)

and
 $\text{trgl } BRF = \text{trgl } AFH$ (III 11 and special case)

and
 $\text{quadr } KLFP = \text{trgl } ALN$ (III 5)

therefore
 $\text{rect } BF, FD \text{ trgl } AFH \quad \text{rect } KL, LX \text{ trgl } ALN$

And
 $\text{trgl } AFH \text{ sq } AF \quad \text{trgl } ALN \text{ sq } AL,$

then
 $\text{rect } BF, FD \text{ sq } FA \quad \text{rect } KL, LX \text{ sq } AL$

PROPOSITION 21

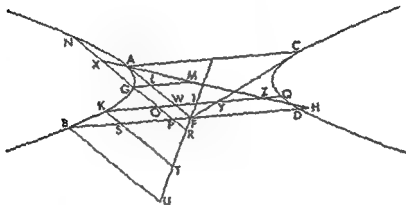
With the same things supposed if two points are taken on the section and through them straight lines are drawn the one parallel to the tangent, the other parallel to the straight line joining the points of contact and cutting each other and the sections then as the rectangle contained by the straight lines drawn from the point

of meeting to cut the sections as to the square on the tangent, so will the rectangle contained by the straight lines between the sections and the point of meeting be to the rectangle contained by the straight lines between the section and the point of meeting

For let there be the same things as before, and let points G and K be taken and through them let $NAGOPR$ and KST be drawn parallel to AF , and GLV and KOH parallel to AC

I say that

$$\text{rect } BF, FD \text{ sq } FA \cdot \text{rect } KO, OQ \text{ rect } NO, OG$$



For since

$$\text{sq } AF \text{ trgl } AFH \text{ sq } AL \text{ trgl } ALM \text{ sq } XO \text{ trgl } XOZ$$

and

$$\text{sq } XO \text{ trgl } XOZ \text{ sq } XG \text{ trgl } XGM,$$

therefore

$$\text{whole sq } XO \text{ whole trgl } XOZ$$

$$\text{part subtracted sq } XG \text{ part subtracted trgl } XGM,$$

therefore also

$$\text{remainder rect } NO, OG \text{ remainder quadr } GOZM \text{ sq } AF \text{ trgl } AFH$$

But

$$\text{trgl } AFH = \text{trgl } BUF \text{ (in 11 and special case),}$$

and

$$\text{quadr } GOZM = \text{quadr } KORT \text{ (in 12),}$$

therefore

$$\text{sq } AF \text{ trgl } BPU \text{ rect } NO, OG \text{ quadr } KORT$$

But it was shown (in the course of III 20) $\text{trgl } BUF \text{ sq } BF$ or $\text{rect } BF, FD$

$$\text{(in 33 38) quadr } KORT \text{ rect } KO, OQ$$

therefore ex aequali

$$\text{sq } AF \text{ rect } BF, FD \text{ rect } NO, OG \text{ rect } KO, OQ$$

And inversely

$$\text{rect } BF, FD \text{ sq } FA \text{ rect } KO, OQ \text{ rect } NO, OG$$

PROPOSITION 22

If two parallel straight lines touch opposite sections and any straight lines are drawn cutting each other and the sections one parallel to the tangent the other parallel to the straight line joining the points of contact, then as the transverse side

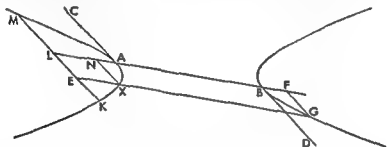
of the figure on the straight line joining the points of contact is to the upright, so the rectangle contained by the straight lines between the sections and the point of meeting will be to the rectangle contained by the straight lines between the section and the point of meeting

Let there be the opposite sections A and B , and let AC and BD be parallel and tangent to them, and let AB be joined. Then let EXG be drawn across parallel to AB and $KELM$ parallel to AC

I say that

AB upright side of the figure rect $GE EY$ rect KE, EM

Let λN and GF be drawn through G and λ parallel to AC



For since AC and BD are parallels tangent to the sections AB is a diameter (II 31), and KL , XN and GF are ordinates to it (I 32), then (I 21)

AB upright side

rect BL, LA sq LK rect $BN NA$ sq $N\lambda$ or sq $L\lambda$

Therefore

whole rect $BL LA$ whole sq LK

part subtracted rect $BN NA$ part subtracted sq LE ,

or

rect BL, LA sq $L\lambda$ rect $FA AN$ sq LE ,

for

$NA = BF$ (I 21),

therefore also

remainder rect $FL LN$ remainder rect KE, EM AB upright

But

rect $FL LN = \text{rect } GE EY$,

therefore

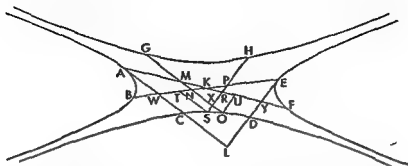
AB the transverse side of figure upright

rect $GE EY$ rect KE, EM

PROPOSITION 23

If in conjugate opposite sections two straight lines touching contrary sections meet in any one section at random and any straight lines are drawn parallel to the tangents and cutting each other and the other opposite sections then as the squares on the tangents are to each other so the rectangle contained by the straight lines between the sections and the point of meeting will be to the rectangle contained by the straight lines similarly taken

Let there be the conjugate opposite sections AB CD , EF and GH , and their center K , and let $AWCL$ and $EYDL$, tangents to the sections AB and EF



meet at L and let AK and EK be joined and produced to B and F , and let $GM\lambda O$ be drawn from G parallel to AL and $HPR\lambda S$ from H parallel to EL

I say that

$$\text{sq } EL \text{ sq } LA \text{ rect } HX, \lambda S \text{ rect } GY \lambda D$$

For let ST be drawn through S parallel to AL , and OU from O parallel to EL . Since then BE is a diameter of the conjugate opposite sections AB CD , EF and GH and EL touches the section and HS has been drawn parallel to it,

$$HP = PS \text{ (II 20, First Def 1 5),}$$

and for the same reasons

$$GM = MO$$

And since

$$\text{sq } EL \text{ trgl } EWL \text{ sq } PS \text{ trgl } PTS \text{ sq } PY \text{ trgl } PNX,$$

also

$$\text{remainder rect } H\lambda, XS \text{ remainder quadr } TN \lambda S \text{ sq } EL \text{ trgl } WLE$$

But

$$\text{trgl } EWL = \text{trgl } ALY \text{ (III 4),}$$

and

$$\text{quadr } TN\lambda S = \text{quadr } XR\lambda O \text{ (III 15) }^1$$

therefore

$$\text{sq } EL \text{ trgl } ALY \text{ rect } H\lambda \lambda S \text{ quadr } \lambda RUO$$

But

$$\text{trgl } AL\lambda \text{ sq } AL \text{ quadr } \lambda RUO \text{ rect } G\lambda \lambda O \text{ (same way),}$$

therefore *ex aequali*

$$\text{sq } EL \text{ sq } AL \text{ rect } H\lambda \lambda S \text{ rect } GY, XO$$

PROPOSITION 21

If in conjugate opposite sections two straight lines are drawn from the center through to the sections and one of them is taken as the transverse diameter and the other as the upright diameter and any straight lines are drawn parallel to the two diameters and meeting each other and the sections and the point of meeting of the straight lines is the place between the four sections then the rectangle contained by

¹This is the case of III 15 where the tangents are one to each of the opposite sections. Compare with the two cases of III 12 and III 18.

For $\text{trgl } TSI = \text{trgl } APR = \text{trgl } ANK$ (III 15)

and $\text{trgl } MOU = \text{trgl } M\lambda K = \text{trgl } ANK$ (III 15)

the segments of the parallel to the transverse diameter together with the rectangle to which the rectangle contained by the segment of the parallel to the upright diameter has the ratio which the square on the upright diameter has to the square on the transverse will be equal to twice the square on the half of the transverse

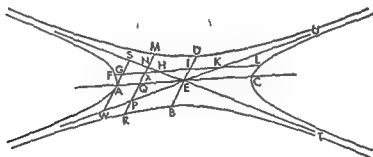
Let there be the conjugate opposite sections A, B, C and D whose center is E , and from E let the transverse diameter ABC and the upright diameter DEB be drawn through and let $FGHIKL$ and $MNXOPR$ be drawn parallel to AC and DB and meeting each other at X , and first let X be within the angle SEW or the angle UET

I say that the rectangle FX, XL together with the rectangle to which the rectangle RX, XM has the ratio which the square on DB has to the square on AC , is equal to twice the square on AE

For let the asymptotes of the sections SET and UEW be drawn and through A , $SGAW$ tangent to the section

Since then

$$\text{rect } SA, AW = \text{sq } DE \text{ (I 60, II 1),}$$



therefore

$$\text{rect } SA, AW \text{ sq } EA \text{ sq } DE \text{ sq } EA$$

And

$$\text{rect } SA, AW \text{ sq } AE \text{ comp } SA \text{ AEWA } AE$$

But

$$SA \text{ AE } NX \text{ XH}$$

and

$$WA \text{ AE } PY \text{ XK}$$

therefore

$$\text{sq } DE \text{ sq } AE \text{ comp } NX \text{ XH, PY } XK$$

But

$$\text{rect } PX \text{ YN } \text{rect } KX \text{ XH } \text{comp } NX \text{ XH } PX \text{ XK,}$$

therefore

$$\text{sq } DE \text{ sq } AE \text{ rect } PX, YN \text{ rect } KY \text{ XH}$$

Therefore also

$$\text{sq } DL \text{ sq } AL \text{ sq } DE + \text{rect } PY, XN \text{ sq } AE + \text{rect } KY \text{ XH}$$

And

$$\text{sq } DE = \text{rect } PM \text{ MN (II 11)} = \text{rect } RN \text{ NM (II 16)}$$

and

$$\text{sq } AE = \text{rect } KF \text{ FH (II 11)} = \text{rect } LH, HF \text{ (II 16),}$$

therefore

$$\text{sq } DE \text{ sq } AE \text{ rect } PY, \lambda N + \text{rect } RN, NM \\ \text{rect } K \lambda, \lambda H + \text{rect } LH, HF$$

And

$$\text{rect } P \lambda \lambda N + \text{rect } RN, NM = \text{rect } RY, XM,^1$$

therefore

$$\text{sq } DE \text{ sq } AE \text{ rect } R \lambda, XM \text{ rect } K \lambda, \lambda H + \text{rect } K F, FH$$

Then it must be shown that

$$\text{rect } F \lambda \lambda L + \text{rect } K \lambda \lambda H + \text{rect } K F, FH = 2 \text{ sq } AE$$

Let the common square AE that is rectangle KF, FH , be subtracted, therefore it remains to be shown that

$$\text{rect } F \lambda \lambda L + \text{rect } K \lambda, \lambda H = \text{sq } AE$$

And this is so, for

$$\text{rect } F \lambda \lambda L + \text{rect } K \lambda \lambda H = \text{rect } LH, HF, \dagger$$

$$\text{rect } F \lambda, \lambda L + \text{rect } K \lambda \lambda H = \text{rect } K F, FH \text{ (II 16)}$$

$$= \text{sq } AE \text{ (II 11)}$$

Then let the straight lines FL and MR meet on one of the asymptotes at H

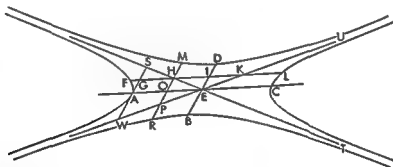
$$\text{rect } FH, HL = \text{sq } AE,$$

and

$$\text{rect } MH, HR = \text{sq } DE \text{ (II 11 16),}$$

therefore

$$\text{sq } DE \text{ sq } AE \text{ rect } MH, ER \text{ rect } FH, HL$$



And so we want twice rectangle FH, HL to equal twice the square on AE

And it does

¹For

$$RP = \lambda M \text{ (II 8)}$$

and
therefore

$$RO = OM \text{ (II 3)}$$

But

$$PO = O \lambda$$

and for the same reasons

$$\text{rect } P \lambda \lambda A + \text{sq } O \lambda = \text{sq } ON \text{ (Eucl II 5)}$$

and

$$\text{rect } RN, \lambda M + \text{sq } O \lambda = \text{sq } OM$$

Hence

$$\text{rect } R \lambda \lambda M + \text{sq } O \lambda = \text{sq } OM$$

and adding equals to equals,

$$\text{rect } R \lambda \lambda M + \text{sq } O \lambda = \text{rect } R \lambda \lambda M + \text{sq } O \lambda$$

$$\text{rect } R \lambda \lambda M + \text{sq } O \lambda + \text{rect } P \lambda \lambda A + \text{sq } O \lambda = \text{rect } R \lambda \lambda M + \text{sq } O \lambda + \text{sq } ON$$

Subtracting the common squares

$$\text{rect } RN, \lambda M + \text{rect } P \lambda \lambda A = \text{rect } RY, XM$$

[†]By the same manner of proof as in the note above but using also Euclid II 6 because of the different position of the point λ

And let the point Y be within the angle SEA or angle WET Then likewise by the composition of ratios

$$\text{sq } DE \text{ sq } AE \text{ rect } P\lambda, \lambda N \text{ rect } K\lambda \lambda H$$

And

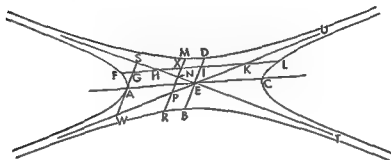
$$\text{sq } DE = \text{rect } PM, RN = \text{rect } RN NM,$$

and

$$\text{sq } AL = \text{rect } FH HL,$$

therefore

$$\text{rect } RN, NM \text{ rect } FH, HL$$



part subtracted rect $P\lambda \lambda N$ part subtracted rect $\lambda\lambda \lambda H$

Therefore also

$$\text{rect } RN NM \text{ rect } FH HL$$

remainder rect $RX XM$ remainder ($\text{sq } AE - \text{rect } K\lambda \lambda H$)

Therefore it must be shown that

$$\text{rect } F\lambda \lambda L + (\text{sq } AE - \text{rect } K\lambda \lambda H) = 2 \text{ sq } AE$$

Let the common square on AE that is rectangle $FH HL$ be subtracted therefore it remains to be shown that

$$\text{rect } \lambda X \lambda H + (\text{sq } AE - \text{rect } K\lambda \lambda H) = \text{sq } AE$$

And this is so for

$$\text{rect } \lambda\lambda \lambda H + \text{sq } AE - \text{rect } \lambda X \lambda H = \text{sq } AE$$

PROPOSITION 25

With the same things supposed let the point of meeting of the parallels to AC and BD be within one of the sections D and B as set out below at X

I say that the rectangle contained by the segments of the parallel to the transverse that is rectangle $OX \lambda N$, will be greater than the rectangle to which the rectangle contained by the segments of the parallel to the upright diameter that is rectangle $RX XM$ has the ratio which the square on the upright diameter has to the square on the transverse by twice the square on the half of the transverse

For for the same reasons

$$\text{sq } DE \text{ sq } AE \text{ rect } PX XH \text{ rect } SY \lambda L$$

and

$$\text{sq } DE = \text{rect } PM MH$$

and

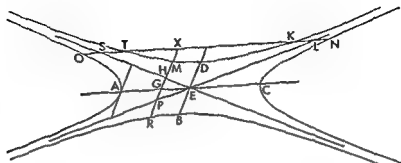
$$\text{sq } AE = \text{rect } LO OS \text{ (II 11)}$$

therefore also

$$\text{sq } DE = \text{sq } AE - \text{rect } PM, MH - \text{rect } LO, OS$$

And since

$$\text{whole rect } PX, \lambda H = \text{whole rect } L\lambda, \lambda S$$



$$\begin{aligned} &\text{part subtracted rect } PM, MH - \text{part subtracted rect } LO, OS \\ &\text{or rect } ST, TL \text{ (II 22),} \end{aligned}$$

therefore also

$$\begin{aligned} &\text{remainder rect } HY, SM : \text{remainder rect } T\lambda, \lambda K \\ &\text{(first note to III 24 II 8)} \quad \text{sq } DE = \text{sq } AE \end{aligned}$$

Therefore it must be shown that

$$\text{rect } O\lambda, \lambda N = \text{rect } T\lambda, \lambda K + 2 \text{ sq } AE$$

Let the common rectangle $TY, \lambda K$ be subtracted, therefore it must be shown that

$$\text{rect } OT, TN \text{ (first note to III 24)} = 2 \text{ sq } AE$$

And it is (II 23)

PROPOSITION 26

And if the point of meeting of the parallels at Y is within one of the sections A and C as set out below then the rectangle contained by the segments of the parallel to the transverse that is rectangle $L\lambda, \lambda F$ will be less than the rectangle to which the rectangle contained by the segments of the other parallel that is rectangle $R\lambda, \lambda G$ has the ratio which the square on the upright diameter has to the square on the transverse by twice the square on half of the transverse

For, since for the same reasons as before

$$\text{sq } DE = \text{sq } AE - \text{rect } W\lambda, \lambda S - \text{rect } K\lambda, \lambda H$$

therefore also

$$\begin{aligned} &\text{whole rect } R\lambda, \lambda G^1 - \text{whole rect } K\lambda, \lambda H + \text{sq } AE \\ &\quad = \text{sq upright diameter} - \text{sq transverse} \end{aligned}$$

Therefore it must be shown that

$$\text{rect } IY, \lambda F + 2 \text{ sq } AL = \text{rect } K\lambda, \lambda H + \text{sq } AE$$

¹For by II 11

$$\text{rect } WG, GS = \text{sq } DE$$

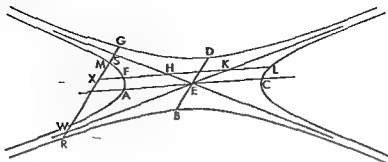
and

$$IW = GS \text{ (II 16)}$$

Therefore by the first note to III 24 and II 16

$$\text{rect } W\lambda, \lambda S + \text{sq } DE = \text{rect } W\lambda, \lambda S + \text{rect } WG, GS = \text{rect } R\lambda, \lambda G$$

Let the common square on AE be subtracted, therefore it remains to be shown that



$$\text{rect } LX \cdot XF + \text{sq } AE = \text{rect } KY, \backslash H$$

or

$$\text{rect } LX, \backslash F + \text{rect } LH \cdot HF = \text{rect } K \backslash \backslash H \quad (\text{II } 16 \text{ } 11)$$

And it is for

$$\text{rect } LH, HF + \text{rect } LX, \backslash F = \text{rect } K \backslash, \backslash H^1$$

PROPOSITION 27

If the conjugate diameters of an ellipse or circumference of a circle are drawn and one of them is called the upright diameter and the other the transverse and two straight lines meeting each other and the line of the section, are drawn parallel to them then the squares on the straight lines cut off on the straight line drawn parallel to the transverse between the point of meeting of the straight lines and the line of the section plus the figures described on the straight lines cut off on the straight line drawn parallel to the upright diameter between the point of meeting of the straight lines and the line of the section figures similar and similarly situated to the figure on the upright diameter will be equal to the square on the transverse diameter

For let there be the ellipse or circumference of a circle $ABCD$, whose center is E and let two of its conjugate diameters be drawn the upright AFC and the transverse BED and let $NGFH$ and $KFLM$ be drawn parallel to AC and BD

I say that the squares on NF and FH plus the figures described on NF and FM , similar and similarly situated to the figure on AC will be equal to the square on BD

Let NX be drawn from N parallel to AF therefore it has been dropped ordinate-wise to BD And let BP be the upright side Now since

$$\frac{BP}{AC} = \frac{AC}{BD} \quad (\text{I } 15),$$

therefore also

$$\frac{BP}{BD} = \frac{\text{sq } AC}{\text{sq } BD}$$

And

$$\text{sq } BD = \text{figure on } AC,$$

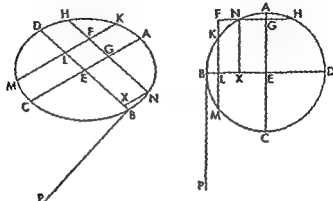
therefore

$$\frac{BP}{BD} = \frac{\text{sq } AC}{\text{figure on } AC}$$

¹ This is another case of the first note to III 21

And

sq AC figure on AC
sq NY figure on NY similar to the figure on AC (Eucl vi 22),



therefore also

$BP \cdot BD$ sq NY figure on NY similar to the figure on AC

And also

$BP \cdot BD$ sq NY rect $BY \cdot \lambda D$ (i 21)

therefore

figure on $N\lambda$ or FL similar to the figure on $AC = \text{rect } BY \cdot \lambda D$

Then likewise we could show that

figure on λL similar to the figure on $AC = \text{rect } BL \cdot LD$

And since the straight line λH has been cut equally at G and unequally at F ,

sq $HF + \text{sq } FN = 2[\text{sq } HG + \text{sq } GF] = 2[\text{sq } NG + \text{sq } GF]$ (Eucl vi 9)

Then for the same reasons also

sq $MF + \text{sq } FK = 2[\text{sq } KL + \text{sq } LF]$

and the figure on MF and FK similar to the figure on AC are double the similar figures on λL and LF

And

figure on λI + figure on $FL = \text{rect } BY \cdot \lambda D + \text{rect } BL \cdot LD$ (above),

and

sq $NG + \text{sq } GF = \text{sq } \lambda E + \text{sq } EL$,

therefore

sq $NI + \text{sq } FH$ + figures on λF and FH similar to the figure on $AC =$
 $2[\text{rect } BY \cdot \lambda D + \text{rect } BL \cdot LD + \text{sq } \lambda E + \text{sq } EL]$

And since the straight line BD has been cut equally at E and unequally at Y ,

rect $BY \cdot \lambda D + \text{sq } \lambda E = \text{sq } BF$ (Eucl ii 5)

And likewise also

rect $BL \cdot LD + \text{sq } LE = \text{sq } BE$,

and so

rect $BY \cdot \lambda D + \text{rect } BL \cdot LD + \text{sq } \lambda E + \text{sq } LE = 2 \text{ sq } BF$

Therefore the squares on NI and FH together with figures on λF and FM similar to the figure on AC are double the square on BF . But also

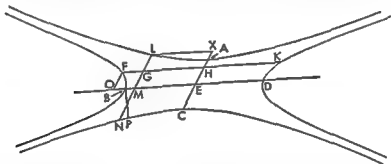
sq $BD = 2 \text{ sq } BE$,

therefore the squares on NI and FH plus the figures on λF and FM similar to the figure on AC are equal to the square on BD

PROPOSITION 28

If in conjugate opposite sections conjugate diameters are drawn, and one of them is called the upright, and the other the transverse and two straight lines are drawn parallel to them and meeting each other and the sections then the squares on the straight lines cut off on the straight line drawn parallel to the upright between the point of meeting of the straight lines and the sections have to the squares on the straight lines cut off on the straight line drawn parallel to the transverse between the point of meeting of the straight lines and the sections the ratio which the square on the upright diameter has to the square on the transverse diameter

Let there be the conjugate opposite sections A, B, C , and D and let AEC be the upright diameter and BED the transverse and let $FGHK$ and $LGMN$



be drawn parallel to them and cutting each other and the sections

I say that

$$\text{sq } LG + \text{sq } GN : \text{sq } FG + \text{sq } GK :: \text{sq } AC : \text{sq } BD$$

For let LY and FO be drawn ordinatewise from F and L , therefore they are parallel to AC and BD . And from B let the upright side for BD BP be drawn, then it is evident that

$$PB : BD :: \text{sq } AC : \text{sq } BD \text{ (I 15)} :: \text{sq } AE : \text{sq } EB$$

$$\text{sq } FO : \text{rect } BO \cdot OD \text{ (I 21)} :: \text{rect } CX, XA : \text{sq } LY \text{ (I 60 21)}$$

Therefore as one of the antecedents is to one of the consequents so are all of the antecedents to all of the consequents (Eucl v 12) therefore

$$\text{sq } AC : \text{sq } BD :: \text{rect } CX, XA + \text{sq } AE + \text{sq } OF : \text{rect } DO, OB + \text{sq } BE, \\ + \text{sq } LX$$

or

$$\text{sq } AC : \text{sq } BD :: \text{rect } CX, XA + \text{sq } AE + \text{sq } EH : \\ \text{rect } DO, OB + \text{sq } BE + \text{sq } ME$$

But

$$\text{rect } CX, XA + \text{sq } AE = \text{sq } YE,$$

and

$$\text{rect } DO, OB + \text{sq } BE = \text{sq } OE \text{ (Eucl II 6),}$$

therefore

$$\text{sq } AC : \text{sq } BD :: \text{sq } YE + \text{sq } EH : \text{sq } OE + \text{sq } EM \\ \text{sq } LM + \text{sq } MG : \text{sq } FH + \text{sq } HG$$

And as has been shown

$$\text{sq } NG + \text{sq } GL = 2[\text{sq } LM + \text{sq } MG]$$

and

$$\text{sq } FG + \text{sq } GK = 2[\text{sq } FH + \text{sq } HG] \text{ (Eucl II 9),}$$

therefore also

$$\text{sq } AC = \text{sq } BD = \text{sq } NG + \text{sq } GL = \text{sq } FG + \text{sq } GK$$

PROPOSITION 29

With the same things supposed, if the parallel to the upright diameter cuts the asymptotes then the squares on the straight lines cut off on the straight line drawn parallel to the upright between the point of meeting of the straight lines and the asymptotes plus the half of the square on the upright diameter has to the squares on the straight lines cut off on the straight line drawn parallel to the transverse between the point of meeting of the straight lines and the sections the ratio which the square on the upright diameter has to the square on the transverse

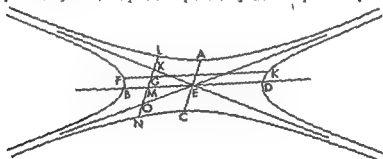
For let there be the same things as before and let NL cut the asymptotes at Y and O

It is to be shown that

$$\text{sq } LG + \text{sq } GO + \text{half sq } AC = \text{sq } FG + \text{sq } GK = \text{sq } AC = \text{sq } BD$$

or

$$\text{sq } LG + \text{sq } GO + 2 \text{ sq } AL = \text{sq } FG + \text{sq } GK = \text{sq } AC = \text{sq } BD$$



For since

$$LY = ON \text{ (II 16),}$$

$$\text{sq } LG + \text{sq } ON + 2 \text{ rect } NY, LY = \text{sq } LG + \text{sq } GO,¹$$

therefore

$$\text{sq } XG + \text{sq } GO + 2 \text{ sq } AE = \text{sq } LG + \text{sq } ON$$

And

$$\text{sq } IG + \text{sq } ON = \text{sq } FG + \text{sq } GK = \text{sq } AC = \text{sq } BD \text{ (III 28),}$$

therefore also

$$\text{sq } LG + \text{sq } GO + 2 \text{ sq } AE = \text{sq } FG + \text{sq } GK = \text{sq } AC = \text{sq } BD$$

¹For

$$OM = ML$$

Therefore as in a lemma of Pappus since

$$2 \text{ rect } NY, LY + 2 \text{ sq } ML = 2 \text{ sq } ML \text{ (Eucl II 5)}$$

adding the common square on CW

$$2 \text{ rect } NY, LY + 2 \text{ sq } MY + 2 \text{ sq } GW = 2 \text{ sq } ML + 2 \text{ sq } GW$$

And

$$2 \text{ sq } ML + 2 \text{ sq } CW = \text{sq } NG + \text{sq } LG$$

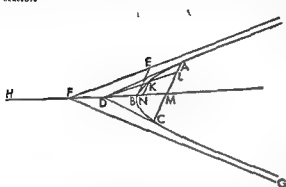
and

$$2 \text{ sq } MX + 2 \text{ sq } GW = \text{sq } OG + \text{sq } GX \text{ (Eucl II 9)}$$

Therefore as above

PROPOSITION 30

If two straight lines touching an hyperbola meet and through the points of contact a straight line is produced and through the point of meeting a straight line is drawn parallel to some one of the asymptotes and cutting both the section and the straight line joining the points of contact then the straight line between the point of meeting and the straight line joining the points of contact will be bisected by the section¹



Let there be the hyperbola *ABC*, and let *AD* and *DC* be tangents and *EF* and *FG* asymptotes and let *AC* be joined and through *D* parallel to *FE* let *DKL* be drawn

I say that

$$DK = KL$$

For let *FDBM* be joined and produced both ways and let *FH* be made equal to *BF* and through the points *B* and *A* let *BE* and *AN* be drawn

parallel to *AC* therefore they have been dropped ordinatewise (II 30 5 7)
And since triangle *BEF* is similar to triangle *DNK*, therefore

$$\text{sq } DN : \text{sq } NK :: \text{sq } BF : \text{sq } BE \quad (\alpha)$$

And

$$\text{sq } BF : \text{sq } BE :: HB : \text{upright} \quad (\pi 1),$$

therefore also

$$\text{sq } DN : \text{sq } NK :: HB : \text{upright}$$

But

$$HB : \text{upright} :: \text{rect } HN, NB : \text{sq } NK \quad (\iota 21),$$

therefore also

$$\text{sq } DN : \text{sq } NK :: \text{rect } MN, NB : \text{sq } NK \quad (\beta)$$

Therefore

$$\text{rect } HN, NB = \text{sq } DN$$

¹The propositions from 30 to 34 inclusive are one special case and propositions 35 and 36 are another special case of proposition 37. The first group takes the line drawn through the intersection of the tangents as parallel to an asymptote. The second group takes one of the tangents as an asymptote. Proposition 34 lying between is special in both these ways.

In proposition 37 we have the line *CF* divided by the section at *D* and *F* and at *E* by the straight line joining the points of contact in such a way that

$$CF : CD :: FE : ED$$

This is the same form of the harmonic proportion as we found in I 34 and *DF* is the harmonic mean between *CF* and *FE*.

If we argue by analogy from this proportion treating infinity as a definite magnitude and two such infinities as would occur here as equal and subject to the general laws of magnitudes we can immediately deduce the special cases of propositions 30 to 36. Thus in the case of the first group *CF* and *FE* both become infinite therefore *CD* is equal to *ED*.

And

$$\text{sq } LG = \text{sq } YE$$

and

$$\text{sq } KF = \text{sq on half of second diameter (II 1),} \\ = \text{rect } CE, ED \text{ (I 38),}$$

therefore

$$\text{sq } XC = \text{sq } XE + \text{rect } CE, ED$$

Therefore the straight line CD has been cut equally at X and unequally at E (Eucl II 5)

And DH is parallel to GY , therefore

$$CG = GH$$

PROPOSITION 32

If two straight lines touching an hyperbola meet, and a straight line is produced through the points of contact and a straight line is drawn through the point of meeting of the tangents parallel to the straight line joining the points of contact and a straight line is drawn through the midpoint of the straight line joining the points of contact parallel to one of the asymptotes, then the straight line cut off between this midpoint and the parallel will be bisected by the section

Let there be the hyperbola ABC , whose center is D , and asymptote DE , and let AF and FC touch and let CA and FD be joined and produced to G and H , then it is evident that

$$AH = HC$$

Then let FK be drawn through F parallel to AC and HLK through H parallel to DE

I say that

$$KL = HL$$

Let LM and BE be drawn through B and L parallel to AC then as has been already shown (III 30 α , β and conclusion)

$$\text{sq } DB = \text{sq } BE = \text{sq } HM = \text{sq } ML = \text{rect } BM, MG = \text{sq } ML$$

therefore

$$\text{rect } GM, MB = \text{sq } MH$$

And also

$$\text{rect } HD, DF = \text{sq } DB$$

because AF touches and AH has been dropped ordinatewise (I 37) therefore

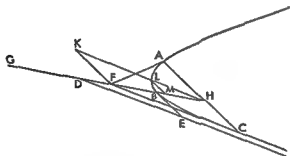
$$\text{rect } GM, MB + \text{sq } DB = \text{rect } HD, DF + \text{sq } MH = \text{sq } DM \text{ (Eucl II 6)}$$

Therefore FH has been bisected at M with DF added And KF and LM are parallel, therefore

$$KL = LH$$

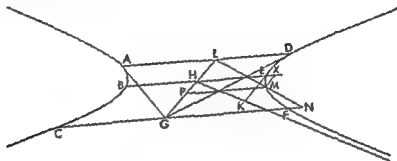
PROPOSITION 33

If two straight lines touching opposite sections meet, and one straight line is produced through the points of contact and another straight line is drawn through the point of meeting of the tangents parallel to the straight line joining the points of



contact and still another straight line is drawn through the midpoint of the straight line joining the points of contact parallel to one of the asymptotes and meeting the section and the parallel drawn through the point of meeting, then the straight line between the midpoint and the parallel will be bisected by the section

Let there be the opposite sections ABC and DEF , and tangents AG and DG and center H , and asymptote AH , and let HG be joined and produced and also let ALD be joined then it is evident that it is bisected at L (II 30) Then let BHE and CGF be drawn through G and H parallel to AD , and LMN through L parallel to HK



I say that

$$LM = MN$$

For let EK and MA be dropped from E and M parallel to GH and MP through M parallel to AD

Since then through things already shown (III 30 α and β)

$$\text{sq } HF = \text{sq } EK = \text{rect } BL, LE = \text{sq } AM$$

therefore

$$\text{sq } HE = \text{sq } EA = \text{rect } BL, LE + \text{sq } HE = \text{sq } AE + \text{sq } AM \quad (\text{Eucl v 12})$$

or

$$\text{sq } HF = \text{sq } EA = \text{sq } HA = \text{sq } AE + \text{sq } AM \quad (\text{Eucl II 6})$$

But it has been shown (I 38, II 1)

$$\text{sq } EA = \text{rect } GH, HL$$

and

$$\text{sq } AM = \text{sq } MP$$

therefore

$$\text{sq } HE = \text{sq } EK = \text{sq } HA = \text{sq } MP = \text{rect } GH, HL + \text{sq } MP$$

And

$$\text{sq } HF = \text{sq } EK = \text{sq } MP = \text{sq } PL \quad (\text{Eucl VI 4})$$

therefore

$$\text{sq } MP = \text{sq } PL = \text{rect } GH, HL + \text{sq } MP$$

Therefore

$$\text{sq } PL = \text{rect } GH, HL + \text{sq } MP$$

Therefore the straight line IG has been cut equally at P and unequally at H (Eucl II 5)

And MP and GN are parallel therefore

$$LM = MN$$

PROPOSITION 34

If one point is taken on one of the asymptotes of an hyperbola, and a straight line from it touches the section and through the point of contact a parallel to the asymptote is drawn then the straight line drawn from the point taken parallel to the other asymptote will be bisected by the section

Let there be the hyperbola AB and the asymptotes CD and DE , and let a point C be taken at random on CD , and through it let CBE be drawn touching the section and through B let FBG be drawn parallel to CD and through C let CAG be drawn parallel to DE

I say that

$$CA = AG$$

For let AH be drawn through A parallel to CD , and BK through B parallel to DE Since then

$$CB = BE \text{ (II 3),}$$

therefore also

$$CK = KD$$

and

$$DF = FE$$

And since

$$\text{rect } KB, BF = \text{rect } CA, AH \text{ (II 12),}$$

and

$$BF = DK = CK,$$

and

$$AH = DC,$$

therefore

$$\text{rect } DC, CA = \text{rect } GC, CK$$

Therefore

$$DC \cdot CK = GC \cdot CA$$

And

$$CD = 2CK,$$

therefore also

$$GC = 2CA$$

Therefore

$$CA = AG$$

PROPOSITION 35

With the same things being so if from the point taken some straight line is drawn cutting the section at two points then as the whole straight line is to the straight line cut off outside so will the segments of the straight line cut off inside be to each other

For let there be the hyperbola AB and the asymptotes CD and DE and CBE touching and HB parallel and through C let some straight line $CALFG$ be drawn across cutting the section at A and F

I say that

$$FC \cdot CA = FL \cdot AL$$

For let CNY KAM $OPBR$ and FU be drawn through C A B and F

parallel to DE , and APS and $TFRMX$ through A and F parallel to CD

Since then

$$AC = FG \text{ (II 8),}$$

therefore also

$$KA = TG \text{ (Eucl VI 4)}$$

But

$$KA = DS,$$

therefore also

$$TG = DS$$

And so also

$$CK = DU$$

And since

$$CK = DU,$$

also

$$DK = CU,$$

therefore

$$DK \quad CK \quad CU \quad CK$$

And

$$CU \quad CK \quad FC \quad AC,$$

and

$$FC \quad AC \quad MK \quad KA,$$

and

$$MK \quad KA \quad \text{p||g } MD \quad \text{p||g } DA \text{ (Eucl VI 1),}$$

and

$$DK \quad CK \quad \text{p||g } HK \quad \text{p||g } KN,$$

therefore also

$$\text{p||g } MD \quad \text{p||g } DA \quad \text{p||g } HK \quad \text{p||g } KN$$

But

$$\text{p||g } DA = \text{p||g } DB \text{ (II 12)} = \text{p||g } ON,$$

for

$$CB = BE \text{ (II 3),}$$

and

$$DO = OC,$$

therefore

$$\text{p||g } MD \quad \text{p||g } ON \quad \text{p||g } HK \quad \text{p||g } KN$$

and

$$\text{remainder p||g } MH \quad \text{remainder p||g } BK \quad \text{whole p||g } MD \quad \text{whole p||g } ON$$

And since

$$\text{p||g } DA = \text{p||g } DB$$

let the common parallelogram DP be subtracted,

therefore

$$\text{p||g } AP = \text{p||g } PH$$

Let the common parallelogram AB be added therefore

$$\text{whole p||g } BK = \text{whole p||g } AH$$

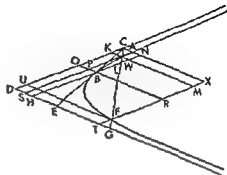
Therefore

$$\text{p||g } MD \quad \text{p||g } DA \quad \text{p||g } MH \quad \text{p||g } AH$$

But

$$\text{p||g } MD \quad \text{p||g } DA \quad MK \quad KA \quad FC \quad AC,$$

and



$\text{p}llg\ MH\ \text{p}llg\ AH\ MW\ WA\ FL\ LA,$

therefore also

$FC\ AC\ FL\ LA$

PROPOSITION 36

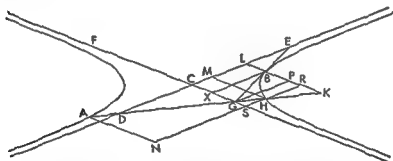
With the same things being so if the straight line drawn across from the point neither cuts the section at two points nor is parallel to the asymptote it will meet the opposite section and as the whole straight line is to the straight line between the section and the parallel through the point of contact so will the straight line between the opposite section and the asymptote be to the straight line between the asymptote and the other section

Let there be the opposite sections A and B whose center is C and asymptotes DE and FG and let some point G be taken on CG and from it let GBE be drawn tangent and GH neither parallel to CE nor cutting the section in two points (I 26)

It has been shown that GH produced meets CD and therefore also section A . Let it meet it at A and let KBL be drawn through B parallel to CG

I say that

$AK\ KH\ AG\ GH$



For let HM and AN be drawn from the points A and H parallel to CG and $BX\ GP$ and $RHSN$ from $B\ G$ and H parallel to DE . Since then

$AD = GH$ (I 16)
 $AG\ GH\ DH\ HG$

But

$AG\ GH\ NS\ SH$

and

$DH\ GH\ CS\ SG$

And therefore

$NS\ SH\ CS\ SG$

But

$NS\ SH\ \text{p}llg\ AC\ \text{p}llg\ CH$

and

$CS\ SG\ \text{p}llg\ RC\ \text{p}llg\ RG$

therefore also

$\text{p}llg\ NC\ \text{p}llg\ CH\ \text{p}llg\ RC\ \text{p}llg\ RG$

And as one is to one so are all to all therefore

And since $\text{pllg } NC \text{ pllg } CH \text{ whole pllg } NL \text{ whole pllg } CH + \text{pllg } RG$

$$EB = BG,$$

also

$$LB = BP$$

and

$$\text{pllg } LA = \text{pllg } BG$$

And

$$\text{pllg } LA = \text{pllg } CH \text{ (II 12),}$$

therefore also

$$\text{pllg } BG = \text{pllg } CH$$

Therefore

$$\text{pllg } NC \text{ pllg } CH \text{ whole pllg } NL \text{ whole pllg } BG + \text{pllg } RG$$

or

$$\text{pllg } NC \text{ pllg } CH \text{ pllg } NL \text{ pllg } RX$$

But

$$\text{pllg } RX = \text{pllg } LH$$

since also

$$\text{pllg } CH = \text{pllg } BC \text{ (II 12),}$$

and

$$\text{pllg } MB = \text{pllg } AH$$

Therefore

$$\text{pllg } NC \text{ pllg } CH \text{ pllg } NL \text{ pllg } LH$$

But

$$\text{pllg } NC \text{ pllg } CH \text{ } NS \text{ } SH \text{ } AG \text{ } GH,$$

and

$$\text{pllg } NL \text{ pllg } LH \text{ } NR \text{ } RH \text{ } AK \text{ } KH,$$

therefore also

$$AK \text{ } KH \text{ } AG \text{ } GH$$

PROPOSITION 37

If two straight lines touching a section of a cone or circumference of a circle or opposite sections meet and a straight line is joined to their points of contact and from the point of meeting of the tangents some straight line is drawn across cutting the line (of the section) at two points then as the whole straight line is to the straight line cut off outside so will the segments produced by the straight line joining the points of contact be to each other

Let there be the section of a cone AB and tangents AC and CB and let AB be joined and let $CDEF$ be drawn across

I say that

$$CF \text{ } CD \text{ } FE \text{ } FD$$

Let the diameters CH and AK be drawn through C and A and through F and D DP FR LFM and NDO parallel to AH and LC Since then LFM is parallel to NDO

$$FC \text{ } CD \text{ } LF \text{ } \Lambda D \text{ } FM \text{ } DO \text{ } LM \text{ } \Lambda O$$

and therefore

$$\text{sq } LM \text{ sq } \Lambda O \text{ sq } FM \text{ sq } DO$$

But

$$\text{sq } LM \text{ sq } \Lambda O \text{ trgl } LMC \text{ trgl } \Lambda CO \text{ (Eucl vi 19),}$$

and

$$\text{sq } FM \text{ sq } DO \quad \text{trgl } FRM \text{ trgl } DPO,$$

therefore also

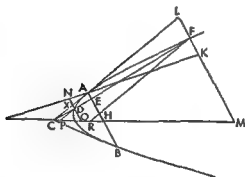
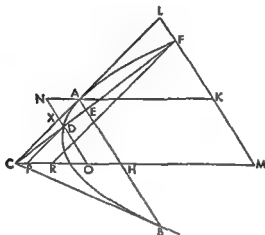
$$\text{trgl } LMC \text{ trgl } \lambda CO \quad \text{trgl } FRM \text{ trgl } DPO \\ \text{remainder quadr } LCRF \quad \text{remainder quadr } \lambda CP$$

But

$$\text{quadr } LCRF = \text{trgl } ALK \text{ (III 2, III 11),}$$

and

$$\text{quadr } \lambda CPD = \text{trgl } ANY \text{ (III 2, III 11),}$$



therefore

$$\text{sq } LM \text{ sq } XO \quad \text{trgl } ALK \text{ trgl } ANX$$

But

$$\text{sq } LM \text{ sq } XO \quad \text{sq } FC \text{ sq } CD,$$

and

$$\text{trgl } ALK \text{ trgl } ANX \quad \text{sq } LA \text{ sq } AX \quad \text{sq } FE \text{ sq } ED,$$

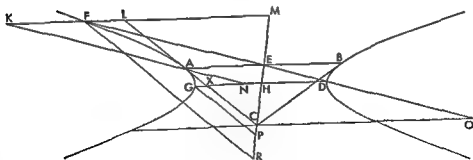
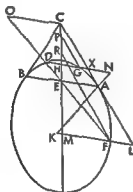
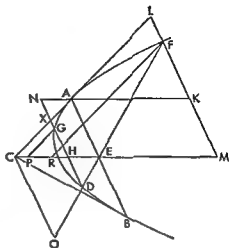
therefore also

$$\text{sq } FC \text{ sq } CD \quad \text{sq } FE \text{ sq } ED$$

And therefore

$$FC \quad CD \quad FE \quad ED$$

For let $LFAM$ and $DHGAN$ be drawn through F and D parallel to AB , and through F and G FR and GP parallel to LC . Then likewise as before (III 37) it will be shown that
 $\text{sq } LM = \text{sq } XH = \text{sq } LA = \text{sq } AX$
 And
 $\text{sq } LM = \text{sq } XH = \text{sq } LC = \text{sq } CX$
 $\text{sq } FO = \text{sq } OD$,
 and
 $\text{sq } LA = \text{sq } AX = \text{sq } FE = \text{sq } ED$,
 therefore
 $\text{sq } FO = \text{sq } OD = \text{sq } FE = \text{sq } ED$,
 and
 $FO = OD = FE = ED$



PROPOSITION 39

If two straight lines touching opposite sections meet and a straight line is produced through the points of contact and a straight line drawn from the point of meeting of the tangents cuts both of the sections and the straight line joining the points of contact then as the whole straight line drawn across to the straight line

and

$$\text{trgl } FRM = \text{trgl } A\backslash N + \text{trgl } \backslash MD \text{ (II 11),}$$

therefore

$$\text{trgl } DHS \text{ trgl } \backslash MD \text{ trgl } ASK + \text{trgl } DHS \text{ trgl } A\backslash N + \text{trgl } \backslash MD,$$

and

$$\text{remainder trgl } ASK \text{ remainder trgl } A\backslash N \text{ trgl } DHS \text{ trgl } \backslash MD$$

But

$$\text{trgl } ASK \text{ trgl } A\backslash N \text{ sq } KA \text{ sq } AN \text{ sq } EG \text{ sq } FG,*$$

and

$$\text{trgl } DHS \text{ trgl } \backslash MD \text{ sq } HD \text{ sq } DM \text{ sq } ED \text{ sq } DF$$

Therefore also

$$EG \text{ } FG \text{ } ED \text{ } DF$$

PROPOSITION 40

With the same things being so if a straight line is drawn through the point of meeting of the tangents parallel to the straight line joining the points of contact and if a straight line drawn from the midpoint of the straight line joining the points of contact cuts both of the sections and the straight line parallel to the straight line joining the points of contact then as the whole straight line drawn across is to the straight line cut off outside between the parallel and the section, so will the straight line's segments produced by the sections and the straight line joining the points of contact be to each other

Let there be the opposite sections A and B whose center $\equiv C$, and tangents AD and DB , and let AB and CDE be joined, therefore

$$AE = EB \text{ (II 39)}$$

And from D let FDG be drawn parallel to AB , and from E , LE at random

I say that

$$HL \text{ } LA \text{ } HE \text{ } EA$$

From H and A let NMH and KOP be drawn parallel to AB and HR and AS parallel to AD , and let $\backslash ACT$ be drawn through

Since then $\backslash AU$ and MAP have been drawn across the parallels $\backslash M$ and AP

$$XA \text{ } AU \text{ } MA \text{ } AP$$

But

$$XA \text{ } AU \text{ } HE \text{ } EA,$$

and

$$HE \text{ } EA \text{ } HN \text{ } KO$$

because of the similarity of the triangles HEN and KEO , therefore

$$HN \text{ } KO \text{ } MA \text{ } AP,$$

therefore also

$$\text{sq } HN \text{ sq } KO \text{ sq } MA \text{ sq } AP$$

For

$$EG \text{ } TG \text{ } KA \text{ } TA$$

and

$$TG \text{ } TF \text{ } TA \text{ } TN$$

and

$$TG - TF \text{ } TG \text{ } TA - TN \text{ } TA$$

therefore ex aequalis

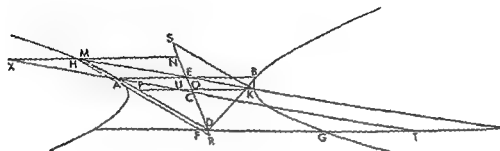
$$EG \text{ } FG \text{ } KA \text{ } AV$$

But

$$sq\ HN \quad sq\ KO \quad trgl\ HRN \quad trgl\ KSO,$$

and

$$sq\ MA \quad sq\ AP \quad trgl\ \lambda MA \quad trgl\ AUP,$$



therefore also

$$trgl\ HRN \quad trgl\ KSO \quad trgl\ \lambda MA \quad trgl\ AUP$$

And

$$trgl\ HNR = trgl\ \lambda MA + trgl\ MND \text{ (iii 11),}$$

and

$$trgl\ KSO = trgl\ AUP + trgl\ DOP \text{ (iii 11),}$$

therefore also

$$trgl\ \lambda MA + trgl\ MND \quad trgl\ AUP + trgl\ DOP \quad trgl\ \lambda MA \quad trgl\ AUP,$$

therefore also

$$\text{remainder } trgl\ NMD \quad \text{remainder } trgl\ DOP \quad \text{whole} \quad \text{whole}$$

But

$$trgl\ \lambda MA \quad trgl\ AUP \quad sq\ \lambda A \quad sq\ AU,$$

and

$$trgl\ NMD \quad trgl\ DOP \quad sq\ MN \quad sq\ PO,$$

therefore also

$$sq\ MN \quad sq\ PO \quad sq\ \lambda A \quad sq\ AU$$

But

$$sq\ MN \quad sq\ PO \quad sq\ ND \quad sq\ OD,$$

and

$$sq\ \lambda A \quad sq\ AU \quad sq\ HF \quad sq\ EK,$$

and

$$sq\ ND \quad sq\ DO \quad sq\ HL \quad sq\ LK,$$

therefore also

$$sq\ HE \quad sq\ FK \quad sq\ HL \quad sq\ LK$$

Therefore

$$HE \quad EK \quad HL \quad LK$$

PROPOSITION 11

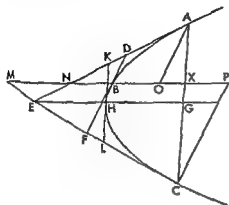
If three straight lines touching a parabola meet each other they will be cut in the same ratio

Let there be the parabola ABC , and tangents ADE FFC and DBF

I say that

$$CF \quad FE \quad ED \quad DA \quad FB \quad BD$$

For let AC be joined and bisected at G



Then it is evident that the straight line from E to G is a diameter of the section (II 29)

If then it goes through B DF is parallel to AC , (I 5) and will be bisected by EG and therefore

$$AD = DE \text{ (I 35)}$$

and

$$CF = FE \text{ (I 35),}$$

and what was sought is apparent

Let it not go through B but through H , and let KHL be drawn through H parallel to AC therefore it will touch the section at H (I 32), and because of things already said (I 35),

$$AK = KE$$

and

$$LC = LE$$

Let $MNB\Upsilon$ be drawn through B parallel to EG and AO and CP through A and C parallel to DF Since then MB is parallel to EH , MB is a diameter (I 40, I 51 end), and DF touches at B , therefore AO and CP have been dropped ordinatewise (II 5, First Def 1 4) And since MB is a diameter, and CM a tangent and CP an ordinate

$$MB = BP \text{ (I 35),}$$

and so also

$$MF = FC$$

And since

$$MF = FC$$

and

$$EL = LC,$$

$$MC \quad CF \quad EC \quad CL,$$

and alternately

$$MC \quad EC \quad CF \quad CL$$

But

$$MC \quad EC \quad XC \quad CG$$

therefore also

$$CF \quad CL \quad XC \quad CG$$

And

$$CL \quad EC \quad CG \quad CA$$

therefore *ex aequali*

$$CA \quad XC \quad EC \quad CF$$

and *convertendo*

$$EC \quad FE \quad CA \quad AX,$$

separando

$$CF \quad FE \quad XC \quad AX$$

Again since MB is a diameter and AN a tangent and AO an ordinate,

$$NB=BO \text{ (I 35),}$$

and

$$ND=DA$$

And also

$$EA=AA,$$

therefore

$$AE \quad AA \quad NA \quad DA,$$

alternately

$$AE \quad NA \quad AA \quad DA$$

But

$$AE \quad NA \quad GA \quad AX,$$

therefore also

$$AA \quad DA \quad GA \quad AX$$

And also

$$AE \quad AA \quad CA \quad GA,$$

therefore, *ex aequali*,

$$AE \quad DA \quad CA \quad AX,$$

separando

$$ED \quad DA \quad AX \quad AX$$

And it was also shown

$$AX \quad AX \quad CF \quad FE,$$

therefore

$$CF \quad FE \quad FD \quad DA$$

Again since

$$AX \quad AX \quad CP \quad AO,$$

and

$$CP=2 BF,$$

and

$$CM=2 MF,$$

and

$$AO=2 BD,$$

and

$$AN=2 ND,$$

therefore

$$XC \quad AX \quad FB \quad BD \quad CF \quad FE \quad ED \quad DA$$

PROPOSITION 42

If in an hyperbola or ellipse or circumference of a circle or opposite sections straight lines are drawn from the vertex of the diameter parallel to an ordinate and some other straight line at random is drawn tangent it will cut off from them straight lines containing a rectangle equal to the fourth part of the figure to the same diameter

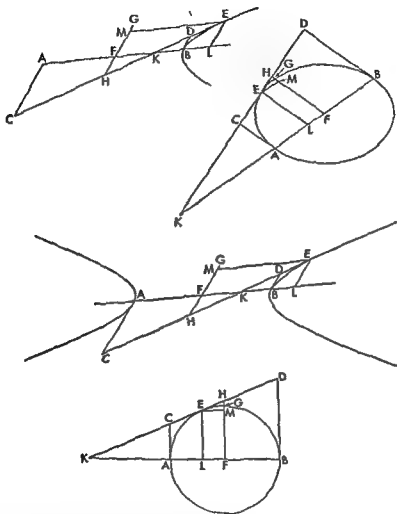
For let there be some one of the aforesaid sections whose diameter is AB and from A and B let AC and DB be drawn parallel to an ordinate, and let some other straight line CED be tangent at E

I say that

$$\text{rect } AC \ BD = \text{fourth part of figure to } AB$$

For let its center be F and through it let FG be drawn parallel to AC and BD . Since then AC and BD are parallel and FG is also parallel, therefore it is the diameter conjugate to AB (First Def, 16), and so

sq FG = fourth part of figure to AB (See Def 13)



If then FG goes through E in the case of the ellipse and circle
 $AC = FG = BD$ (17)

and it is immediately evident that

rect $AC \cdot BD =$ sq FG or fourth part of figure to AB

Then let it not go through it and let DC and BA produced meet at K and let EL be drawn through E parallel to AC and FM parallel to AB . Since then

rect $KF, FL =$ sq AF (137)
 $KF \cdot AF = AF \cdot FL$

and

$$KA \ AL \ KF \ AF \text{ or } FB \text{ (Eucl v 19),}$$

inversely

$$FB \ KF \ AL \ KA,$$

componendo or separando

$$BK \ KF \ LK \ KA$$

Therefore also

$$DB \ FH \ EL \ CA$$

Therefore

$$\begin{aligned} \text{rect } DB, CA &= \text{rect } FH, EL, \\ &= \text{rect } HF, FM \end{aligned}$$

But

$$\begin{aligned} \text{rect } HF, FM &= \text{sq } FG \text{ (I 38),} \\ &= \text{fourth figure to } AB \text{ (Sec Def 1 11),} \end{aligned}$$

therefore also

$$\text{rect } DB \ CA = \text{fourth figure to } AB$$

PROPOSITION 43

If a straight line touch an hyperbola it will cut off from the asymptotes beginning with the center of the section straight lines containing a rectangle equal to the rectangle contained by the straight lines cut off by the tangent at the section's vertex at its axis

Let there be the hyperbola AB , and asymptotes CD and DE , and axis BD , and let FBG be drawn through B tangent, and some other tangent at random CAH

I say that

$$\text{rect } FD \ DG = \text{rect } CD \ DH$$

For let AK and BL be drawn from A and B parallel to DG and AM and BN parallel to CD Since then CAH touches

$$CA = AH \text{ (II 3),}$$

and so

$$CH = 2AH$$

and

$$CD = 2AM$$

and

$$DH = 2AK$$

Therefore

$$\text{rect } CD \ DH = 4 \text{ rect } KA \ AM$$

Then likewise it could be shown

$$\text{rect } FD \ DG = 4 \text{ rect } LB \ BN$$

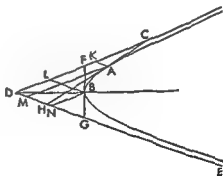
But

$$\text{rect } KA \ AM = \text{rect } LB \ BN \text{ (II 12)}$$

Therefore also

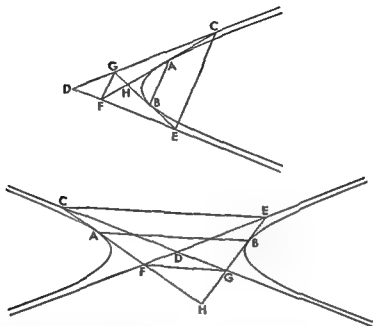
$$\text{rect } CD \ DH = \text{rect } FD \ DG$$

Then likewise it could be shown even if DB were some other diameter and not the axis



PROPOSITION 44

If two straight lines touching an hyperbola or opposite sections meet the asymptotes then the straight lines drawn to the sections will be parallel to the straight line joining the points of contact



For let there be either the hyperbola or the opposite sections AB and asymptotes CD and DE and tangents $CAHF$ and $EBHG$, and let AB FG , and CE be joined

I say that they are parallel

For since

$$\text{rect } CD \text{ } DF = \text{rect } GD, DE \text{ (III 43),}$$

therefore

$$\frac{CD}{DE} = \frac{GD}{DF},$$

therefore CE is parallel to FG And therefore

$$\frac{HF}{FC} = \frac{HG}{GE}$$

And

$$\frac{FC}{AF} = \frac{GE}{GB},$$

for each is double (II 3) therefore *ex aequali*

$$\frac{HG}{GB} = \frac{HF}{FA}$$

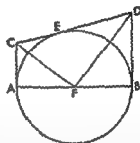
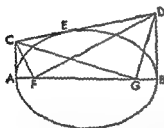
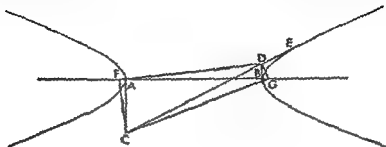
Therefore FG is parallel to AB

PROPOSITION 45

If in an hyperbola or ellipse or circumference of a circle or opposite sections straight lines are drawn from the vertex of the axis at right angles and a rectangle equal to the fourth part of the figure is applied to the axis on each side and exceed

ing by a square figure in the case of the hyperbola and opposite sections but deficient in the case of the ellipse and some straight line is drawn tangent to the section and meeting the perpendicular straight lines then the straight lines drawn from the points of meeting to the points produced by the application make right angles at the aforesaid points¹

Let there be one of the sections mentioned whose axis is AB , and AC and BD



¹ The points of application are in modern terminology the foci of the conics. The circle is seen here as an ellipse whose two foci or focal points coincide with the center. This theory is of course a special application of Euclid vi 28 and 29 two theorems on which depends one whole side of Greek geometry.

Apollonius never speaks of the focus of the parabola but it can be found by analogy with the ellipse.

Thus in the ellipse above

$$\text{rect } AF \cdot FB = \text{fourth rect } AB \cdot R$$

where R is the parameter. Or

$$\text{rect } 1F (AB - AF) = \text{fourth rect } AB \cdot R$$

or

$$1F \text{ fourth } AB (AB - AF)$$

Then if we consider the ellipse as its axis AB gets as large as we please we can think of it as approaching as near as we please to a parabola with parameter R . The ratio $1B : (1B - 1F)$ approaches as near as we please to equality and hence also the ratio $AF : \text{fourth } R$. At the limit we can think of the ellipse as the parabola its axis AB as infinite and AB as equal to $AB - 1F$. Then $1F$ will be equal to a fourth R . Thus the focus of a parabola will be defined as the point on its axis at a distance from the vertex equal to one quarter of the parameter. Then many of the properties of the foci of the ellipse can be proved analogously for the parabola. Thus in the case of this proposition, FD will become parallel to CE . Hence any straight line from the focus of a parabola parallel to a tangent will make a right angle with the straight line drawn from the focus to the intersection of the tangent and the perpendicular to the axis at the vertex.



at right angles and CED tangent and let the rectangle $AF FB$ and the rectangle $4G GB$ equal to the fourth part of the figure be applied on each side (Eucl vi 28 29), as has been said and let CF CG , DF , and DG be joined

I say that angle CFD and angle CGD are each a right angle

For since it has been shown

$$\text{rect } AC BD = \text{fourth figure on } AB \text{ (III 42),}$$

and since also

$$\text{rect } AF FB = \text{fourth figure on } AB,$$

therefore

$$\text{rect } AC, BD = \text{rect } AF, FB$$

Therefore

$$AC \cdot AF = FB \cdot BD$$

And the angles at points A and B are right therefore

$$\text{angle } ACF = \text{angle } BFD \text{ (Eucl vi 6),}$$

and

$$\text{angle } AFC = \text{angle } FDB$$

And since angle CAF is right, therefore

$$\text{angle } ACF + \text{angle } AFC = 1 \text{ right angle}$$

And it has also been shown that

$$\text{angle } ACF = \text{angle } DFB,$$

therefore

$$\text{angle } AFC + \text{angle } DFB = 1 \text{ right angle}$$

Therefore

$$\text{angle } DFC = 1 \text{ right angle}$$

Then likewise it could also be shown

$$\text{angle } CGD = 1 \text{ right angle}$$

PROPOSITION 46

With the same things being so the straight lines joined make equal angles with the tangents

For with the same things supposed I say that

$$\text{angle } ACF = \text{angle } DCG$$

and

$$\text{angle } CDF = \text{angle } BDG$$

For since it has been shown that both angle CFD and angle CGD are right angles (III 45) the circle described about CD as a diameter will pass through points F and G therefore

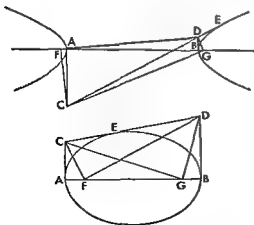
$$\text{angle } DCG = \text{angle } DFG$$

for they are on the same segment of the circle And it was shown angle $DFG = \text{angle } ACF$ (III 45), and so

$$\text{angle } DCG = \text{angle } ACF$$

And likewise also

$$\text{angle } CDF = \text{angle } BDG$$



PROPOSITION 47

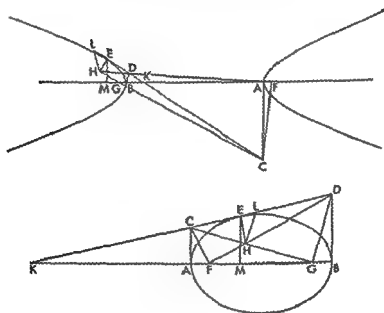
With the same things being so the straight line drawn from the point of meeting of the joined straight lines to the point of contact will be perpendicular to the tangent

For let the same things as before be supposed and let CG and FD meet each other at H and let CD and BA produced meet at K , and let EH be joined

I say that EH is perpendicular to CD

For if not let HL be drawn from H perpendicular to CD Since then
 $\text{angle } CDF = \text{angle } GBD$ (III 46),
 and also

$$\text{rt angle } DBC = \text{rt angle } DLH,$$



therefore triangle DGB is similar to triangle LHD Therefore

$$GD : DH :: BD : DL$$

But

$$GD : DH :: FC : CH$$

because the angles at F and G are right angles (III 45) and the angles at H are equal but

$$FC : CH :: AC : CL$$

because of the similarity of the triangles AFC and LCH (III 46), therefore also

$$BD : DL :: AC : CL$$

Alternately

$$BD : AC :: DL : CL$$

But

$$BD : AC :: BK : KA$$

therefore also

$$DL : CL :: BK : KA$$

Let EM be drawn from E parallel to AC , therefore it will have been dropped ordinate to AB (II 7), and

$$BK : KA = BM : MA$$

And

$$BM : MA = DE : EC$$

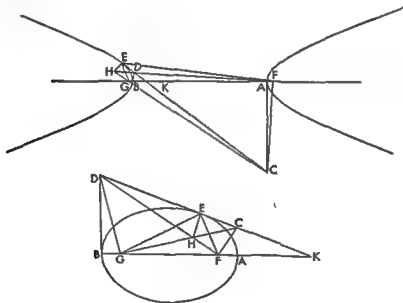
therefore also

$$DL : CL = DE : EC,$$

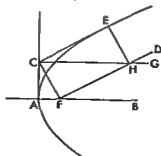
and this is absurd. Therefore HL is not perpendicular nor is any other straight line except HE *

PROPOSITION 48

With the same things being so, it must be shown that the straight lines drawn from the point of contact to the points produced by the application make equal angles with the tangent



*There is the analogous theorem for the parabola FD becomes a straight line parallel to



CE and CG a straight line parallel to AB . Again HE is perpendicular to CE and this can be proved rigorously as well as understood by analogy

For let the same things be supposed, and let EF and EG be joined
I say that

$$\text{angle } CEF = \text{angle } GED$$

For since angles DGH and DEH are right angles (in 45 47), the circle described about DH as a diameter will pass through the points E and G (Eucl in 31), and so

$$\text{angle } DHG = \text{angle } DEG \text{ (Eucl in 21),}$$

for they are in the same segment Likewise then also

$$\text{angle } CEF = \text{angle } CHF$$

But

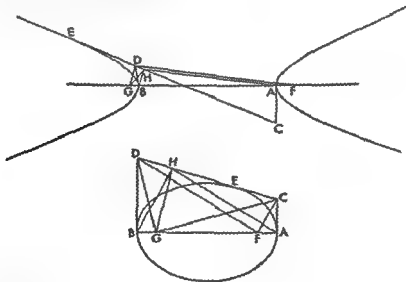
$$\text{angle } CHF = \text{angle } DHG,$$

for they are vertical angles, therefore also

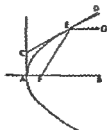
$$\text{angle } CEF = \text{angle } DEG *$$

PROPOSITION 49

With the same things being so if from one of the points (of application) a perpendicular is drawn to the tangent then the straight lines from that point to the ends of the axis make a right angle



Here there is another and important analogous theorem for the parabola EG becomes parallel to AB and



$$\text{angle } DEG = \text{angle } CEF$$

For let the same things be supposed, and let the perpendicular GH be drawn from G to CD , and let AH and BH be joined

I say that angle AHB is a right angle

For since angle DBG is a right angle and also angle DHG the circle described about DG as a diameter will pass through H and B , and
 angle $BHG = \text{angle } BDG$

But it was shown

angle $AGC = \text{angle } BDG$ (in 46),

therefore also

angle $BHG = \text{angle } AGC = \text{angle } AHC$ (Eucl III 21)

And so also

angle $CHG = \text{angle } AHB$

But angle CHG is a right angle, therefore also angle AHB is a right angle

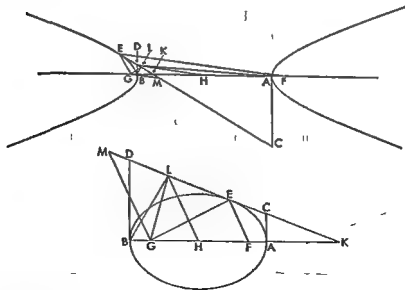
PROPOSITION 50

With the same things being so if from the center of the section there falls to the tangent a straight line parallel to the straight line drawn through the point of contact and one of the points (of application), then it will be equal to one half the axis

For let there be the same things as before and let H be the center, and let EF be joined and let DC and BA meet at K and through H let HL be drawn parallel to EF

I say that

$$HL = HB$$



For let EG , AL , LG be joined and through G let GM be drawn parallel to EF . Since then

$$\text{rect } AF \cdot FB = \text{rect } AG \cdot GB \text{ (See III 45),}$$

therefore

$$AF = GB$$

But also

$$AH = HB,$$

therefore also

$$FH = HG$$

And so also

$$EL = LM$$

And since it was shown (III 48)

$$\text{angle } CEF = \text{angle } DEG,$$

and

$$\text{angle } CEF = \text{angle } EMG,$$

therefore also

$$\text{angle } EMG = \text{angle } DEG$$

And therefore

$$EG = GM$$

But it was also shown

$$EL = LM$$

therefore GL is perpendicular to EM And so through what was shown before (III 49) angle ALB is a right angle and the circle described about AB as a diameter will pass through L And

$$HA = HB$$

therefore also since HL is a radius of the semicircle,

$$HL = HB$$

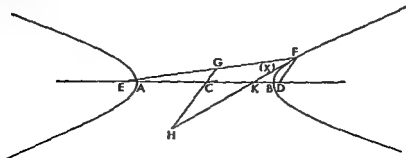
PROPOSITION 51

If a rectangle equal to the fourth part of the figure is applied from both sides to the axis of an hyperbola or opposite sections and exceeding by a square figure and straight lines are deflected from the resulting points of application to either one of the sections, then the greater of the two straight lines exceeds the less by exactly as much as the axis

For let there be an hyperbola or opposite sections whose axis is AB and center C and let each of the rectangles AD DB and AE , EB be equal to the fourth part of the figure and from points E and D let the straight lines EF and FD be deflected to the line of the section

I say that

$$EF = FD + AB$$



Let FKH be drawn tangent through F , and GCH through C parallel to FD therefore

$$\text{angle } KHG = \text{angle } KFD,$$

for they are alternate And

$$\text{angle } KFD = \text{angle } GFH \text{ (III 48),}$$

therefore

$$GF = GH$$

But

$$GF = GE,$$

since also

$$AE = BD$$

and

$$AC = CB$$

and

$$EC = CD,$$

and therefore

$$GH = EG$$

And so

$$FE = 2GH$$

And since it has been shown (III 50)

$$CH = CB,*$$

therefore

$$FE = 2(GC + CB)$$

But

$$FD = 2GC,$$

and

$$AB = 2CB,$$

therefore

$$FE = FD + AB$$

And so EF is greater than FD by AB

PROPOSITION 52

If in an ellipse a rectangle equal to the fourth part of the figure is applied from both sides to the major axis and deficient by a square figure, and from the points resulting from the application straight lines are deflected to the line of the section then they will be equal to the axis

Let there be an ellipse whose major axis is AB and let each of the rectangles $AC \cdot CB$ and $AD \cdot DB$ be equal to the fourth of the figure and from C and D let the straight lines CE and ED have been deflected to the line of the section

I say that

$$CE + ED = AB$$

*For

$$GF = GH$$

and by III 50 a line $C(\lambda)$ drawn parallel to GF is equal to CB But also

$$C(\lambda) = CH$$

Hence

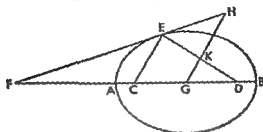
$$CH = CB$$

Let FEH be drawn tangent and G be center and through it let GKH be drawn parallel to CE . Since then

$$\text{Angle } CEF = \text{angle } HFA \text{ (in 48),}$$

and

$$\text{angle } CEF = \text{angle } EHA,$$



therefore also

$$\text{angle } EHK = \text{angle } HEK$$

Therefore also

$$HK = KE$$

And since

$$AG = GB$$

and

$$AC = DB,$$

therefore also

$$CG = GD,$$

and so also

$$FA = KD$$

And for this reason

$$ED = 2HK,$$

and

$$EC = 2KG$$

and

$$ED + EC = 2GH$$

But also

$$AB = 2GH \text{ (in 50),}$$

therefore

$$AB = FD + EC$$

PROPOSITION 53

If in an hyperbola or ellipse or circumference of a circle or opposite sections straight lines are drawn from the vertex of a diameter parallel to an ordinate, and straight lines drawn from the same ends to the same point on the line of the section cut the parallels then the rectangle contained by the straight lines cut off is equal to the figure on that same diameter

Let there be one of the aforesaid sections ABC whose diameter is AC , and let AD and CF be drawn parallel to an ordinate and let ABE and CBD be drawn across

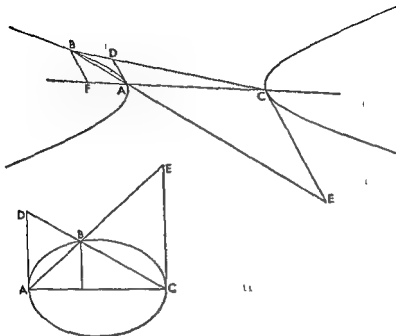
I say that

$$\text{rect } AD \cdot FC = \text{figure on } AC$$

For let BF be drawn from B parallel to an ordinate

Therefore

rect AF, FC sq FB transverse side upright side
sq AC the figure (r 21)



But

rect AF, FC sq FB comp AF, FB, FC, FB ,

therefore

figure sq AC comp FB, AF, FB, FC

But

AF, FB, AC, CE

and

FC, FB, AC, AD

therefore

figure sq AC comp CE, AC, AD, AC

And also

rect AD, CE sq AC comp CE, AC, AD, AC ,

therefore

figure sq AC rect AD, CE sq AC

Therefore

rect AD, CE = figure on AC

PROPOSITION 54

If two tangents to a section of a cone or to a circumference of a circle meet and through the points of contact parallels to the tangents are drawn and from the points of contact to the same point of the line of the section straight lines are drawn across cutting the parallels then the rectangle contained by the straight lines cut

off to the square on the straight line joining the points of contact has a ratio compounded of the ratio which the inside segment line joining the point of meeting of the tangents and the midpoint of the straight line joining the points of contact has in square to the remainder and of the ratio which the rectangle contained by the tangents has to the fourth part of the square on the straight line joining the points of contact

Let there be a section of a cone or circumference of a circle ABC and tangents AD and CD , and let AC be joined and bisected at E , and let DBE be joined, and let AF be drawn from A parallel to CD , and CG from C parallel to AD and let some point H on the section be taken and let the straight lines AH and CH be joined and produced to G and F

I say that

$$\text{rect } AF, CG \text{ sq } AC \text{ comp sq } EB \text{ sq } BD, \text{ rect } AD, DC \\ \text{fourth sq } AC \text{ or rect } AE, EC$$

For let $KHO \& L$ be drawn from H parallel to AC and from B , MBN parallel to AC , then it is evident that MN is tangent (II 29 5, 6) Since then

$$AE = EC,$$

also

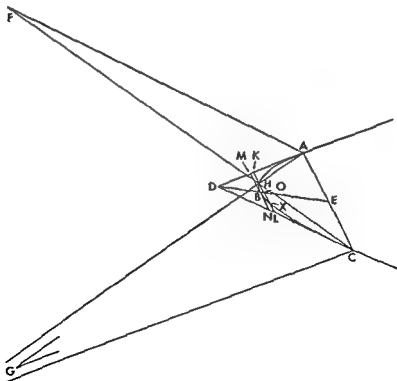
$$MB = BN$$

and

$$KO = OL$$

and

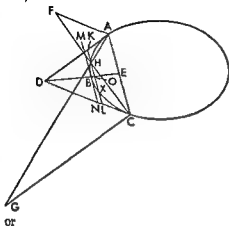
$$HO = OX \text{ (II 7)}$$



and

$$KH=VL$$

Since then MB and MA are tangents and KHL has been drawn parallel to MB ,


$$\begin{array}{l} \text{sq } AM \quad \text{sq } MB \quad \text{sq } AK \quad \text{rect} \\ XA \quad KH \text{ (III 16)} \end{array}$$

or

$$\begin{array}{l} \text{sq } AM \text{ rect } MB, BV \quad \text{sq } AK \\ \text{rect } LH, HK \end{array}$$

And

rect NC, AM sq AM ; rect LC ,
 AK sq AK (Eucl vi 2, v, 18),
 therefore *ex aequali*

rect NC, AM rect MB, BN
rect LC, AK rect LH, HK

But

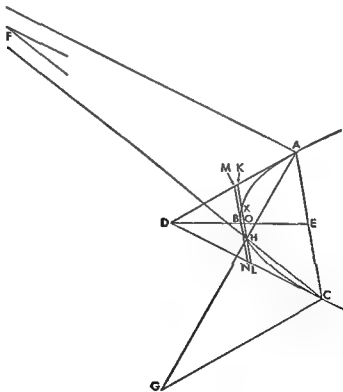
$$\begin{array}{l} \text{rect } LC \text{ } AK \quad \text{rect } LH, HK \text{ comp} \\ LC \text{ } LH, AK \quad HK \end{array}$$

or

$$\text{rect } LC, AK \quad \text{rect } LH, HA \quad \text{comp } FA \quad AC, GC \quad CA$$

which is the same as

rect GC FA sq CA



Therefore

$$\text{rect } NC, AM \text{ rect } MB, HN \text{ rect } GC, FA \text{ sq } CA$$

But with the rectangle ND, DM taken as a mean

$$\text{rect } NC, AM \text{ rect } MB, BN \text{ comp}$$

therefore $\text{rect } NC, AM \text{ rect } ND, DM \text{ rect } ND, DM \text{ rect } MB, BN,$

$$\text{rect } NC, AM \text{ rect } ND, DM \text{ rect } ND, DM \text{ rect } MB, BN$$

But

$$\text{rect } NC, AM \text{ rect } ND, DM \text{ sq } EB \text{ sq } BD,$$

and

$$\text{rect } ND, DM \text{ rect } NB, BW \text{ rect } CD, DA \text{ rect } CE, EA,$$

therefore

$$\text{rect } GC, FA \text{ sq } CA \text{ comp sq } BE \text{ sq } BD, \text{ rect } CD, DA \text{ rect } CE, EA$$

PROPOSITION 55

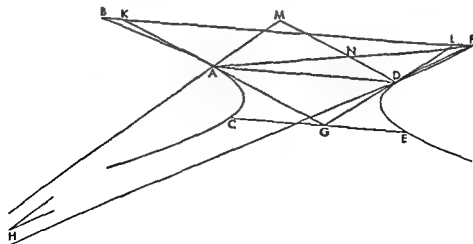
If two straight lines touching opposite sections meet and through the point of meeting a straight line is drawn parallel to the straight line joining the points of contact and from the points of contact parallels to the tangents are drawn across, and straight lines are produced from the points of contact to the same point of one of the sections cutting the parallels then the rectangle contained by the straight lines cut off will have to the square on the straight line joining the points of contact the ratio which the rectangle contained by the tangents has to the square on the straight line drawn through the point of meeting parallel to the straight line joining the points of contact as far as the section.

Let there be the opposite sections ABC and DEF , and tangents to them AG and GD and let AD be joined and from G let CGE be drawn parallel to AD , and from A AM parallel to DG and from D DM parallel to AG and let some point I be taken on the section DF , and let ANF and FDH be joined

I say that

$$\text{sq } CG \text{ rect } AG, GD \text{ sq } AD \text{ rect } HA, DN$$

For let ILK be drawn through I parallel to AD



Since then it has been shown that

$$\text{sq } EG \text{ sq } GD \text{ rect } BL \text{ } LF \text{ sq } DL \text{ (III 20),}$$

and

$$CG = EG \text{ (II 38),}$$

and

$$BK = LF \text{ (II 38),}$$

therefore

$$\text{sq } CG \text{ sq } GD \text{ rect } KF, FL \text{ sq } DL$$

And also

$$\text{sq } GD \text{ rect } AG \text{ } GD \text{ sq } DL \text{ rect } DL \text{ } AK \text{ (Eucl VI 2, 1),}$$

therefore *ex aequali*

$$\text{sq } GC \text{ rect } AG, GD \text{ rect } KF, FL \text{ rect } DL, AK$$

But

$$\text{rect } KF, FL \text{ rect } DL, AK \text{ comp } KF \text{ } AK, FL \text{ } DL$$

But

$$KF \text{ } AK \text{ } AD \text{ } DN,$$

and

$$FL \text{ } DL \text{ } AD \text{ } HA,$$

therefore

$$\text{sq } CG \text{ rect } AG, GD \text{ comp } AD \text{ } DN, AD \text{ } HA$$

And also

$$\text{sq } AD \text{ rect } HA, DN \text{ comp } AD \text{ } DN, AD \text{ } HA,$$

therefore

$$\text{sq } CG \text{ rect } AG \text{ } GD \text{ sq } AD \text{ rect } HA, DN$$

PROPOSITION 56

If two straight lines touching one of the opposite sections meet and parallels to the tangents are drawn through the points of contact and straight lines cutting the parallels are drawn from the points of contact to the same point of the other section, then the rectangle contained by the straight lines cut off will have to the square on the straight line joining the points of contact the ratio compounded of the ratio which of the straight line joining the point of meeting and the midpoint that part between the midpoint and the other section has in square to that part between the same section and the point of meeting and of the ratio which the rectangle contained by the tangents has to the fourth part of the square on the straight line joining the points of contact

Let there be the opposite sections AB and CD whose center is O and tangents $A'EFG$ and $BEHK$ and let AB be joined and bisected at L and let LE be joined and drawn across to D , and let AM be drawn from A parallel to BE and BN from B parallel to AE and let some point C be taken on the section CD and let CBM and CAN be joined

I say that

$$\text{rect } MA \text{ } BN \text{ sq } AB \text{ comp sq } LD \text{ sq } DE, \text{ rect } AE \text{ } EB \\ \text{fourth sq } AB \text{ or rect } AL \text{ } LB$$

For let GCK and HDF be drawn from C and D parallel to AB then it is evident that

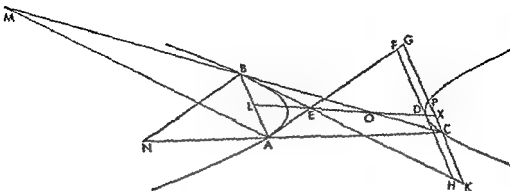
$$HD = DF$$

and

$$KY = YG$$

and also

$$XC \approx XP.$$



and so also

CK-GP

And since AB and DC are opposite sections and BEH and HD are tangents, and AG is parallel to DH therefore

$\text{sq } BH \text{ sq } HD \text{ sq } BK \text{ rect } PK, KC \text{ (III 18 note)}$

But

$$\begin{aligned} \text{sq } HD &= \text{rect } HD \text{ DF}, \\ \text{rect } PA, AC &= \text{rect } AC \text{ CG} \end{aligned}$$

Therefore

$$\text{sq } BH \text{ rect } HD, DF \quad \text{sq } BA \text{ rect } AC, CG$$

And also

rect FA BH sq BH rect GA BA sq BK .

therefore *ex aequali*

rect $FA BH$ rect $HD DF$ rect $GA BA$ rect $AC CG$

And with rectangle HE EF taken as a mean

rect $FA BH$ rect $HD DF$ comp
rect $FA HB$ rect $HE EF$ rect $HE EF$ rect $HD DF$.

and

rect FA HB rect HE EP sq LD sq DE

end

$$\text{rect } HE EF \quad \text{rect } HD DF \quad \text{rect } AE EB \quad \text{rect } AL LB$$

therefore

$$\text{rect } GA \ BH \quad \text{rect } AC, CG \text{ comp sq } LD \quad \text{sq } DE \text{ rect } AF, FB$$

$$\text{rect } AL \ LB$$

And

rect GA BK rect AC CG comp BK AC,GA CG

But

$BA \quad AC \quad MA \quad AB$

and

GA CG BN AB

therefore

$$\begin{array}{l} \text{rect } MA \quad BV \quad sq \quad AB \text{ comp } MA \quad AB \quad BN \quad AB \\ \text{comp } sq \quad LD \quad sq \quad DE \text{ rect } AL \quad LB \text{ rect } AL \quad LB \end{array}$$

TRANSLATOR'S APPENDIX ON THREE and FOUR LINE LOCI

The three-line locus property of conics is easily deduced for the ellipse hyperbola parabola and circle from III. 54, and for the opposite sections from III. 55 and 56. The three-line locus property of conics can be stated thus: Any conic section or circle or pair of opposite sections can be considered as the locus of points whose distances from three given fixed straight lines (the distances being either perpendicular or at a given constant angle to each of the given straight lines) although the constant angle may be different for each of the three straight lines) are such that the square of one of the distances is always in a constant ratio to the rectangle contained by the other two distances.

It is shown in III. 54 that in the case of conic sections and circles $\text{rect } AF \cdot CG = \text{sq } AC \text{ comp } \text{sq } EB = \text{sq } BD$, $\text{rect } AD, DC = \text{fourth sq } AC$. Now if we consider the straight lines AD, DC and AC as fixed and given and therefore straight line DE fixed and given as bisecting AC , then it is evident that the straight lines AC, EB, BD, AD, DC , and therefore the squares on them and the rectangles contained by them are also fixed and given. Then although as the point H is taken at different points along the conic the straight lines AF and CG change in magnitude nevertheless the magnitude of the rectangle $AF \cdot CG$ because of the above proportion remains constant.

For let HX be drawn parallel to BE and HY to AD and HZ to DC . Then HX is the distance from H to AC at a given angle and AY because of parallels represents the distance from H to AD at another given angle and ZC represents the distance from H to DC at another given angle. Then by similar triangles

$$\frac{CZ}{AY} = \frac{ZH}{YH} = \frac{AC}{AC} = \frac{AF}{CG},$$

therefore compounding
 $\text{rect } CZ, AY = \text{rect } ZH, YH = \text{sq } AC = \text{rect } AF, CG$

Now we have seen that the rectangle $AF \cdot CG$ is a constant magnitude as the point H changes and the square on AC is constant therefore their ratio is constant.

Therefore
 $\text{rect } CZ, AY = \text{rect } ZH, YH$ is a constant ratio. (1)

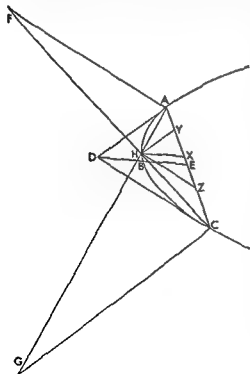
Again by similar triangles

$$\frac{ZH}{YH} = \frac{HY}{HX} = \frac{CD}{AD} = \frac{DE}{DE},$$

therefore compounding

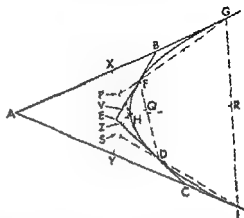
$$\text{rect } ZH, YH = \text{sq } HX = \text{rect } CD, AD = \text{sq } DE$$

But rectangle CD, AD and the square on DE are constant magni-

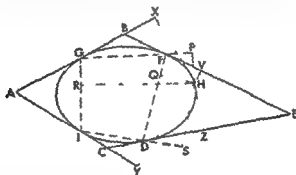


tudes as the point H changes therefore their ratio is constant Therefore
 $\text{rect } ZH, \lambda H \text{ sq } HX \text{ is a constant ratio}$ (2)
 Compounding (1) and (2) we get a constant ratio that is
 $\text{rect } CZ, \lambda AY \text{ sq } HX \text{ is a constant ratio}$

In other words, as the point H changes the rectangle contained by the distances from H to two of the given straight lines (at given angles to those straight lines) has a constant ratio to the square on the distance to the third straight line (at a given angle to that straight line) And it can easily be proved by means of similar triangles that if any other three angles are chosen for the distances than those chosen here for the demonstration then the corresponding ratio will be constant although not equal



The four line locus property can be easily deduced from the three-line If to any conic section we construct four tangents $AG, BE, AI,$ and EC and the straight lines FG, GI, ID and DF , joining the points of contact, and draw the distances from any point H on the conic to these straight lines at any given angles (perpendiculars are convenient) then by the three line locus property with respect to triangle FBG for any point H on the conic
 $\text{rect } H\lambda, H\gamma \text{ sq } HP \text{ is constant,}$ (a)



with respect to triangle AIG

$$\text{rect } H\lambda, H\gamma \text{ sq } HR \text{ is constant} \quad (\beta)$$

with respect to triangle DCI

$$\text{rect } H\gamma, H\delta \text{ sq } HS \text{ is constant} \quad (\gamma)$$

with respect to triangle IFD

$$\text{rect } H\delta, H\epsilon \text{ sq } HQ \text{ is constant} \quad (\delta)$$

It will be noticed that we have taken in succession a pair of adjacent tangents and the straight line joining their points of contact. It will also be noticed that the rectangles in the four ratios present a cyclic arrangement so that if the

inverse of (α) is compounded with (β), and the inverse of (γ) with (δ) we would have two constant ratios

$$\text{pllpd } HY, HP \text{ HP } \text{pllpd } HV, HR, HR, \quad (\epsilon)$$

$$\text{pllpd } HV, HS \text{ HS } \text{pllpd } HY, HQ, HQ \quad (\zeta)$$

Again compounding the first of these with the second we would have finally
rect HP, HS rect HQ, HR , a constant ratio

And this is the property of the four line locus namely the locus of points H such that the rectangle contained by the distances from points H to any two given fixed straight lines FG and ID has to the rectangle contained by the distances from H to two other fixed straight lines IG, FD , a constant ratio

The rigorous method of effecting these compoundings is as follows For inverting (α), by Eucl xi 32 we have the constant ratios

$$\text{sq } HP \text{ rect } H\lambda, HV \text{ pllpd } HP \text{ HP}, HY \text{ pllpd } HX, HV, HY,$$

$$\text{rect } H\lambda, HY \text{ sq } HR \text{ pllpd } HX \text{ HY}, HV \text{ pllpd } HR, HR \text{ HV}$$

Hence, by definition the ratio (a constant one) compounded of these two is
pllpd $H\lambda, HP, HP$ pllpd HV, HR, HR

And in the same way we find the constant ratio compounded of the inverse of (γ) and (δ) Now

$$\text{pllpd } HY \text{ HP}, HP \text{ pllpd } HV, HR, HR \text{ comp } HY \text{ HV}, \text{sq } HP \text{ sq } HR$$

$$\text{pllpd } HV, HS \text{ HS } \text{pllpd } HY, HQ, HQ \text{ comp } HV \text{ HY}, \text{sq } HS \text{ sq } HQ$$

If then we take two lines M and N such that

$$\frac{HP}{HS} = \frac{HR}{HQ} = \frac{HR}{HQ} = \frac{M}{N}, \quad (\eta)$$

$$\frac{HS}{HS} = \frac{HQ}{HQ} = \frac{HQ}{HQ} = \frac{N}{N}, \quad (\theta)$$

then

$$\text{sq } HP \text{ sq } HR \text{ HP } M,$$

$$\text{sq } HS \text{ sq } HQ \text{ HS } N$$

Hence

$$\text{ratio comp } HY \text{ HV}, \text{sq } HP \text{ sq } HR \text{ ratio comp } HY \text{ HV}, HP \text{ M}$$

$$\text{ratio comp } HV \text{ HY}, \text{sq } HS \text{ sq } HQ \text{ ratio comp } HV \text{ HY}, HS \text{ N}$$

But

$$\text{rect } HY \text{ HP} \text{ rect } HV \text{ M} \text{ comp } HY \text{ HV}, HP \text{ M},$$

$$\text{rect } HV, HS \text{ rect } HY \text{ N} \text{ comp } HV \text{ HY}, HS \text{ N}$$

and

$$\text{pllpd } HY \text{ HP}, HS \text{ pllpd } HV \text{ M}, HS \text{ rect } HY \text{ HP} \text{ rect } HY \text{ M},$$

$$\text{pllpd } HV \text{ HS}, M \text{ pllpd } HY \text{ N} \text{ M} \text{ rect } HV, HS \text{ rect } HY \text{ N},$$

and these are constant ratios Hence compounding we get the constant ratio

$$\text{pllpd } HY, HP, HS \text{ pllpd } HY, N, M,$$

which is the same as the constant ratio

$$\text{rect } HP, HS \text{ rect } N, M$$

Now taking L and O as some constants

$$\text{rect } HP \text{ HS} \text{ rect } N \text{ M} \text{ L} \text{ O}$$

and

$$\text{rect } HP \text{ HS} \text{ rect } HR \text{ HQ} \text{ rect } HR \text{ HQ} \text{ rect } M \text{ N}$$

by compounding (η) and (θ) But equal ratios have equal duplicate ratios (Heath's note to Euclid vi 22) and hence

$$\text{rect } HP \text{ HS} \text{ rect } HR \text{ HQ} \text{ is constant}$$

In the case of opposite sections it is shown in III 56

$$\text{rect } MA \text{ BV} \text{ sq } AB \text{ comp sq } LD \text{ sq } DE \text{ rect } AE \text{ FB} \text{ fourth sq } AB$$

Then it is evident for the same reasons as before that for different points C the

magnitudes MA and BN may change, but the rectangle $MA BN$ is a constant magnitude

For as before, let CY be drawn parallel to DE , CY to EA , CZ to EB By similar triangles

$$\frac{AY}{BZ} = \frac{YC}{ZC} = \frac{AB}{MA} = \frac{BN}{MA},$$

therefore compounding

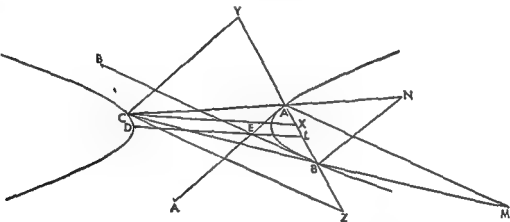
$$\text{rect } AY, BZ = \text{rect } YC, ZC \propto \text{rect } MA, BN$$

Since rectangle MA, BN is constant as C changes, and also the square on AB is constant therefore

$$\text{rect } AY, BZ = \text{rect } YC, ZC \text{ is a constant ratio} \quad (1)$$

Again by similar triangles

$$\frac{ZC}{CY} = \frac{EB}{EL},$$



$$\frac{YC}{CY} = \frac{EA}{EL}$$

therefore compounding

$$\text{rect } YC, ZC \propto \text{rect } EA, EL$$

Hence

$$\text{rect } YC, ZC \propto \text{rect } EA, EL \quad (2)$$

Compounding (1) and (2) we have a constant ratio

$$\frac{AY}{BZ} = \frac{YC}{ZC} = \frac{EA}{EL}$$

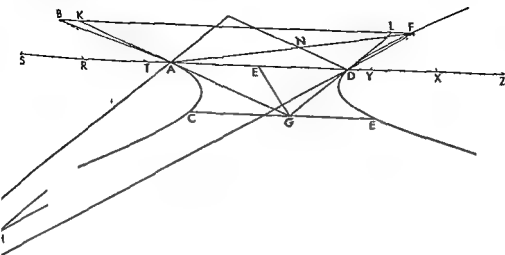
But AY and BZ are equal to CA and CB the distances from C This is the property of the three line locus of section C with respect to the straight lines EA and FB tangents to the other section and LB the straight line joining their points of contact And so one opposite section is a three line-locus to the tangents to the other of the opposite sections That it is also a four line locus could be shown in the same way as before

Again from III 55 we can conclude that both of the opposite sections together are a three line locus to the triangle formed by a tangent to each of the sections and the straight line joining their points of contact For by III 55

$$\text{rect } HA, DV \propto \text{rect } AD, AG \propto \text{rect } AG, GD \propto \text{rect } CG$$

Now since the three last terms of this proportion are evidently constants as the point F changes therefore also although HA and DA change with F yet rectangle HA, DV remains constant in magnitude Then reproducing the figure of

in 55 we drop YF parallel to DL and FZ to AA , and FV to GE , where E is the midpoint of AD . Then by similar triangles



YD FY AD HA,
AZ FZ AD DN

therefore compounding

rect *YD AZ* rect *FY, FZ* sq *AD* rect *HA, DN*

But the last two terms are constant, therefore

rect $YD \cdot AZ$ rect $FY \cdot FZ$ is a constant ratio (1)

Again by similar triangles

| | | | |
|-----------|-----------|-----------|------------|
| <i>FL</i> | <i>FY</i> | <i>DG</i> | <i>LG</i> |
| <i>FZ</i> | <i>FY</i> | <i>AG</i> | <i>EG.</i> |

therefore compounding

$$\text{rect } FY, FZ \quad \text{sq } FY \quad \text{rect } DG, AG \quad \text{rect } ED, EG$$

But the last two terms are constant therefore

rect $FY FZ$ sq FY is a constant ratio (2)

Compounding (1) and (2), we see that

rect $YD \cdot AZ$ sq FV is a constant ratio

But this is the definition of a three line locus that the rectangle contained by the distances from any point on the locus to two fixed straight lines have to the square on the distance to a third fixed straight line a constant ratio. But

$$DY = LF$$

$$AZ = \mathbf{1}F$$

and FY is the distance from F to AD . And so the ratio fulfills the definition

Furthermore if we consider B the point of intersection of the straight line ΛF drawn parallel to AD with the other opposite section and draw BS parallel to F_1 , BR to F_2 , and BT to F_3 since they are parallels between parallels,

$$BR = FV$$

$$KF = AZ$$

$$TA = B\Lambda$$

magnitudes MA and BN may change, but the rectangle MA, BV is a constant magnitude

For as before let CX be drawn parallel to DE , CY to EA , CZ to EB By similar triangles

$$\frac{AY}{BZ} = \frac{YC}{ZC} = \frac{AB}{AB} = \frac{BN}{MA},$$

therefore compounding

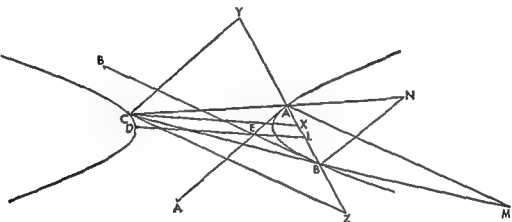
$$\text{rect } AY, BZ = \text{rect } YC, ZC = \text{sq } AB = \text{rect } MA, BN$$

Since rectangle MA, BN is constant as C changes and also the square on AB is constant therefore

$$\text{rect } AY, BZ = \text{rect } YC, ZC \text{ is a constant ratio} \quad (1)$$

Again by similar triangles

$$\frac{ZC}{CX} = \frac{EB}{EL},$$



$$\frac{YC}{CX} = \frac{EA}{EL}$$

therefore compounding

$$\text{rect } YC, ZC = \text{sq } CX = \text{rect } EB, EA = \text{sq } EL$$

Hence

$$\text{rect } YC, ZC = \text{sq } CX \text{ is a constant ratio} \quad (2)$$

Compounding (1) and (2) we have a constant ratio

$$\text{rect } AY, BZ = \text{sq } CX$$

But AY and BZ are equal to CA' and CB the distances from C This is the property of the three line locus of section C with respect to the straight lines EA and EB tangents to the other section and EB the straight line joining their points of contact And so one opposite section is a three line-locus to the tangents to the other of the opposite sections That it is also a four line locus could be shown in the same way as before

Again from III 55 we can conclude that both of the opposite sections together are a three line locus to the triangle formed by a tangent to each of the sections and the straight line joining their points of contact For by III 55

$$\text{rect } HA, DV = \text{sq } AD = \text{rect } AG, GD = \text{sq } CG$$

Now since the three last terms of this proportion are evidently constants as the point F changes therefore also although HA and DA change with F , yet rectangle HA, DV remains constant in magnitude Then reproducing the figure of

INTRODUCTION TO ARITHMETIC

But it was shown in the course of III 55 that

$$BK = LF,$$

$$BL = AF$$

Hence

$$TA = BK = YD = LF,$$

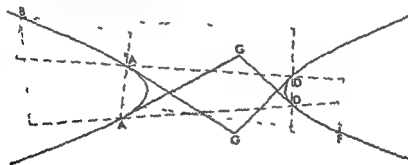
$$AZ = AF = BL$$

Therefore

$$\text{rect } LF, AF \text{ sq } FY \quad \text{rect } BK, BL \text{ sq } BR$$

Hence any point B on one opposite section fulfill the same constant ratio with respect to its distances from the three fixed lines AD , GD , and AG as any point F on the other opposite section

It can be similarly deduced that the opposite sections are together a four line locus with respect to any four fixed straight lines joining points, two on each section (the points being four points of contact of four tangents and the straight lines the straight lines joining them)



To sum up ■ parabola ellipse circle and hyperbola are three-line loci with respect to any two tangents to them and a straight line joining the points of contact. One opposite section is also a three line locus with respect to any two tangents to the other section together with the straight line joining their points of contact. The two opposite sections together are a three line-locus with respect to two tangents each to one of the sections together with the straight line joining their points of contact.

The parabola ellipse circle and hyperbola are four line loci with respect to any inscribed quadrilateral. One opposite section is also a four line locus with respect to any quadrilateral inscribed in the other section. The two opposite sections together are a four line locus to any four straight lines joining four points, two lying on each opposite section.

INTRODUCTION TO ARITHMETIC

BIOGRAPHICAL NOTE

NICOMACHUS, fl c AD 100

NICOMACHUS OF GERASA flourished around the end of the first century of our era. In one of his surviving books the *Introduction to Harmonics*, he mentions a certain Thrasyllus presumably Thrasyllus of Mendes, a writer on music who lived in the reign of Tiberius. Another book by Nicomachus, the *Introduction to Arithmetic* was translated into Latin by Apuleius under the Antonines. This places the life of Nicomachus somewhere between the middle of the first century and the middle of the second century. Perhaps the fact that Ptolemy, whose recorded astronomical observations were made between AD 127 and 151, is not mentioned in the *Introduction to Harmonics* makes it probable that he was not yet famous at the time Nicomachus was writing.

The manuscripts of Nicomachus' books and the scholia call him "of Gerasa." The best known city of that name was in Palestine and was primarily Greek. However, it can hardly be supposed that Nicomachus received all of his philosophical and mathematical education at Gerasa. He probably studied at Alexandria at this time the center of mathematical studies and of Neo-Pythagoreanism. Jamblichus says of Nicomachus: "The man is great in mathematics, and had as instructors those that were most skilled in the subject."

Nothing is known of the personal life of Nicomachus except what is said or implied in the dedication of the *Introduction to Harmonics* to an unknown lady: "But I must spur on all my zeal most noble and august lady since it is you that bid me. And if the gods are willing just as soon as I shall have leisure and a rest from my journeyings I will compile for you a better and more detailed *Introduction* dealing with this very subject and so that you may the more easily follow the argument I will take my beginning say from the same point as that at which I began your instruction when I was expounding the subject to you."

Nicomachus appears to have been an important member of the Neo-Pythagorean group though his extant writings would seem to indicate that he was a popularizer and a compiler of manuals and not the head of a school. Besides the *Introduction to Arithmetic* and the *Introduction to Harmonics* he also wrote a book on the mystical doctrine of number called *Theologoumena Arithmeticae*, which is one of the best sources on Neo-Pythagoreanism. Extracts and paraphrases of this work survive in a later anonymous work of the same name and in the *Bibliotheca* a collection of extracts from ancient works made in the ninth century by Photius, patriarch of Constantinople. Nicomachus also wrote an *Introduction to Geometry* and a *Life of Pythagoras* which have not survived and a larger work on music possibly that promised in the dedication to the *Introduction to Harmonics* of which we have only fragments. He may have written a book on the interpretation of Plato though the evidence for it is slight and also an *Introduction to Astronomy* thereby completing the quadrivial series.

The success of the *Introduction to Arithmetic* must have been immediate. It

was used as a text book throughout later antiquity and in the Latin paraphrase of Boethius throughout the Middle Ages. It had a host of commentators. In the *Philopatrias* attributed to Lucian, a character says "You reckon like Nicomachus." This remark lends itself to more than one interpretation, but in any case it is evidence of his fame. Nicomachus also appears to have been considered one of the 'golden chain' or succession of true philosophers, for Proclus the fifth century Neo-Platonist who belonged to that 'chain,' claimed, on the basis of a dream, that he had within him the soul of Nicomachus.

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BOOK ONE

CHAPTER I

[1] The ancients who under the leadership of Pythagoras first made science systematic defined philosophy as the love of wisdom. In deed the name itself means this and before Pythagoras all who had knowledge were called wise indiscriminately—a carpenter for example ■ cobbler ■ helmsman and in a word anyone who was versed in any art or handicraft Pythagoras however restricting the title so as to apply to the knowledge and comprehension of reality and calling the knowledge of the truth in this the only wisdom naturally designated the desire and pursuit of this knowledge a philosophy as being desire for wisdom.

[2] He is more worthy of credence than those who have given other definitions since he makes clear the sense of the term and the thing defined. This wisdom he defined as the knowledge or science of the truth in real things conceiving science to be a steadfast and firm apprehension of the underlying substance and real things to be those which continue uniformly and the same in the universe and never depart even briefly from their existence these real things would be things immaterial by sharing in the substance of which everything else that exists under the same name and is so called is said to be this particular thing and exists.

[3] For bodily material things are to be sure forever involved in continuous flow and change—in imitation of the nature and peculiar quality of that eternal matter and substance which has been from the beginning and which was all changeable and variable throughout. The bodiless things however of which we conceive in connection with or together with matter such as qualities quantities configurations largeness smallness equality relations actualities dispositions places times all those things in a word whereby the qualities found in each body are comprehended—all these are of themselves immovable and unchangeable but accidentally they share in and partake of the affections of the body to which they belong.

[4] Now it is with such things that 'wisdom' is particularly concerned but accidentally also with things that share in them that is bodies.

CHAPTER II

[1] Those things however are immaterial eternal without end and it is their nature to persist ever the same and unchanging abiding by their own essential being and each one of them is called real in the proper sense. But what are involved in birth and destruction growth and diminution all kinds of change and participation are seen to vary continually, and while they are called real things by the same term as the former so far as they partake of them they are not actually real by their own nature for they do not abide for even the shortest moment in the same condition but are always passing over in all sorts of changes. [2] To quote the words of Timaeus in Plato¹

What is that which always is and has no birth and what is that which is always becoming but never is? The one is apprehended by the mental processes with reasoning and is ever the same the other can be guessed at by opinion in company with unreasoning sense a thing which becomes and passes away but never really is.

[3] Therefore if we crave for the goal that is worthy and fitting for man namely happiness of life²—and this is accomplished by philosophy alone and by nothing else and philosophy as I said means for us desire for wisdom and wisdom the science of the truth in things and of things some are properly so called others merely share the name—it is reasonable and most necessary to distinguish and systematize the accidental qualities of things.

[4] Things then both those properly so called and those that simply have the name are some of them unified and continuous for example an animal the universe a tree and

¹Timaeus 27

²The word used by Nicomachus *εὐδαιμονία* once employed by Aristotle in the *Ethics* 1038^a 20 ff.

the like which are properly and peculiarly called magnitudes others are discontinuous in a side-by-side arrangement and, as it were in heaps which are called multitudes a flock for instance a people a heap a chorus and the like

[5] Wisdom then must be considered to be the knowledge of these two forms. Since however all multitude and magnitude are by their own nature of necessity infinite—for multitude starts from a definite root and never ceases in increasing and magnitude when division beginning with a limited whole is carried on cannot bring the dividing process to an end but proceeds therefore to infinity—and since sciences are always sciences of limited things and never of infinities it is accordingly evident that a science dealing either with magnitude *per se* or with multitude *per se* could never be formulated for each of them is limitless in itself multitude in the direction of the more and magnitude in the direction of the less. A science however would arise to deal with something separated from each of them with quantity set off from multitude and size set off from magnitude.

CHAPTER III

[1] Again to start afresh since of quantity one kind is viewed by itself having no relation to anything else as even odd perfect and the like and the other is relative to something else and is conceived of together with its relationship to another thing like double greater smaller half one and one-half time one and one-third times and so forth it is clear that two scientific methods will lay hold of and deal with the whole investigation of quantity arithmetic absolute quantity and music relative quantity.¹

[2] And once more inasmuch as part of size is in a state of rest and stability and another part in motion and revolution two other sciences in the same way will accurately treat of size geometry the part that abides and is at rest astronomy that which moves and revolves.

[3] Without the aid of these then it is not possible to deal accurately with the forms of being nor to discover the truth in things knowledge of which is wisdom and evidently not even to philosophize properly for just as painting contributes to the manual arts and correctness of theory so in truth lines numbers harmonic intervals and the revolutions of circles

bear aid to the learning of the doctrines of wisdom says the Pythagorean Androcydes [4] Likewise Archytas of Tarentum at the beginning of his treatise *On Harmony* says the same thing in about the same words. It seems to me that they do well to study mathematics and it is not at all strange that they have correct knowledge about each thing what it is. For if they knew rightly the nature of the whole, they were also likely to see well what is the nature of the parts. About geometry indeed and arithmetic and astronomy they have handed down to us a clear understanding and not least about music. For these seem to be sister sciences for they deal with sister subjects the first two forms of being.

[5] Plato too at the end of the thirteenth book of the *Laws* to which some give the title *The Philosopher* because he investigates and defines in it what sort of man the real philosopher should be in the course of his summary of what had previously been fully set forth and established adds. Every diagram system of numbers every scheme of harmony and every law of the movement of the stars ought to appear one to him who studies rightly and what we say will properly appear if one studies all things looking to one principle for there will be seen to be one bond for all the things and if any one attempts philosophy in any other way he must call on Fortune to assist him. For there is never a path without these this is the way these the studies be they hard or easy by this course must one go and not neglect it. The one who has attained all these things in the way I describe him I for my part call wisest and this I maintain through thick and thin. [6] For it is clear that the studies are like ladders and bridges that carry our minds from things apprehended by sense and opinion to those comprehended by the mind and understanding and from those material physical things our foster brethren known to us from childhood to the things with which we are unacquainted foreign to our senses but in their immateriality and eternity more akin to our souls and above all to the reason which is in our souls.

[7] And likewise in Plato's *Republic* when the interlocutor of Socrates appears to bring certain plausible reasons to bear upon the mathematical sciences to show that they are useful to human life arithmetic for reckoning distributions contributions exchanges and partnerships geometry for sieges the founding of cities and sanctuaries and the partition

¹Cf. Aristotle *Met.* IV 15 1070^b 26

of land music for festivals entertainment and the worship of the gods and the doctrine of the spheres or astronomy, for farming navigation and other undertakings, revealing beforehand the proper procedure and suitable season Socrates, reproaching him, says 'You amuse me because you seem to fear that these are useless studies that I recommend but that is very difficult nay impossible For the eye of the soul blinded and buried by other pursuits is rekindled and aroused again by these and these alone and it is better that thus be saved than thousands of bodily eyes for by it alone is the truth of the universe beheld'

CHAPTER IV

[1] Which then of these four methods must we first learn? Evidently the one which naturally exists before them all is superior and takes the place of origin and root and as it were of mother to the others [2] And thus is arithmetic not solely because we said that it existed before all the others in the mind of the creating God like some universal and exemplary plan relying upon which as a design and archetypal example the creator of the universe sets in order his material creations and makes them attain to their proper ends but also because it is naturally prior in birth inasmuch as it abolishes other sciences with itself but is not abolished together with them For example animal is naturally antecedent to man for abolish animal and man is abolished but if man be abolished it no longer follows that animal is abolished at the same time And again man is antecedent to schoolteacher for if man does not exist neither does 'schoolteacher' but if schoolteacher is nonexistent it is still possible for man to be Thus since it has the property of abolishing the other ideas with itself it is likewise the older

[3] Conversely that is called younger and posterior which implies the other thing with itself but is not implied by it like musician for this always implies man Again take horse animal is always implied along with horse but not the reverse for if animal exists it is not necessary that horse should exist nor if man exists must musician also be implied

Republic 527 ff

*Plato Rep 522

*Cf below II 22 3 Cf Aristotle, *Met* 1019^a 1 ff

*Cf Aristotle e.g. *Top* VI 143^b 17 also *Top* II 4 111 25 ff

[4] So it is with the foregoing sciences if geometry exists arithmetic must also needs be implied for it is with the help of this latter that we can speak of triangle quadrilateral octahedron, icosahedron double eightfold or one and one-half times or anything else of the sort which is used as a term by geometry, and such things cannot be conceived of without the numbers that are implied with each one For how can 'triple' exist, or be spoken of unless the number 3 exists beforehand or eightfold without 8? But on the contrary 3 4 and the rest might be without the figures existing to which they give names [5] Hence arithmetic abolishes geometry along with itself but is not abolished by it and while it is implied by geometry it does not itself imply geometry

CHAPTER V

[1] And once more is this true in the case of music not only because the absolute is prior to the relative as great to greater and rich to richer and man to 'father' but also because the musical harmonies diatessaron diapente and diapason are named for numbers similarly all of their harmonic ratios are arithmetical ones for the diatessaron is the ratio of 4 3 the diapente that of 3 2 and the diapason the double ratio and the most perfect the didiapason is the quadruple ratio

[2] More evidently still astronomy attains through arithmetic the investigations that pertain to it not alone because it is later than geometry in origin—for motion naturally comes after rest—nor because the motions of the stars have a perfectly melodious harmony but also because risings settings progressions retrogressions increases and all sorts of phases are governed by numerical cycles and quantities

[3] So then we have rightly undertaken first the systematic treatment of this as the science naturally prior more honorable and more venerable and as it were mother and nurse of the rest and here we will take our start for the sake of clearness

CHAPTER VI

[1] All that has by nature with systematic method been arranged in the universe seems both in part and as a whole to have been determined and ordered in accordance with number by the forethought and the mind of him that

*Cf Plato Rep 528

created all things for the pattern was fixed like a preliminary sketch by the domination of number preexistent in the mind of the world creating God number conceptual only and immaterial in every way, but at the same time the true and the eternal essence so that with reference to it as to an artistic plan should be created all these things time motion the heavens the stars all sorts of revolutions

[2] It must needs be then that scientific number being set over such things as these should be harmoniously constituted in accordance with itself not by any other but by itself

[3] Everything that is harmoniously constituted is knit together out of opposites and of course out of real things for neither can non-existent things be set in harmony nor can things that exist but are like one another nor yet things that are different but have no relation one to another It remains accordingly that those things out of which a harmony is made are both real different and things with some relation to one another

[4] Of such things therefore scientific number consists for the most fundamental species in it are two embracing the essence of quantity different from one another and not of a wholly different genus odd and even and they are reciprocally woven into harmony with each other inseparably and uniformly, by a wonderful and divine Nature as straightway we shall see

CHAPTER VII

[1] Number is limited multitude or a combination of units or a flow of quantity made up of units and the first division of number is even and odd

[2] The even¹ is that which can be divided into two equal parts without a unit intervening in the middle and the odd is that which cannot be divided into two equal parts because of the afore-said intervention of a unit

[3] Now this is the definition after the ordinary conception by the Pythagorean doctrine however the even number is that which admits of division into the greatest and the smallest parts at the same operation greatest in size and smallest in quantity in accordance with the natural contrariety of these two genera and the odd is that which does not allow this to be done to it but is divided into two unequal parts

[4] In still another way by the ancient defini-

tion the even is that which can be divided alike into two equal and two unequal parts except that the dyad which is its elementary form admits but one division that into equal parts and in any division whatsoever it brings to light only one species of number, however it may be divided, independent of the other The odd is a number which in any division whatsoever, which necessarily is a division into unequal parts shows both the two species of number together never without intermixture one with another but always in one another's company

[5] By the definition in terms of each other, the odd is that which differs by a unit from the even in either direction that is toward the greater or the less and the even is that which differs by a unit in either direction from the odd that is, is greater by a unit or less by a unit

CHAPTER VIII

[1] Every number is at once half the sum of the two on either side of itself and similarly half the sum of those next but one in either direction and of those next beyond them and so on as far as it is possible to go [2] Unity alone because it does not have two numbers on either side of it is half merely of the adjoining number hence unity is the natural starting point of all number

[3] By subdivision of the even there are the even times even the odd times even and the even times odd The even times even and the even times odd are opposite to one another like extremes and the odd times even is common to them both like a mean term

[4] Now the even times even² is a number which is itself capable of being divided into two equal parts in accordance with the properties of its genus and with each of its parts similarly capable of division and again in the same way each of their parts divisible into two equals until the division of the successive subdivisions reaches the naturally indivisible unit [5] Take for example 64 one half of this is 32 and of this 16 and of this the half is 8 and of this 4 and of this 2 and then finally unity is half of the latter and this is naturally indivisible and will not admit of a half

[6] It is a property of the even times even that whatever part of it be taken it is always

¹ Euclid's definition is The even times even number is that which is measured by an even number an even number of times *Elements VII Def 8*

² Cf. Euclid, VII Def 8

even times even in designation and at the same time by the quantity of the units in it even times even in value and that neither of these two things will ever share in the other class [7] Doubtless it is because of this that it is called even times even because it is itself even and always has its parts and the parts of its parts down to unity even both in name and in value in other words every part that it has is even times even in name and even times even in value

[8] There is a method of producing the even times even so that none will escape but all successively fall under it if you do as follows [9] As you proceed from unity as from a root by the double ratio to infinity as many terms as there are will all be even times even and it is impossible to find others besides these for instance 1 2 4 8 16 32 64 128 256 512

[10] Now each of the numbers set forth was produced by the double ratio, beginning with unity and is in every respect even times even and every part that it may be found to have is always named from some one of the numbers before it in the series and the sum of units in this part is the same as one of the numbers before it, by a system of mutual correspondence indeed and interchange If there is an even number of terms of the double ratio from unity, not one mean term can be found but always two from which the correspondence and interchange of factors and values values and factors will proceed in order going first to the two on either side of the means then to the next on either side until it comes to the extreme terms so that the whole will correspond in value to unity and unity to the whole For example if we set down 128 as the largest term the number of terms will be even for there are eight in all up to this number and they will not have one mean term for this is impossible with an even number but of necessity two 8 and 16 These will correspond to each other as factors for of the whole 128 16 is one eighth and conversely 8 is one sixteenth Thence proceeding in either direction we find that 32 is one fourth and 4 one thirty second and again 64 is one half and 1 one sixty fourth and finally at the extremes unity is one one hundred twenty eighth and conversely 128 is the whole to correspond with unity

[11] If however the series consists of an odd number of terms seven for example and we deal with 64 there will be of necessity one mean term in accordance with the nature of the odd the mean term will correspond to it-

self because it has no partner and those on either side of it in turn will correspond to one another until this correspondence ends in the extremes Unity for example will be one sixty fourth and 64 the whole corresponding to unity 32 is one half and 1 one thirty-second 16 is one fourth and 4 one sixteenth and 8 the eighth part with nothing else to correspond to it

[12] It is the property of all these terms when they are added together successively to be equal to the next in the series lacking one unit so that of necessity their summation in any way whatsoever will be an odd number, for that which fails by a unit of being equal to an even number is odd [13] This observation will be of use to us very shortly in the construction of perfect numbers But to take an example the terms from unity preceding 256 in the series when added together, are within 1 of equaling 256 and all the terms before 128 the term immediately preceding are similarly equal to 128 save for one unit and to the next terms the sums of those below them are similarly related Thus unity itself is within one unit of equaling the next term which is 2 and these two together fail by 1 of equaling the next and the three together are within 1 of the next in order and you will find that this goes on with out interruption to infinity

[14] This too it is very needful to recall If the number of terms of the even times even series dealt with is even the product of the extremes will always be equal to the product of the means if there is an odd number of terms the product of the extremes will be equal to the square of the mean For in the case of an even number of terms 1 times 128 is equal to 8 times 16 and further to 2 times 64 and again to 4 times 32 and this is so in every case and with an odd number of terms 1 times 64 equals 2 times 32 and this equals 4 times 16 and this again equals 8 times 8 the mean term alone multiplied by itself

CHAPTER IX

[1] The even times odd number is one which is by its genus itself even but is specifically opposed to the aforesaid even times even It is a number of which though it admits of the division into two equal halves after the fashion the genus common to it and the even the halves are not

¹See Chapter 16

²Cf on I 8 7

into two equals for example 6 10 14 18 22 26 and the like for after these have been divided their halves are found to be indivisible.

[2] It is the property of the even times odd that whatever factor it may be discovered to have is opposite in name to its value and that the quantity of every part is opposite in value to its name and that the numerical value of its part never by any means is of the same genus as its name. To take a single example the number 18 its half with an even name is 9 odd in value its third part again with an odd designation is 6 even in value conversely, the sixth part is 3 and the ninth part 2 and in other numbers the same peculiarity will be found.

[3] It is possibly for this reason that it received such a name that is because although it is even its halves are at once odd.

[4] This number is produced from the series beginning with unity with a difference of 2, namely the odd numbers set forth in proper order as far as you like and then multiplied by 2. The numbers produced would be in order the 6 10 14 18 22 26 30 and so on as far as you care to proceed. The greater terms all ways differ by 4 from the next smaller ones the reason for which is that their original basic forms the odd numbers exceed one another by 2 and were multiplied by 2 to make this series and 2 times 2 makes 4.

[5] Accordingly in the natural series of numbers the even times odd numbers will be found fifth from one another exceeding one another by a difference of 4 passing over three terms and produced by the multiplication of the odd numbers by 2.

[6] They are said to be opposite in properties to the even times even because of these the greatest extreme term alone is divisible while of these former the smallest only proved to be indivisible and in particular because in the former case the reciprocal arrangement of parts from extremes to mean term or terms makes the product of the former equal to the square or product of the latter but in this case by the same correspondence and comparison the mean term is one half the sum of the extremes or if there should be two means their sum equals that of the two extremes.

CHAPTER X

[1] The odd times even number is the one which displays the third form of the even belonging in common to both the previously men-

tioned species like a single mean between two extremes for in one respect it resembles the even times even and in another the even times odd and that property wherein it varies from the one it shares with the other, and by that property which it shares with the one it differs from the other.

[2] The odd times even number is an even number which can be divided into two equal parts whose parts also can so be divided and sometimes even the parts of its parts but it cannot carry the division of its parts as far as unity. Such numbers are 24 28 40 for each of these has its own half and indeed the half of its half and sometimes one is found among them that will allow the halving to be carried even farther among its parts. There is none however that will have its parts divisible into halves as far as the naturally indivisible unit.

[3] Now in admitting more than one division the odd times even is like the even times even and unlike the even times odd but in that its subdivision never ends with unity it is like the even times odd and unlike the even times even.

[4] It alone has at once the proper qualities of each of the former two and then again properties which belong to neither of them for of them one had only the highest term divisible, and the other only the smallest indivisible but this neither for it is observed to have more divisions than one in the greater term, and more than one indivisible in the lesser.

[5] Furthermore there are in it certain parts whose names are not opposed to their values nor of the opposite genus after the fashion of the even times even and there are also always other parts of a name opposite and contrary in kind to their values after the fashion of the even times odd. For example in 24 there are parts not opposed in name to their values the fourth part 6 the half 12 the sixth 4 and the twelfth 2 but the third part 8 the eighth 3 and the twenty fourth 1 are opposed and it is with the rest.

[6] This number is produced by a somewhat complicated method and shows after a fashion even in its manner of production that it is a mixture of both other kinds for whereas the even times even is made from even numbers the doubles from unity to infinity and the even times odd from the odd numbers from 3 progressing to infinity this must be woven

together out of both classes as being common to both [] Let us then set forth the odd numbers from 3 by themselves in due order in one series

3 5 7 9 11 13 15 17 19

and the even times even, beginning with 4 again one after another in a second series after their own order

4 8 16 32 64 128 256

as far as you please [8] Now multiply by the first number of either series—it makes no difference which—from the beginning and in order all those in the remaining series and note down the resulting numbers then again multiply by the second number of the same series the same numbers once more as far as you can and write down the results then with the third number again multiply the same terms anew and how ever far you go you will get nothing but the odd times even numbers

[9] For the sake of illustration let us use the first term of the series of odd numbers and multiply by it all the terms in the second series in order thus 3×4 3×8 3×16 3×32 and so on to infinity The results will be 12 24 48 96 which we must note down in one line Then taking a new start do the same thing with the second number 5×4 5×8 5×16 5×32 The results will be 20 40 80 160 Then do the same thing once more with 7 the third number 7×4 7×8 7×16 7×32 The results are 28 56

term or their product should there be two Thus this one species has the peculiar properties of them both because it is a natural mixture of them both

CHAPTER VI

[1] Again while the odd is distinguished over against the even in classification and has nothing in common with it since the latter is divisible into equal halves and the former is not thus divisible nevertheless there are found three species of the odd¹ differing from one another of which the first is called the prime and incomposite² that which is opposed to it the secondary and composite and that which is midway between both of these and is viewed as a mean among extremes namely the variety which in itself is secondary and composite, but relatively is prime and incomposite

[2] Now the first species the prime and incomposite is found whenever an odd number admits of no other factor save the one with the number itself as denominator which is always unity for example 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 None of these numbers will by any chance be found to have a fractional part with a denominator different from the number itself but only the one with this as denominator and this part will be unity in each case for 3 has only a third part, which has the same denominator as the number and is of course uni-

| Odd numbers | 3 | 5 | 7 | 9 | 11 | 13 | 15 |
|-----------------|----|----|-----|-----|-----|------|------|
| Even times even | 4 | 8 | 16 | 32 | 64 | 128 | 256 |
| Breadth | 12 | 24 | 48 | 96 | 192 | 384 | 768 |
| | 20 | 40 | 80 | 160 | 320 | 640 | 1280 |
| | 28 | 56 | 112 | 224 | 448 | 896 | 1792 |
| | 36 | 72 | 144 | 288 | 576 | 1152 | 2304 |
| | 44 | 88 | 176 | 352 | 704 | 1408 | 2816 |
| Length | | | | | | | |

Odd times even numbers

112 224 and in the same way as far as you care to go you will get similar results

[10] Now when you arrange the products of multiplication by each term in its proper line making the lines parallel in marvelous fashion there will appear along the breadth of the table the peculiar property of the even times odd that the mean term is always half the sum of the extremes if there should be one mean and the sum of the means equals the sum of the extremes if two But along the length of the table the property of the even times even will appear for the product of the extremes is equal to the square of the mean should there be one mean

ty 5 a fifth 7 a seventh and 11 only an eleventh part and in all of them these parts are unity

[3] It has received this name because it can be measured only by the number which is first and common to all unity and by no other moreover because it is produced by no other number combined with itself save unity alone for 5 is 5×1 and 7 is 7×1 and the others in accordance with their own quantity To be sure when they are combined with themselves other

¹Cf Euclid Elem VII D ff 11 14

²Cf Euclid Elem VII Def 11 Aristotle VIII 3 157 39

numbers might be produced originating from them as from a fountain and a root wherefore they are called 'prime' because they exist beforehand as the beginnings of the others. For every origin is elementary and incomposite in to which every thing is resolved and out of which everything is made but the origin itself cannot be resolved into anything or constituted out of anything

CHAPTER XII

[1] The secondary composite number¹ is an odd number indeed because it is distinguished as a member of this same class but it has no elementary quality, for it gets its origin by the combination of something else. For this reason it is characteristic of the secondary number to have in addition to the fractional part with the number itself as denominator yet another part or parts with different denominators the former always as in all cases unity the latter never unity but always either that number or those numbers by the combination of which it was produced. For example 9 15 21 25 27 33 39 each one of these is measured by unity as other numbers are and like them has a fractional part with the same denominator as the number itself by the nature of the class common to them all but by exception and more peculiarly they also employ a part or parts with a different denominator 9 in addition to the ninth part has a third part besides 15 a third and a fifth besides a fifteenth 21 a seventh and a third besides a twenty first and 25 in addition to the twenty fifth which has as a denominator 25 itself also a fifth with a different denominator

[2] It is called secondary then because it can employ yet another measure along with unity and because it is not elementary but is produced by some other number combined with itself or with something else in the case of 9 3 in the case of 15 5 or by Zeus 3 and those following in the same fashion. And it is called composite for this or some such reason that it may be resolved into those numbers out of which it was made since it can also be measured by them. For nothing that can be broken down is incomposite but by all means composite

CHAPTER XIII

[1] Now while these two species of the odd are opposed to each other a third one² is con-

ceived of between them deriving as it were its specific form from them both namely the number which is in itself secondary and composite but relatively to another number is prime and incomposite. This exists when a number, in addition to the common measure unity is measured by some other number and is therefore able to admit of a fractional part or parts with denominator other than the number itself as well as the one with itself as denominator. When this is compared with another number of similar properties it is found that it cannot be measured by a measure common to the other nor does it have a fractional part with the same denominator as those in the other. As an illustration let 9 be compared with 25. Each in itself is secondary and composite but relatively to each other they have only unity as a common measure and no factors in them have the same denominator for the third part in the former does not exist in the latter nor is the fifth part in the latter found in the former

[2] The production of these numbers is called by Eratosthenes the sieve because we take the odd numbers mingled together and indiscriminate and out of them by this method of production separate as by a kind of instrument or sieve the prime and incomposite by themselves and the secondary and composite by themselves and find the mixed class by themselves

[3] The method of the sieve is as follows. I set forth all the odd numbers in order beginning with 3 in as long a series as possible and then starting with the first I observe what ones it can measure and I find that it can measure the terms two places apart as far as we care to proceed. And I find that it measures not at chances and at random but that it will measure the first one that is the one two places removed by the quantity of the one that stands first in the series that is by its own quantity for it measures it 3 times and the one two places from this by the quantity of the second in order for this it will measure 5 times and again the one two places further on by the quantity of the third in order or 7 times and the one two places still farther on by the quantity of the fourth in order or 9 times and so ad infinitum in the same way

[4] Then taking a fresh start I come to the second number and observe what it can measure and find that it measures all the terms four places apart the first by the quantity of the first in order or 3 times the second by that of the second or 5 times the third by that of

¹Cf Euclid *Elements* VII Def 14

²Cf Euclid *Elements* VII Def 13

the third or 7 times and in this order *ad infinitum*

[5] Again as before the third term 7 taking over the measuring function will measure terms six places apart and the first by the quantity of 3 the first of the series the second by that of 5 for this is the second number and the third by that of 7, for this has the third position in the series

[6] And analogously throughout this process will go on without interruption so that the numbers will succeed to the measuring function in accordance with their fixed position in the series the interval separating terms measured is determined by the orderly progress of the even numbers from 2 to infinity or by the doubling of the position in the series occupied by the measuring term and the number of times a term is measured is fixed by the orderly advance of the odd numbers in series from 3

[7] Now if you mark the numbers with certain signs you will find that the terms which succeed one another in the measuring function neither measure all the same number—and sometimes not even two will measure the same one—nor do absolutely all of the numbers set forth submit themselves to a measure but some entirely avoid being measured by any number whatsoever some are measured by one only and some by two or even more [8] Now these that are not measured at all but avoid it are primes and incomposite sifted out as it were by a sieve those measured by only one measure in accordance with its own quantity will have but one fractional part with denominator different from the number itself in addition to the part with the same denominator and those which are measured by one measure only but in accordance with the quantity of some other number than the measure and not its own, or are measured by two measures at the same time will have several fractional parts with other denominators besides the one with the same as the number itself these will be secondary and composite

[9] The third division the one common to both the former which is in itself secondary and composite but primary and incomposite in relation to another will consist of the numbers produced when some prime and incomposite number measures them in accordance with its own quantity if one thus produced be compared to another of similar origin For example if 9 which was produced by 3 measuring by its own quantity for it is 3 times 3 be compared with 20 which was produced from 5 measuring by

its own quantity for it is 4 times 5 these numbers have no common measure except unity

[10] We shall now investigate how we may have a method of discerning whether numbers are prime and incomposite or secondary and composite, relatively to each other since of the former unity is the common measure but of the latter some other number also besides unity and what this number is

[11] Suppose there be given us two odd numbers and some one sets the problem and directs us to determine whether they are prime and incomposite relatively to each other or secondary and composite and if they are secondary and composite what number is their common measure We must compare the given numbers and subtract the smaller from the larger as many times as possible then after this subtraction subtract in turn from the other as many times as possible for this changing about and subtraction from one and the other in turn will necessarily end either in unity or in some one and the same number which will necessarily be odd [12] Now when the subtractions terminate in unity they show that the numbers are prime and incomposite relatively to each other and when they end in some other number odd in quantity and twice produced then say that they are secondary and composite relatively to each other and that their common measure is that very number which twice appears

For example if the given numbers were 23 and 45 subtract 23 from 45 and 22 will be the remainder subtracting this from 23 the remainder is 1 subtracting this from 22 as many times as possible you will end with unity Hence they are prime and incomposite to one another and unity which is the remainder is their common measure

[13] But if one should propose other numbers 21 and 49 I subtract the smaller from the larger and 28 is the remainder Then again I subtract the same 21 from this for it can be done and the remainder is 7 This I subtract in turn from 21 and 14 remains from which I subtract 7 again for it is possible and 7 will remain But it is not possible to subtract 7 from 7 hence the termination of the process with a repeated 7 has been brought about and you may declare the original numbers 21 and 49 secondary and composite relatively to each

This mode of determining common factors is found in Euclid (VII 1 & 2) and is commonly termed the Euclidean method of finding the greatest common divisor of numbers

other and 7 their common measure in addition to the universal unit.

CHAPTER XIV

[1] To make again a fresh start of the simple even numbers some are superabundant some deficient like extremes set over against each other, and some are intermediary between them and are called perfect. [2] Those which are said to be opposites to one another the superabundant and deficient are distinguished from one another in the relation of inequality¹ in the directions of the greater and the less for apart from these no other form of inequality could be conceived nor could evil² disease disproportion unseemliness nor any such thing save in terms of excess or deficiency. I or in the realm of the greater³ there arise excesses overreaching and superabundance and in the less need deficiency privation and lack but in that which lies between the greater and the less, namely the equal are virtues wealth moderation propriety beauty and the like to which the aforesaid form of number the perfect is most akin.

[3] Now the superabundant number is one which has over and above the factors which belong to it and fall to its share others in addition just as if an animal should be created with too many parts or limbs with ten tongues as the poet says⁴ and ten mouths or with nine lips or three rows of teeth or a hundred hands or too many fingers on one hand. Similarly if when all the factors in a number are examined and added together in one sum it proves upon investigation that the number's own factors exceed the number itself this is called a superabundant number for it oversteps the symmetry which exists between the perfect and its own parts. Such are 12 24 and certain others for 12 has a half 6 a third 4 a fourth 3 a sixth 2 and a twelfth 1 which added together make 16 which is more than the original 12 its parts therefore are greater than the whole itself. [4] And 24 has a half a third fourth sixth eighth twelfth and twenty fourth, which are 12 8 6 4 3 2 1. Added together they make 36 which compared to the original number 24 is found to be greater than it, although made up solely of its factors. Hence in this case also the parts are in excess of the whole.

¹Cf I 17 2 4 6 also I 23 4

²Cf Arist. *Ethics* II 6 1106^b 33

³Cf Arist. *Ethics* II 6 1106^b 21 33

⁴Homer *Odyssey* XII 85 ff

CHAPTER XV

[1] The deficient number is one which has qualities the opposite of those pointed out and whose factors added together are less in comparison than the number itself. It is as if some animal should fall short of the natural number of limbs or parts or as if a man should have but one eye as in the poem "And one round orb was fixed in his brow", or as though one should be one-handed or have fewer than five fingers on one hand or lack a tongue or some such member. Such a one would be called deficient and so to speak maimed after the peculiar fashion of the number whose factors are less than itself such as 3 or 14. For 3 has the factors half fourth and eighth which are 4, 2, and 1 and added together they make 7 and less than the original number. The parts therefore fall short of making up the whole. [2] Again 14 has a half a seventh a fourteenth, 7 2 and 1 respectively and all together they make 10, less than the original number. So this number also is deficient in its parts with respect to making up the whole out of them.

CHAPTER XVI

[1] While these two varieties are opposed after the manner of extremes the so-called perfect number⁵ appears as a mean, which is discovered to be in the realm of equality and neither makes its parts greater than itself, added together nor shows itself greater than its parts but is always equal to its own parts. For the equal is always conceived of as in the mid-ground between greater and less and is as it were moderation between excess and deficiency and that which is in tune between pitches too high and too low.

[2] Now when a number comparing with itself the sum and combination of all the factors whose presence it will admit neither exceeds them in multitude nor is exceeded by them then such a number is properly said to be perfect as one which is equal to its own parts. Such numbers are 6 and 28 for 6 has the factors half third and sixth 3 2 and 1, respectively and these added together make 6 and are equal to the original number and neither more nor less. Twenty-eight has the factors half fourth seventh fourteenth and twenty-eighth which are 14 7 4 2 and 1 these added together make 28 and so neither are the

⁵Fuelst's definition *Flem* VII 22 is "A perfect number is one that is equal to its own parts."

parts greater than the whole nor the whole greater than the parts but their comparison is in equality which is the peculiar quality of the perfect number

[5] It comes about that even as fair and excellent things are few and easily enumerated while ugly and evil ones are widespread so also the superabundant and deficient numbers are found in great multitude and irregularly placed—for the method of their discovery is irregular—but the perfect numbers are easily enumerated and arranged with suitable order for only one is found among the units 6 only one other among the tens 28 and a third in the rank of the hundreds 496 alone and a fourth within the limits of the thousands that is below ten thousand 8128 And it is their accompanying characteristic to end alternately in 6 or 8 and always to be even

[4] There is a method of producing them neat and unailing which neither passes by any of the perfect numbers nor fails to differentiate any of those that are not such which is carried out in the following way

You must set forth the even times even numbers from unity advancing in order in one line as far as you please 1 2 4 8 16 32 64 128 2 6 512 1024 2048 4096 Then you must add them together one at a time and each time you make a summation observe the result to see what it is If you find that it is a prime incomposite number multiply it by the quantity of the last number added and the result will always be a perfect number If however the result is secondary and composite do not multiply but add the next and observe again what the resulting number is if it is secondary and composite again pass it by and do not multiply but add the next but if it is prime and incomposite multiply it by the last term added and the result will be a perfect number and so on to infinity In similar fashion you will produce all the perfect numbers in succession overlooking none

For example to 1 I add 2 and observe the sum and find that it is 3 a prime and incomposite number in accordance with our previous demonstrations for it has no factor with denominator different from the number itself but only that with denominator agreeing Therefore I multiply it by the last number to be taken into the sum that is 2 I get 6 and thus I declare to be the first perfect number in actuality and to have those parts which are beheld

in the numbers of which it is composed For it will have unity as the factor with denominator the same as itself, that is its sixth part and 3 as the half which is seen in 6 and conversely 2 as its third part

[6] Twenty-eight likewise is produced by the same method when another number 4 is added to the previous ones For the sum of the three 1 2 and 4 is 7, and is found to be prime and incomposite for it admits only the factor with denominator like itself the seventh part Therefore I multiply it by the quantity of the term last taken into the summation and my result is 28, equal to its own parts and having its factors derived from the numbers already adduced a half corresponding to 2 a fourth to 7 a seventh to 4 a fourteenth to effect the half and a twenty-eighth in accordance with its own nomenclature which is 1 in all numbers

[6] When these have been discovered, 6 among the units and 28 in the tens you must do the same to fashion the next [7] Again add the next number 8 and the sum is 15 Observing this I find that we no longer have a prime and incomposite number but in addition to the factor with denominator like the number itself it has also a fifth and a third with unlike denominators Hence I do not multiply it by 5 but add the next number 16 and 31 results As this is a prime incomposite number of necessity it will be multiplied in accordance with the general rule of the process by the last number added 16 and the result is 496 in the hundred and then comes 8128 in the thousands and so on as far as it is convenient for one to follow

[8] Now unity is potentially a perfect number but not actually for taking it from the series as the very first I observe what sort it is according to the rule and find it prime and incomposite for it is so in very truth not by participation like the rest but it is the primary number of all and alone incomposite [9] I multiply it therefore by the last term taken into the summation that is by itself and my result is 1 for 1 times 1 equals 1 [10] Thus unity is perfect potentially for it is potentially equal to its own parts the others actually

CHAPTER XVII

[1] Now that we have given a preliminary systematic account of absolute quantity we come in turn to relative quantity

[2] Of relative quantity then the highest general divisions are two equality and inequality

[1] Cf. Euclid (IX 36)

ity for everything viewed in comparison with another thing is either equal or unequal, and there is no third thing besides these

[3] Now the equal is seen when of the things compared one neither exceeds nor falls short in comparison with the other for example 100 compared with 100 10 with 10 2 with 2, a mina with a mina a talent with a talent, a cubit with a cubit and the like, either in bulk length weight or any kind of quantity [4] And as a peculiar characteristic also this relation is of itself not to be divided or separated, as being most elementary, for it admits of no difference For there is no such thing as this kind of equality and that kind, but the equal exists in one and the same manner [5] And that which corresponds to an equal thing to be sure, does not have a different name from it but the same like friend neighbor comrade so also equal for it is equal to an equal

[6] The unequal on the other hand is split up by subdivisions and one part of it is the greater the other the less which have opposite names and are antithetical to one another in their quantity and relation For the greater is greater than some other thing and the less again is less than another thing in comparison, and their names are not the same but they each have different ones for example father and son striker and struck, teacher and pupil and the like

[7] Moreover of the greater separated by a second subdivision into five species one kind is the multiple another the superparticular another the superpartient another the multiple superparticular and another the multiple superpartient [8] And of its opposite the less there arise similarly by subdivision five species opposed to the foregoing five varieties of the greater the submultiple subsuperparticular subsuperpartient submultiple superparticular and submultiple-superpartient for as whole answers to whole smaller to greater so also the varieties correspond each to each in the aforesaid order with the prefix sub-

CHAPTER XVIII

[1] Once more then the multiple is the species of the greater first and most original by nature as straightway we shall see and it is a number which when it is observed in comparison with another contains the whole of that number more than once For example compared with unity all the successive numbers beginning with 2 generate in their proper order the regular forms of the multiple for 2 in the

first place is and is called the double 3 triple, 4 quadruple and so on for 'more than once' means twice, or three times, and so on in succession as far as you like

[2] Answering to this is the submultiple which is itself primary in the smaller division of inequality It is the number which when it is compared with a larger is able to measure it completely more than once and more than once starts with twice and goes on to infinity

[3] If then it measures the larger number that is being compared twice only, it is properly called the subdouble as 1 is of 2 if thrice subtriple, as 1 of 3 if four times subquadruple as 1 of 4 and so on in succession

[4] While each of these the multiple and the submultiple is generically infinite the varieties by subdivision and the species also are observed naturally to make an infinite series For the double beginning with 2 goes on through all the even numbers as we select alternate numbers out of the natural series and these will be called doubles in comparison with the even and odd numbers successively placed beginning with unity [5] All the numbers from the beginning two places apart and third in order are triples for example 3 6 9 12 15, 18 21 24 It is their property to be alternately odd and even and they themselves in the regular series from unity are triples of all the numbers in succession as far as one wishes to go on with the process

[6] The quadruples are those in the fourth places three apart for instance 4 8 12 16, 20, 24 28 32 and so on These are the quadruples of the regular series of numbers from unity going on as far as one finds it convenient to follow It belongs to them all to be even, for one needs only to take the alternate terms out of the even numbers already selected Thus necessarily it is true that the even numbers with no further designation are all doubles the alternate ones quadruples those two places apart sextuples and those three places apart octuples and this series will go on on this same analogy indefinitely

[7] The quintuples will be seen to be those four places apart placed fifth from one another and them elies the quintuples of the successive numbers beginning with unity Alternately they are odd and even like the triples

CHAPTER XIX

[1] The superparticular the second species of the greater both naturally and in order, is a

number that contains within itself the whole of the number compared with it and some one factor of it besides

[9] If this factor is half the greater of the terms compared is called specifically sesquialter and the smaller sub-sesquialter if it is a third sesquitercian and sub sesquitercian and as you go on throughout it will always thus agree so that these species also will progress to infinity even though they are species of an unlimited genus

For it comes about that the first species the sesquialter ratio has as its consequents the even numbers in succession from 2 and no other at all and as antecedents the triples in succession from 3 and no other [9] These must be joined together regularly first to first, second to second third to third—3 2 6 4 9 6 12 8—and the analogous numbers to the ones corresponding to them in position

[4] If we care to investigate the second species of the superparticular the sesquitercian (for the fraction naturally following after the half is the third) we shall have this definition of it—a number which contains the whole of the number compared and a third of it in addition to the whole We may have examples of it in the proper order in the successive quadruples beginning with 4 joined to the triples from 3 each term with the one in the corresponding position in the series for example, 4 3 8 6 12 9 and so on to infinity [5] It is plain that that which corresponds to the sesquitercian but is called with the prefix sub- sub-sesquitercian is the number the whole of which is contained and a third part in addition for example 3 4 6 8 9 12 and the similar pairs of numbers in the same position in the series

[6] And we must observe the never failing corollary of all this that the first forms in each series the so-called root numbers are next to one another in the natural series the next after the root forms show an interval of only one number the third two the fourth three the fifth four and so on as far as you like [7] Furthermore that the fraction after which each of the superparticulars is named is seen in the lesser of the root numbers never in the greater

[8] That by nature and by no disposition of ours the multiple is a more elementary and an older form than the superparticular we shall shortly learn through a somewhat intricate process And here for a simple demonstration we must prepare in regular and parallel lines the multiples specified above according to their

varieties first the double in one line then in a second the triple then the quadruple in a third and so on as far as the tenfold multiples so that we may detect their order and variety, their regulated progress and which of them is naturally prior and indeed other corollaries delightful in their exactness Let the diagram be as follows [9]

| | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|-----|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
| 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 |
| 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
| 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 |
| 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 |
| 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 |
| 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 | 90 |
| 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |

[10] Let there be set forth in the first row the natural series from unity and then in order those species of the multiple which we were bidden to insert

[11] Now then in comparison with the first rows beginning with unity if we read both across and up and down in the form of the letter gamma the next rows both ways themselves in the form of a gamma beginning with 4 are multiples according to the first form of the multiple for they are doubles The first differs by unity from the first the second from the second by 2 the third from the third by 3 the next by 4 those following by 5 and you will find that this follows throughout

The third rows in both directions from 9, their common origin will be the triples of the terms in that same first row according to the second form of the multiple the cross lines like the letter chi ending in the term 3 in either direction are to be taken into consideration [12] The difference for these numbers will progress after the series of the even numbers being 2 for the first 4 for the next 6 for the third and this difference Nature has of her own accord interpolated for us between these rows that are being examined as is evident in the diagram

[13] The fourth row whose common origin in both directions is 16 and whose cross-lines end with the terms 4 exhibits the third species of multiple the quadruple when it is compared with that same first row according to corresponding positions first term with first second with second third with third and so on Again the differences of these numbers are 6 then 9 then 12 and the quantities that progress by steps of 3 These numbers are detected in the

structure of the diagram in places just above the quadruples and in the subsequent forms of the multiple the analogy will hold throughout.

[14] In comparison with the second line reading either way which begins with the common origin 4 and runs over in cross lines to the term 2 in each row, the lines which are next in order beneath display the first species of the superparticular that is, the sesquialter between terms occupying corresponding places. Thus by divine nature not by our convention or agreement the superparticulars are of later origin than the multiples. For illustration 3 is the sesquialter of 2 6 of 4 9 of 6 12 of 8 15 of 10 and throughout thus. They have as a difference the successive numbers from unity like those before them.

[15] The sesquiterterians the second species of superparticular proceed with a regular even advance from 4 3 8 ■ 12 9 16 12 and so on having also a regular increase of their differences. [16] And in the other multiple and superparticular relations you will see that the results are in harmony and not by any means inconsistent as you go on to infinity.

[17] The following feature of the diagram moreover is of no less exactness. The terms at the corners are units the one at the beginning a simple unit that at the end the unit of the third course and the other two units of the second course appearing twice so that the product (of the first two) is equal to the square (of the last). [18] Furthermore in reading either way there is an even progress from unity to the tens and again on the opposite sides two other progressions from 10 to 100.

[19] The terms on the diagonal from 1 to 100 are all square numbers the products of equals by equals and those flanking them on either side are all heteromecic unequal and the products of sides of which one is greater than the other by unity and so the sum of two successive squares and twice the heteromecic numbers between them is always a square and conversely a square is always produced from the two heteromecic numbers on the sides and twice the square between them.

[20] An ambitious person might find many other pleasing things displayed in this diagram upon which it is not now the time to dwell for we have not yet gained recognition of them from our Introduction and so we must turn to the next subject. For after these two generic relations of the multiple and the superparticular and the other two of pönte to them with the prefix sub- the submultiple and the sub-

superparticular there are in the greater division of inequality the superpartient and in the less its opposite the sub-superpartient.

CHAPTER XX

[1] It is the superpartient relation when a number contains within itself the whole of the number compared and in addition more than one part of it and "more than one" starts with 2 and goes on to all the numbers in succession. Thus the root-form of the superpartient is naturally the one which has in addition to the whole two parts of the number compared and as a species will be called superbi-partient after this the one with three parts besides the whole will be called supertri-partient as a species then comes the superquadri-partient the superquintipartient, and so forth.

[2] The parts have their root and origin with the third for it is impossible in this case to begin with the half. For if we assume that any number contains two halves of the compared number besides the whole of it we shall inadvertently be setting up a multiple instead of a superpartient because each whole plus two halves of it, added together makes double the original number. Thus it is most necessary to start with two thirds then two fifths two sevenths and after these two ninths, following the advance of the odd numbers for two quarters, for example again are a half, two sixths a third and thus again superparticulars will be produced instead of superpartients which is not the problem laid before us nor in accord with the systematic construction of our science.

[3] After the superpartient the sub-superpartient immediately is produced whenever a number is completely contained in the one compared with it, and in addition several parts of it 2 3 4 or 5 and so on.

CHAPTER XXI

[1] The regular arrangement and orderly production of both species are disclosed when we set forth the successive even and odd numbers beginning with 3 and compare with them simple series of odd numbers only from 5 in succession first to first—that is 5 to 3—second to second—that is 7 to 4—third to third—that is 9 to 5—fourth to fourth—that is 11 to 6—and so on in the same order as far as you like. In this way the forms of the superpartient and the sub-superpartient in due order will be disclosed through the root forms of each species the superbi-partient first then the supertri-partient superquadri-partient and super

quintupartient and further in succession in similar manner, for after the root forms of each species the ones which follow them will be produced by doubling or tripling both the terms and in general by multiplying after the regular forms of the multiple

nerically, it will have in its subdivisions according to species a sort of diversification and change of names proper both to the first part of the name and to the second. For instance in the first part that = the multiple it will have double triple, quadruple quintuple, and

TABLE OF THE SUPERPARTIVITS

| Root-forms | 5 | 3 | 7 | 4 | 9 | 5 | 11 | ■ | 13 | 7 |
|------------|----|----|----|----|----|----|----|----|-----|----|
| | 10 | 6 | 14 | 8 | 18 | 10 | 22 | 12 | 26 | 14 |
| | 15 | 9 | 21 | 12 | 27 | 15 | 33 | 18 | 39 | 21 |
| | 20 | 12 | 28 | 16 | 36 | 20 | 44 | 24 | 52 | 28 |
| | 25 | 15 | 35 | 20 | 45 | 25 | 55 | 30 | 65 | 35 |
| | 30 | 18 | 42 | 24 | 54 | 30 | 66 | 36 | 78 | 42 |
| | 35 | 21 | 49 | 28 | 63 | 35 | 77 | 42 | 91 | 49 |
| | 40 | 24 | 56 | 32 | 72 | 40 | 88 | 48 | 104 | 56 |
| | 45 | 27 | 63 | 36 | 81 | 45 | 99 | 54 | 117 | 63 |

[7] It must be observed that from the two parts in addition to the whole which are contained in the greater term we are to understand third in the case of three parts, fourth with four parts fifth with five sixth and so on so that the order of nomenclature = something like this superbipartient supertripartient superquadrupartient then superquintupartient and similarly with the rest

[8] Now the simple uncompounded relations of relative quantity are these which have been enumerated. Those which are compounded of them and as it were woven out of two into one are the following of which the antecedents are the multiple superparticular and multiple superpartient and the consequents the ones that immediately arise in connection with each of the former named with the prefix sub together with the multiple superparticular the submultiple superparticular and with the multiple superpartient the submultiple superpartient. In the subdivision of the genera the species of the one will correspond to those of the other these also having names with the prefix sub

CHAPTER XXII

[1] Now the multiple superparticular is a relation in which the greater of the compared terms contains within it self the lesser term more than once and in addition some one part of it whatever this may be

[2] As = compound such a number is doubly diversified after the peculiarities of nomenclature of its components on either side for inasmuch as the multiple superparticular is compounded of the multiple and superparticular ge-

nerically, it will have in its subdivisions according to species a sort of diversification and change of names proper both to the first part of the name and to the second. For instance in the first part that = the multiple it will have double triple, quadruple quintuple, and

so forth and in the second part generically from the superparticular, its specific forms in due order the sesquialter sesquitercian sesquiquartan sesquiquintan and so on so that the combination will proceed in somewhat this order

Double sesquialter double sesquitercian, double sesquiquartan double sesquiquintan, double sesquiseptan and analogously

Beginning once more triple sesquialter triple sesquitercian triple sesquiquartan triple sesquiquintan

Again quadruple sesquialter quadruple sesquitercian quadruple sesquiquartan quadruple sesquiquintan

Again quintuple sesquialter quintuple sesquitercian quintuple sesquiquartan quintuple sesquiquintan and the forms analogous to these *ad infinitum*. Whatever number of times the greater contains the whole of the smaller by this quantity the first part of the ratio of the terms joined together in the multiple superparticular is named and whatever may be the factor in addition to the whole several times contained that is in the greater term from this is named the second kind of ratio of which the multiple superparticular is compounded

[3] Examples of it are these 5 is the double sesquialter of 2 7 the double sesquitercian of 3 9 the double sesquiquartan of 4 11 the double sesquiquintan of 5 You will furthermore always produce them in regular order in this fashion by comparing with the successive even and odd numbers from 2 the odd numbers, exclusively from 5 first with first second third with third and the with the one in the same position

The successive terms beginning with 5 and differing by 5 will be without exception double sesquialters of all the successive even numbers from 2 on, when terms in the same position in the series are compared and beginning with 3, if all those with a difference of 3 be set forth as 3 6 9 12 15 18 21 and in another series there be set forth those that differ by 7 to infinity as 7 14 21 28 35 42 49 and the greater be compared with the smaller first to first, second to second third to third fourth to fourth and so on the second species will appear the double sesquiquartan disposed in its proper order

[4] Then again, to take a fresh start if the simple series of quadruples be set forth, 4 8 12 16 20 24 28 32 and then there be placed beside it in another series the successive numbers beginning with 9 and increasing by 9 as 9 18 27 36 45 54 we shall have revealed once more the multiple superparticular in a specific form that is the double sesquiquartan in its proper order and any one who desires can contrive this to an unlimited extent

[5] The second kind begins with the triple sesquialter such as 7 2 14 4 and in general the numbers that advance by steps of 7 compared with the even numbers in order from 2

[6] Then once more 10 is the first triple sesquiquartan 20 is the second and in a word the multiples of 10 in succession compared with the successive triples. This indeed we can observe with greater exactitude and clearness in the table studied above for in comparison with the first row the succeeding rows in order¹ compared as whole rows display the forms of the multiple in regular order up to infinity when they are all compared in each case to the same first row and when each row is compared to all those above it in succession the second row being taken as our starting point all the forms of the superparticular are produced in their proper order and if we start with the third row all of those beginning with the fifth that are odd in the series when they are compared with this same third row and those following it will show all the forms of the superpartient in proper order. In the case of the multiple superparticular the comparisons will have a natural order of their own if we start with the second row and compare the terms from the fifth first to first, second to second third to third and so on and then the terms of the seventh row to the third, those of the ninth to the

fourth and follow the corresponding order as far as we are able to go [7] It is plain that here too the smaller terms have names corresponding to the larger ones with the prefix sub- according to the nomenclature given them all

CHAPTER XXIII

[1] The multiple superpartient is the remaining relation of number. This and the relation called by a corresponding name with the prefix sub-, exist when a number contains the whole of the number compared more than once (that is twice, thrice or any number of times) and certain parts of it more than one either two three or four and so on, besides [2] These parts² are not halves for the reasons mentioned above but either thirds fourths or fifths and so on

[3] From what has already been said it is not hard to conceive of the varieties of this relation for they are differentiated in the same way as and consistently with those that precede double superbi-partient, double supertriple-partient double superquadruple-partient and so on. For example 8 is the double superbi-partient of 3 16 of 6 and in general the numbers beginning with 8 and differing by 8 are double superbi-partients of those beginning with 3 and differing by 3 when those in corresponding places in the series are compared and in the case of the other varieties one could ascertain their proper sequence by following out what has already been said. In this case too we must conceive that the nomenclature of the number compared goes along and suffers corresponding changes with the addition of the prefix sub-

[4] Thus we come to the end of our speculation upon the ten arithmetical relations for a first Introduction. There is however a method very exact and necessary for all discussion of the nature of the universe which very clearly and indisputably presents to us the fact that that which is fair and limited and which subjects itself to knowledge³ is naturally prior to the unlimited incomprehensible and ugly and furthermore that the parts and varieties of the infinite and unlimited are given shape and boundaries by the former and through it attain to their fitting order and sequence and like objects brought beneath some real or measure all gain a share of likeness to it and similarity of name when they fall under its influence. For thus it is reasonable that the rational

¹Referring to the table in Chapter 19

²See 70 2 above

³Cf I 2 5

part of the soul will be the agent which puts in order the irrational part and passion and appetite which find their places in the two forms of inequality will be regulated by the reasoning faculty as though by a kind of equality and sameness [5] And from this equalizing process there will properly result for us the so-called ethical virtues sobriety,¹ courage gentleness self control fortitude and the like

[6] Let us then consider the nature of the principle that pertains to these universal matters It is capable of proving that all the complex species of inequality and the varieties of these species are produced out of equality, first and alone as from a mother and root

[7] Let there be given us equal numbers in three terms first units then two in another group of three then three next four five and so on as far as you like For them as the setting forth of these terms has come about by a divine and not human contrivance may by Nature herself, multiples will first be produced and among these the double will lead the way the triple after the double the quadruple next and then the quintuple and following the order we have previously recognized *ad infinitum* second the superparticular and here again the first form, the sesquialter will lead and the next after it the sesquitercian will follow and after them the next in order the sesquiquartan the sesquiquintan the sesquiseptan and so on *ad infinitum* third the superpartient which once more the superbi-partient will lead the supertripartient will follow immediately upon it and then will come the superquadrupartient the superquintupartient and according to the foregoing as far as one may proceed

[8] Now you must have certain rules like invariable and inviolable natural laws following which the whole aforesaid advance and progress from equality may go on without failure These are the directions Make the first equal to the first the second equal to the sum of the first and second and the third to the sum of the first twice the second and the third For if you fashion according to these rules you would get first all the forms of the multiple in order out of the three given terms of the equality as it were sprouting and growing without your paying any heed or offering any aid From equality you will first get the double from the double the triple from the triple successively

the quadruple and from this the quintuple in due order and so on [9] From the same multiples in their regular order reversed there are immediately produced by a sort of natural necessity through the agency of the same three rules the superparticulars and these not as it chances and irregularly but in their proper sequence for from the first the double reversed comes the first the sesquialter and from the second the triple the second in this class the sesquitercian then the sesquiquartan from the quadruple and in general each one from the one of similar name [10] And with a fresh start if the superparticulars are set forth in the order of their production but with terms reversed the superpartients which naturally follow them are brought to light the superbi-partient from the sesquialter, the supertripartient from the sesquitercian the superquadrupartient from the sesquiquartan and so on *ad infinitum* [11] If, however the superparticulars are set forth with terms not in reverse but in direct order there are produced through the three rules the multiple superparticulars, the double sesquialter out of the first the sesquialter the double sesquitercian from the second the sesquitercian, the double sesquiquartan from the third the sesquiquartan and so on [12] From those produced by the reversal of the superparticular that is the superpartients and from those produced without such reversal the multiple superparticulars there are once more produced in the same way and by the same rules both when the terms are in direct or reverse order the numbers that show the remaining numerical relations

[13] The following must suffice as illustrations of all that has been said hitherto the production of these numbers and their sequence and the use of direct and of reverse order [14] From the relation and proportion in terms of the sesquialter reversed so as to begin with the largest term there arises a relation in superpartient ratios the superbi-partient and from it in direct order beginning with the smallest term a multiple superparticular relation the double sesquialter For example from 9 6 4 we get either 9 15 25 or 4 10 25 From the relation in terms of sesquitercians beginning with the greatest term is derived a superpartient the supertripartient beginning with the smallest term a double sesquitercian For example from 16 12 8 comes either 16 28 49 or 9 21 49 And from the relation in terms sesquiquartans when it is arranged to with the largest term is derived a s

¹Cf Aristotle *Ethics* 1107^b 4 ff See *ibid* 1108 4 ff 1145^b 8 ff

tient the superquadrupartient when it starts with the smallest term ■ multiple superparticular, the double sesquiquintan for instance from 25 20, 16 comes either 25, 45, 81 or 16 36 81

[16] In the case of all the e relations that are thus differentiated and of the one from which both of the differentiated ones are derived the last term is always the same and ■ square the first term becomes the smallest and invariably the extremes are squares

[16] Moreover the multiple superpartients and superpartients of other kinds are made to appear in yet another way out of the superpartients for example from the superbipartient relation arranged so as to begin with the small

est term comes the double superbipartient but arranged so as to start with the greater, the superpartient ratio of 8 5 Thus from ■ 15 25 comes either 9 24 64 or 25 40, 64 From the supertripartient beginning with the smallest term we have the double supertripartient, and beginning with the largest the ratio of 11 7 Thus from 16 28 49 comes either 16, 44, 121 or 49 77, 121 [17] Again from the superquintipartient as for example, 25, 45 81 beginning with the lesser term we derive the double superquintipartient in the terms 25 70 196 but beginning with the greater ■ superpartient again, the ratio of 14 9 in the terms 81 120 196 And you will find the results analogous and in agreement with the foregoing in all successive cases to infinity

BOOK TWO

CHAPTER I

[1] An element \equiv said to be and is the smallest thing which enters into the composition of an object and the least thing into which it can be analyzed. Letters for example are called the elements of literate speech for out of them all articulate speech is composed and into them finally it is resolved. Sounds are the elements of all melody for they are the beginning of its composition and into them it is resolved. The so-called four elements of the universe in general are simple bodies fire water air and earth for out of them in the first instance we account for the constitution of the universe and into them finally we conceive of it as being resolved.

We wish also to prove that equality is the elementary principle¹ of relative number for of absolute number number per se unity and the dyad are the most primitive elements the least things out of which it is constructed even to infinity by which it has its growth and with which its analysis into smaller terms comes to an end. [2] We have however demonstrated that in the realm of inequality advance and increase have their origin in equality and go on to absolutely all the relations with a certain regularity through the operation of the three rules.² It remains then in order to make it an element in very truth to prove that analyses also finally come to an end in equality. Let this then be considered our procedure.

CHAPTER II

[1] Suppose then you are given three terms in any relation whatsoever and in any ratio whether multiple superparticular superpartient or a compound of these multiple superparticular or multiple superpartient provided only that the mean term is seen to be in the same ratio to the lesser as the greater to the mean and vice versa. Subtract always from

the mean the lesser term whether it be first or last in order, and set down the lesser term itself as the first term of your new series then put as your second term what remains from the second after the subtraction then after having subtracted the sum of the new first term and twice the new second term from the remaining number—that \equiv the greater of the numbers originally given you—make the remainder your third term and the resulting numbers will be in some other ratio naturally more primitive. [2] And if again in the same way you subtract the remainder from these same terms it will be found that your three terms have passed back into three others more primitive and you will find that this always takes place as a consequence until they are reduced to equality whence by every necessity it appears evident that equality \equiv the elementary principle of relative quantity.

[3] There follows upon this speculation a most elegant principle extremely useful in its application to the Platonic psychogony³ and the problem of all harmonic intervals for in the Platonic passage we are frequently bidden, for the sake of the argument to set up series of intervals of two three four five or an infinite number of sesquialter ratios or two sesquiterbans sesquiquartans sesquioctaves or superparticulars of any kind whatsoever, and in each case three four or five of them or as many as may be directed. [4] It is reasonable that we should do this not in an unscientific untelligent fashion it may be even blunderingly but artistically surely and quickly by the following procedure.

CHAPTER III

[1] Every multiple will stand at the head⁴ of as many superparticular ratios corresponding in name with itself as it itself chances to be removed from unity and no more nor less under any circumstances.

¹See on I 23 4

²See on I 23 8.

³See Plato *Timaeus* 35 ff

⁴Refers to the table in :

[2] The doubles then, will produce¹ sesqui alters the first one the second two the third three the fourth four, the fifth five, the sixth six and neither more nor less, but by every necessity when the superparticulars that are generated attain the proper number, that is, when their number agrees with the multiples that have generated them at that point by a divine device as it were there is found the

Sixteen the fourth double will stand at the head of four sesquialters 24, 36, 54 and finally 81, so that they may of necessity be equal in number to what generated them for 81 by its nature is not divisible by 16. And this as you go on you will find holds true in similar fashion to infinity

[4] For the sake of illustration let there be set down the table of the doubles thus

The double ratio in the breadth of the table

| | | | | | | | | |
|---------------------------------------|---|---|---|----|----|-----|-----|------------------------------------|
| | 1 | 2 | 4 | 8 | 16 | 32 | 64 | |
| | | 3 | 6 | 12 | 24 | 48 | 96 | |
| The triple ratio along the hypotenuse | | | 9 | 18 | 36 | 72 | 144 | The sesquialter ratio in the depth |
| | | | | 27 | 54 | 108 | 216 | |
| | | | | | 81 | 162 | 324 | |
| | | | | | | 243 | 486 | |
| | | | | | | | 729 | |

number which terminates them all because it naturally is not divisible by that factor whereby the progression of the superparticular ratios went on

CHAPTER IV

[1] We must make a similar table in illustration of the triple

The triple ratio in the breadth

| | | | | | | | | |
|---------------------------------------|---|---|----|----|-----|------|------|--------------------------------------|
| | 1 | 3 | 9 | 27 | 81 | 243 | 729 | |
| | | 4 | 12 | 36 | 108 | 324 | 972 | |
| The quadruple ratio on the hypotenuse | | | 16 | 48 | 144 | 432 | 1296 | The sesquitercian ratio in the depth |
| | | | | 64 | 192 | 576 | 1728 | |
| | | | | | 256 | 768 | 2304 | |
| | | | | | | 1024 | 3072 | |
| | | | | | | | 4096 | |

From the triples all the sesquitercians will proceed likewise equal in number to the number of the generating terms and coming to an end after the independence of their advance is lost in numbers not divisible by 3. Similarly the sesquiquartans come from the quadruples reaching a culmination after their independent progression in a number that is not divisible by 4.

[3] As an example since doubles generate sesquialters corresponding to them in number the first row of multiples will be 1 2 4 8 16 32 64. Now since 2 is the first after unity this will be the origin of one sesquialter only 3 which number is not divisible by 2 so that another sesquialter might arise out of it. The first double therefore is productive of but one sesquialter, and the second 4 of two. For it produces its own sesquialter 6 and that of 8 9 but there is none for 9 because it has no half. Eight which is the third double is father to three sesquialters one its own 12 the second 18 the sesquialter of 12 and third 27 that of 18 there is no fourth one however because of the general rule for 27 is not divisible by 2

¹See I 19 2

In the foregoing table we shall observe that in the same way the first triple 3 stands at the head of but one sesquitercian ratio 4 its own sesquitercian which immediately shuts off the development of another like it for 4 is not divisible by 3, and hence will not have a sesquitercian. The second triple is 9 and hence will begin a series of only two sesquitercian ratios 12 its own and 16 that of 12 but 16 cuts off further progress for it is not divisible by 3 and hence will not have a sesquitercian. [2] Next in order is the triple 27 three times removed from 1 for the triples progress thus 1 3, 9 27. Therefore this number will stand at the head of three sesquitercian ratios and no more. The first is its own 36 the second the sesquitercian of 36 48 the third that of the last 64 and this no longer has a third part and therefore will not admit of a sesquitercian. The fourth leads a series of four sesquitercians and the fifth of course five.

[3] Such then is the illustration and for the other multiples let the manner of your tables be the same. Observe that likewise here as we found to be true in our previous discus-

non Nature shows us that the doubles are more nearly original than the triples the triples than the quadruples the latter than the quintuples and so on throughout For the highest rows of figures across the breadth of the tables if they are doubles will have doubles lying parallel to them and the numbers lying diagonally on the hypotenuse will be of the next succeeding variety greater by 1 that is triples seen also in a series of parallel lines If however there are triples across the breadth the diagonals will by all means be quadruples if the former are quadruples then the latter are quintuples and so forth

CHAPTER V

[1] It remains after we have explained what other ratios are produced by combination of ratios to pass on to the succeeding topics of the *Introduction*

[2] Now the first two ratios of the superparticular combined produce the first ratio of the multiple namely the double for every double is a combination of sesquialter and sesquitercian and every sesquialter and sesquitercian combined will invariably produce a double

For example since 3 is the sesquialter of 2 and 4 the sesquitercian of 3 4 will be the double of 2 and is a combination of sesquialter and sesquitercian Again as 6 is the double of 3 we shall find between them some number that will of necessity preserve the sesquitercian ratio to the one and the sesquialter to the other and indeed 4 lying between 6 and 3 gives the sesquitercian ratio to 3 and the sesquialter to 6

[3] It was rightly said then that the double when resolved is resolved into the sesquialter and the sesquitercian and that when sesquialter and sesquitercian are combined there arises the double and that the first two forms of the superparticular combined make the first form of the multiple

[4] But again to take another start this first form of the multiple which has thus been produced together with the first form of the superparticular will produce the next form of the same class that is the second multiple the triple for from every multiple and sesquialter combined a triple of necessity arises For example as the double of 6 is 12 and the sesquialter of this is 18 then immediately 18 is the triple of 6 and to take another method if I do not care to make 12 the mean term but rather 9 the sesquialter of 6 the same result will come about without deviation and harmoniously

for while 18 is the double of 9 it will preserve the triple ratio to 6 Hence from the sesquialter and the double, the first forms of the superparticular and the multiple there arises by combination the second form of the multiple, the triple and into them it is always resolved [6] For look you 6 which is the triple of 2 will have a mean term 3 which will exhibit two ratios the sesquialter with regard to 2 and the double ratio of 3 to itself

But if this triple ratio likewise the second form of the multiple is combined with the sesquitercian which is the second form of the superparticular there would be produced from them the next form of the multiple namely the quadruple and this also will of necessity be resolved into them after the same fashion as the cases previously set forth and the quadruple taking into combination the sesquitercian will make the quintuple and once more the latter with the sesquitercian will make the sextuple and so on to the end Thus the multiples in regular order from the beginning with the superparticulars in regular order from the beginning will be found to produce the next larger multiples For the double with the sesquialter makes the triple the triple with the sesquitercian the quadruple the quadruple with the sesquitercian the quintuple and as far as you wish to proceed no contrary result will appear

CHAPTER VI

[1] Up to this point then we have sufficiently discussed relative number by a process of selection measuring out what is easily comprehended and appropriate to the nature of the matters thus far introduced Whatever remains to be said on this topic will be filled in after we have put it aside and have first discussed certain subjects which involve a more serviceable inquiry having to do with the properties of absolute number not relative For mathematical speculations are always to be interlocked and to be explained one by means of another The subjects which we must first survey and observe are concerned with linear plane and solid numbers cubical and sphenical equilateral and scalene bricks beams wedges and the like the tradition concerning which to be sure since they are more closely related to magnitude is properly given in the *Geometrical Introduction* Yet the germs of these ideas are taken over into arithmetic, as the science which is the mother of geometry and more elementary than it For we recall that a short time ago we saw that arithmetic abolishes the other science

with itself but is not abolished by them, and conversely is of necessity implied by them but does not itself imply them

[2] First however, we must recognize that each letter by which we indicate a number, such as iota the sign for 10 kappa for 20 and omega for 800 designates that number by man's convention and agreement not by nature. On the other hand the natural unsartificial and therefore simplest indication of numbers would be the setting forth one beside the other of the units contained in each. For example the writing of one unit by means of one alpha will be the sign for 1 two units side by side that is, a series of two alphas will be the sign for 2 when three are put in a line it will be the character for 3 four in a line for 4 five for 5 and so on. For by means of such a notation and indication alone could the schematic arrangement of the plane and solid numbers mentioned be made clear and evident thus

| | |
|--------------|-----------|
| The number 1 | a |
| The number 2 | a a |
| The number 3 | a a a |
| The number 4 | a a a a |
| The number 5 | a a a a a |

and further in similar fashion

[3] Unity then occupying the place and character of a point will be the beginning of intervals and of numbers but not itself an interval or a number, just as the point is the beginning of a line or an interval but is not itself line or interval. Indeed when a point is added to a point it makes no increase for when a non-dimensional thing is added to another non-dimensional thing it will not thereby have dimension just as if one should examine the sum of nothing added to nothing which makes nothing. We saw a similar thing also in the case of equality among the relatives for a proportion is preserved—as the first is to the second so the second is to the third—but no interval is generated in the relation of the extremes to each other as there is in all the other relations with the exception of equality. In exactly the same way unity alone out of all number when it multiplies itself produces nothing greater than itself.

Unity, therefore, is non-dimensional and elementary and dimension first is found and seen in 2 then in 3 then in 4 and in succession in the following numbers for dimension is that

which is conceived of as between two limits

[4] The first dimension is called 'line' for 'line' is that which is extended in one direction. Two dimensions are called 'surface' for a surface is that which is extended in two directions. Three dimensions are called 'solid' for a solid is that which is extended in three directions and it is by no means possible to conceive of a solid which has more than three dimensions depth breadth and length. By these are defined the six directions which are said to exist in connection with every body and by which motions in space are distinguished forward backward up down right and left for of necessity two directions opposite to each other follow upon each dimension up and down upon one, forward and backward upon the second and right and left upon the third.

[5] The statement also as it happens can be made conversely thus. If a thing is solid it has by all means three dimensions length depth and breadth and conversely if it has the three dimensions it is always a solid, and nothing else.

[6] That which has but two dimensions therefore will not be a solid but a surface for the latter admits of but two dimensions. Here too it is possible similarly to reverse the statement directly stated a surface is that which has two dimensions and conversely that which has two dimensions is always a surface.

[7] The surface then, is exceeded by the solid by one dimension and the line is exceeded by the surface by one for the line is that which is extended in but one direction and has only one dimension and it falls short of the solid by two dimensions. The point falls short of the latter by one dimension and hence it has already been stated that it is non-dimensional since it falls short of the solid by three dimensions of the surface by two and of the line by one.

CHAPTER VII

[1] The point then is the beginning of dimension but not itself a dimension and likewise the beginning of a line but not itself a line the line is the beginning of surface but not surface and the beginning of the two-dimensional but not itself extended in two directions. [2] Naturally too surface is the beginning of body, but not itself body and likewise the beginning of the three-dimensional but not itself extended in three directions.

[3] Exactly the same in numbers unity is the beginning of all number that advances unit by unit in one direction linear number is the beginning of plane number which spreads out like a plane in one more dimension and plane number is the beginning of solid number which possesses a depth¹ in the third dimension besides the original ones To illustrate and classify, linear numbers are all those which begin with 1 and advance by the addition of 1 in one and the same dimension and plane numbers are those² that begin with 3 as their most elementary root and proceed through the next succeeding numbers They receive their names also in the same order for there are first the triangles then the squares the pentagons after these then the hexagons the heptagons and so on indefinitely and as we said they are named after the successive numbers beginning with 3

[4] The triangle therefore is found to be the most original and elementary form of the plane number Thus we can see from the fact that, among plane figures³ graphically represented if lines are drawn from the angles to the centers each rectilinear figure will by all means be resolved into as many triangles as it has sides but the triangle itself if treated like the rest will not change into anything else but itself Hence the triangle is elementary among these figures for everything else is resolved in to it but it into nothing else From it the others likewise would be constituted but it from no other It is therefore the element of the others and has itself no element [5] Likewise as the argument proceeds in the realm of numerical forms it will confirm this statement

CHAPTER VIII

[1] Now a triangular number is one which when it is analyzed into units shapes into triangular form the equilateral placement of its parts in a plane 3 6 10 15 21 28 and so on are examples of it for their regular formations expressed graphically will be at once triangular and equilateral As you advance you will find that such a numerical series as far as you like takes the triangular form if you put as the most elementary form the one that arises from unity so that unity may appear to be potentially a triangle and 1 the first actually

[2] Their sides will increase by the succes-

sive numbers for the side of the one potentially first is unity that of the one actually first that is 3 is 2 that of 6 which is actually second 3 that of the third 4 the fourth, 5 the fifth 6, and so on

[3] The triangular number is produced from the natural series of number set forth in a line and by the continued addition of successive terms one by one, from the beginning for by the successive combinations and additions of another term to the sum the triangular numbers in regular order are completed For example from this natural series 1 2 3 4 5 6 7 8 9 10 11 12 13, 14 15 I take the first term and have the triangular number which is potentially first, 1,



then adding the next term I get the triangle actually first for 2 plus 1 equals 3 In its graphic representation it is thus made up Two units side by side are set beneath one unit and the number three is made a triangle



Then when next after these the following number 3 is added simplified into units and joined to the former it gives 6 the second triangle in actuality and furthermore it graphically represents this number



Again the number that naturally follows 4 added in and set down below the former reduced to units gives the one in order next after the aforesaid 10 and takes a triangular form



5 after this then 6 then 7 and all the numbers in order are added so that regularly the sides of each triangle will consist of as many

¹Cf Plato *Timaeus* 53

²But cf Euclid in *Elements* VII Def 17

³Cf Plato *Timaeus* 53 ff

numbers as have been added from the natural series to produce it



Side 5



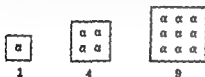
Side 6



Side 7

CHAPTER IX

[1] The square is the next number after this which shows us no longer 3 like the former but 4 angles in its graphic representation but none the less equilateral. Take for example 1 4 9 16 25 36 49 64 81, 100 for the representations of these numbers are equilateral, square figures as here shown, and it will be similar as far as you wish to go



1

4

9



16

25

[2] It is true of these numbers as it was also of the preceding that the advance in their sides progresses with the natural series. The side of the square potentially first 1 is 1 that of 4 the first in actuality 2 that of 9 actually the second 3 that of 16 the next actually the

third 4 that of the fourth 5 of the fifth, 6, and so on in general with all that follow

[3] This number also is produced if the natural series is extended in a line increasing by 1, and no longer the successive numbers are added to the numbers in order, as was shown before but rather all those in alternate places that is, the odd numbers. For the first 1 is potentially the first square the second 1 plus 3 is the first in actuality, the third 1 plus 3 plus 5 is the second in actuality the fourth 1 plus 3 plus 5 plus 7 is the third in actuality the next is produced by adding 9 to the former numbers the next by the addition of 11 and so on

[4] In these cases also it is a fact that the side of each consists of as many units as there are numbers taken into the sum to produce it

CHAPTER X

[1] The pentagonal number is one which likewise upon its resolution into units and depiction as a plane figure assumes the form of an equilateral pentagon 1, 5, 12, 22, 35, 51, 70 and analogous numbers are examples [2] Each side of the first actual pentagon 5 is 2 for 1 is the side of the pentagon potentially first 1 3 is the side of 12 the second of those listed 4 that of the next 22 5 that of the next in order 35 and 6 of the succeeding one, 51 and so on. In general the side contains as many units as are the numbers that have been added together to produce the pentagon chosen out of the natural arithmetical series set forth in a row. For in a like and similar manner there are added together to produce the pentagonal numbers the terms beginning with 1 to any extent whatever that are two places apart that is those that have a difference of 3

Unity is the first pentagonal potentially and is thus depicted



5 made up of 1 plus 4 is the second similarly represented



12 the third is made up out of the two former numbers with 7 added to them so that it may have 3 as a side as three numbers have been

added to make it. Similarly the preceding pentagon 5 was the combination of two numbers and had 2 as its side. The graphic representation of 12 is this:



The other pentagonal numbers will be produced by adding together one after another in due order the terms after 7 that have the difference 3, as for example 10 13 16 19 22 25 and so on. The pentagons will be 35 51 70 92, 117 and so forth.

CHAPTER XI

[1] The hexagonal, heptagonal and succeeding numbers will be set forth in their series by following the same process: if from the natural series of number there be set forth series with their differences increasing by 1. For as the triangular number was produced by admitting into the summation the terms that differ by 1 and do not pass over any in the series, as the square was made by adding the terms that differ by 2 and are one place apart, and the pentagon similarly by adding terms with a difference of 3 and two places apart (and we have demonstrated these by setting forth examples both of them and of the polygonal numbers made from them), so likewise the hexagons will have as their root numbers those which differ by 4 and are three places apart in the series which added together in succession will produce the hexagons. For example 1 5 9 13 17 21 and so on, so that the hexagonal numbers produced will be 1 15 28 45 66 and so on as far as one wishes to go.

[2] The heptagonals which follow these have as their root numbers terms differing by 5 and four places apart in the series like 1 6 11 16 21 26 31 36 and so on. The heptagons that thus arise are 1 7 18 34 55 81 112 148 and so forth.

[3] The octagonals increase after the same fashion with a difference of 6 in their root-

numbers and corresponding variation in their total constitution.

[4] In order that as you survey all cases you may have a rule generally applicable, note that the root-numbers of any polygonal differ by 2 less than the number of the angles shown by the name of the polygonal—that is by 1 in the triangle 2 in the square 3 in the pentagon 4 in the hexagon 5 in the heptagon and so on with similar increase.

CHAPTER XII

[1] Concerning the nature of plane polygonals this is sufficient for a first *Introduction*. That however the doctrine of these numbers is to the highest degree in accord with their geometrical representation and not out of harmony with it would be evident not only from the graphic representation in each case but also from the following. Every square figure diagonally divided is resolved into two triangles and every square number is resolved into two consecutive triangular numbers and hence is made up of two successive triangular numbers. For example 1 3 6 10 15 21 28 36 45 55 and so on are triangular numbers and 1 4 9 16 25 36 49 64 81 100 squares. [2] If you add any two consecutive triangles that you please you will always make a square, and hence whatever square you resolve you will be able to make two triangles of it.

Again any triangle joined to any square figure makes a pentagon; for example the triangle 1 joined with the square 4 makes the pentagon 5; the next triangle 3 of course with the next square makes the pentagon 12; the next 6 with the next square 16 gives the next pentagon 22; 10 and 25 give 35 and so on.

[3] Similarly if the triangles are added to the pentagons following the same order they will produce the hexagonals in due order and again the same triangles with the latter will make the heptagonals in order, the octagonals after the heptagonals and so on to infinity.

[4] To remind us let us set forth rows of the polygonals written in parallel lines as follows. The first row triangles, the next squares, after them pentagonals, then hexagonals, then heptagonals, then if one wishes the succeeding polygonals.

| | | | | | | | | | | |
|-------------|---|---|----|----|----|----|-----|-----|-----|-----|
| Triangles | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 | 55 |
| Squares | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 |
| Pentagonals | 1 | 5 | 12 | 22 | 35 | 51 | 70 | 92 | 117 | 145 |
| Hexagonals | 1 | 6 | 15 | 28 | 45 | 66 | 91 | 120 | 153 | 190 |
| Heptagonals | 1 | 7 | 18 | 34 | 55 | 81 | 112 | 148 | 189 | 235 |

You can also set forth the succeeding polygonals in similar parallel lines

[5] In general you will find that the squares are the sum of the triangles above those that occupy the same place in the series, plus the numbers of that same class in the next place back for example 4 equals 3 plus 1 9 equals 3 plus 3 16 equals 10 plus 3 25 equals 15 plus 10, 36 equals 21 plus 15 and so on

The pentagons are the sum of the squares above them in the same place in the series, plus the elementary triangles that are one place further back in the series for example 5 equals 4 plus 1 12 equals 9 plus 3 22 equals 16 plus 6, 35 equals 25 plus 10 and so on

[6] Again the hexagonals are similarly the sums of the pentagons above them in the same place in the series plus the triangles one place back for instance, 6 equals 5 plus 1 15 equals 12 plus 3, 28 equals 24 plus 4 45 equals 35 plus 10 and as far as you like

[7] The same applies to the heptagonals for 7 is the sum of 6 and 1 18 equals 15 plus 3 34 equals 28 plus 6 and so on Thus each polygonal number is the sum of the polygonal in the same place in the series with one less angle plus the triangle in the highest row one place back in the series

[8] Naturally then the triangle is the element of the polygon both in figures and in numbers and we say this because in the table reading either up and down or across the successive numbers in the rows are discovered to have as differences the triangles in regular order

CHAPTER XIII

[1] From this it is easy to see what the solid number is and how its series advances with equal sides for the number which in addition to the two dimensions contemplated in graphic representation in a plane length and breadth has a third dimension which some call depth others thickness and some height that number would be a solid number extended in three directions and having length depth and breadth

[2] This first makes its appearance in the so-called pyramids These are produced from rather wide bases narrowing to a sharp apex first after the triangular form from a triangular base second after the form of the square from a square base, and succeeding these after the pentagonal form from a pentagonal base then similarly from the hexagon heptagon octagon and so on indefinitely

[3] Exactly so among the geometrical solid

figures if one imagines three lines from the three angles of an equilateral triangle equal in length to the sides of the triangle converging in the dimension height to one and the same point a pyramid would be produced, bounded by four triangles equilateral and equal one to the other one the original triangle and the other three bounded by the aforesaid three line [4] And again if one conceives of four lines starting from a square equal in length to the sides of the square each to each and again converging in the dimension height to one and the same point a pyramid would be completed with a square base and diminishing in square form, bounded by four equilateral triangles and one square the original one [5] And starting from a pentagon hexagon heptagon and however far you care to go lines equal in number to the angles erected in the same fashion from the angles and converging to one and the same point will complete a pyramid named from its pentagonal hexagonal, or heptagonal base, or similarly

[6] So likewise among numbers each linear number increases from unity as from a point as for example 1, 2, 3, 4, 5 and successive numbers to infinity and from the same numbers which are linear and extended in one direction combined in no random manner the polygonal and plane numbers are fashioned—the triangles by the combination of root numbers immediately adjacent the square by adding every other term the pentagons every third term and so on [7] In exactly the same way, if the plane polygonal numbers are piled one upon the other and as it were built up the pyramids that are akin to each of them are produced the triangular pyramid from the triangles the square pyramid from the squares the pentagonal from the pentagons the hexagonal from the hexagons and so on throughout

[8] The pyramids with a triangular base then in their proper order are these 1, 4 10 20 35 56 84 and so on and their origin is the piling up of the triangular numbers one upon the other first 1, then 1 3 then 1 3 6 then 10 in addition to these and next 15 together with the foregoing then 21 besides these next 28 and so on to infinity

[9] It is clear that the greatest number is conceived of as being lowest for it is discovered to be the base the next succeeding one is on top of it and the next on top of that until unity appears at the apex and so to speak tapers off the completed pyramid into a point

CHAPTER XIV

[1] The next pyramids in order are those with a square base which rise in this shape to one and the same point. These are formed in the same way as the triangular pyramids of which we have just spoken. For if I extend in series the square numbers in order beginning with unity thus 1 4, 9 16 25 36 49 64 81 100 and again set the successive terms as in a pile one upon the other in the dimension height when I put 1 on top of 4 the first actual pyramid with square base 5 is produced for here again unity is potentially the first. [2] Once more I put this same pyramid entire composed of 8 units just as it is upon the square 9 and there is made up for me the pyramid 14 with square base and side 3—for the former pyramid had the side 2 and the one potentially first 1 as a side 1 or here too each side of any pyramid whatsoever must consist of as many units as there are polygonal numbers piled together to create it.

[3] Again I place the whole pyramid 14 with the square 9 as its base upon the square 16 and I have 30 the third actual pyramid of those that have a square base and by the same order and procedure from a pentagonal hexagonal or heptagonal base, and even going on farther we shall produce pyramids by piling upon one another the corresponding polygonal numbers starting with unity as the smallest and going on to infinity in each case.

[4] From this too it becomes evident that triangles are the most elementary for absolutely all of the pyramids that are exhibited and shown, with the various polygonal bases are bounded by triangles up to the apex.

[5] But lest we be heedless of truncated bi truncated and tri truncated pyramids the names of which we are sure to encounter in scientific writings you may know that if a pyramid with any sort of polygon as its base tri angle square pentagon or any of the succeeding polygons of the kind when it increases by this process of piling up does not taper off into unity it is called simply truncated when it is left without the natural apex that belongs to all pyramids for it does not terminate in the potential polygon unity as in some one point but in another polygon and an actual one and unity is not its apex but its upper boundary becomes a plane figure with the same number of angles as the base. If however in addition to the failure to terminate in unity it does not even terminate in the polygon next to unity and

the first in actuality, such a pyramid is called bi truncated and if, still further it does not have the second actual polygon at its upper limit but only the one next beneath it will be called tri truncated yes even four times truncated if it does not have the next one as its limit, or five times truncated at the next step and so on as far as you care to carry the nomenclature.

CHAPTER XV

[1] While the origin advance increase and nature of the equilateral solid numbers of pyramidal appearance is the foregoing, with its seed and root in the polygonal numbers and the piling up of them in their regular order there is another series of solid numbers of a different kind consisting of the so-called cubes, beams, bricks, wedges, spheres and parallelepipeds which has the order of its progress somewhat as follows.

[2] The foregoing squares 1 4 9 16 25 36 49 64 and so on which are extended in two directions and in their graphic representation in a plane have only length and breadth will take on yet a third dimension and be solids and extended in three directions if each is multiplied by its own side 4 which is 2 times 2 is again multiplied by 2 to make 8 9 which is 3 times 3 is again increased by 3 in another dimension and gives 27 16 which is 4 times 4 is multiplied by its own side 4 and 64 results and so on with the succeeding squares throughout.

[3] Here too the sides will be composed of as many units as were in the sides of the squares from which they arose in each case the sides of 8 will be 2 like those of 4 those of 27 3, like those of 9 those of 64 4 like those of 16 and so on so that likewise the side of unity the potential cube will be 1 which is the side of the potential square 1.

In general each square is a single plane and has four angles and four sides while each several cube having increased out of some one square multiplied by its own side will have all ways six plane surfaces each equal to the original square and twelve edges each equal to and containing exactly the same number of units as each side of the original square and eight solid angles each of which is bounded by three edges like in each case to the sides of the original square.

CHAPTER XVI

[1] Now since the cube is a solid figure with equal sides in all dimensions in length depth,

and breadth and is equally extended in all the six so called directions¹ it follows that there is opposed to it that which has its dimensions in no case equal to one another but its depth unequal to its breadth and its length unequal to either of these for example 2 times 3 times 4 or 2 times 4 times 8, or 3 times 5 times 12 or a figure which follows some other scheme of inequality

[2] Such solid figures in which the dimensions are everywhere unequal one to another, are called *scalene* in general. Some however, using other names call them 'wedges' for carpenters house builders and blacksmiths wedges and those used in other crafts having unequal sides in every direction are fashioned so as to penetrate they begin with a sharp end and continually broaden out unequally in all the dimensions. Some also call them *sphekislos* wasps because wasps bodies also are very like them compressed in the middle and showing the resemblance mentioned. From this also the *sphekoma* point of the helmet must derive its name for where it is compressed it imitates the waist of the wasp. Others call the same numbers altars using their own metaphor for the altars of ancient style particularly the Ionic do not have the breadth equal to the depth nor either of these equal to the length nor the base equal to the top but are of varied dimensions everywhere

[3] Now whereas the two kinds of numbers cube and scalene are extremes the one equally extended in every dimension the other unequally the so-called parallelepipedons are solid numbers like means between them. The plane surfaces of these are heteromecic numbers² just as in the case of the cubes the faces were squares as has been shown

CHAPTER XVII

[1] Again then to take a fresh start a number is called heteromecic if its representation when graphically described in a plane is quadrilateral and quadrangular to be sure but the sides are not equal one to another nor is the length equal to the breadth but they differ by 1. Examples are 2 6 12 20 30 42 and so on for if one represents them graphically he will always construct them thus 1 times 2 equals 2 2 times 3 equals 6 3 times 4 equals 12 and the succeeding ones similarly 4 times 5 5 times 6, 6 times 7, 7 times 8 and thus indefinitely

provided only that one side is greater than the other by 1 and by no other number. If however the sides differ otherwise than by 1 for instance by 2 3 4 or succeeding numbers as in 2 times 4 3 times 6 4 times 8 or however else they may differ then no longer will such a number be properly called a heteromecic but an oblong number. For the ancients of the school of Pythagoras and his successors saw the other³ and otherness primarily in 2 and 'the same and sameness in 1 as the two beginnings of all things and these two are found to differ from each other only by 1. Thus the other⁴ is fundamentally other by 1, and by no other number, and for this reason customarily other is used among those who speak correctly of two things and not of more than two

[2] Moreover, it was shown that all odd number is given its specific form⁵ by unity and all even number by 2. Hence we shall naturally say that the odd partakes of the nature of the same and the even of that of the other for indeed there are produced by the successive additions of each of these—naturally and not by our decree—by the addition of the odd numbers from 1 to infinity the class of the squares and by the addition of the evens from 2 to infinity that of the heteromecic numbers⁶

[3] There is accordingly every reason to think that the square once more shares in the nature of the same for its sides display the same ratio alike unchanging and firmly fixed in equality to themselves while the heteromecic number partakes of the nature of the other for just as 1 is differentiated from 2 differing by 1 alone thus also the sides of every heteromecic number differ from one another, one differing from the other by 1 alone

To illustrate if I have set out before me the successive numbers in series beginning with 1 and select and arrange by themselves the odd numbers in the line and the even by themselves in another there are obtained these two series

1 3 5 7 9 11 13 15 17 19 21 23 25 27
2 4 6 8 10 12 14 16 18 20 22 24 26 28

[4] Now then the beginning of the odd series is unity which is of the same class as the series and possesses the nature of the same and so whether it multiplies itself in two dimensions

¹Cf Plato *Timaeus* 35 ff

²Cf I 7 2

³Cf II 18 2 and 20 3

¹Cf II 6 4

²See the following chapter

or in three it is not made different nor yet does it make any other number depart from what it was originally¹ but keeps it just as it was. Such a property it is impossible to find in any other number. [6] Of the other series the beginning is 2 which is similar in kind to this series and imitates otherness for whether it multiplies itself or another number it causes a change² for example 2 times 2 2 times 3

[6] But in cases like 8 times 8 times 8 or 8 times 8 times 3 such solid forms are called 'bricks,' the product of a number by itself and then by a smaller number if however a greater height is joined to the square as in 3 times 3 times 7, 8 times 3 times 8 or 3 times 3 times 9 or however many times the square be taken provided only it be a greater number of times than the square it self then the number is a beam the product of a number by itself and then by a larger number. The wedges to be sure were the products of three unequal numbers and cubes of three equal ones.

[7] Among the cubes some of them in addition to being the product of three equal numbers have the further property of ending at every multiplication in the same number as that from which they began these are called spherical and also recurrent. Such indeed are those with sides 5 or 6 for however many times I increase each one of these it will by all means end each time in the same figure the derivative of 6 in 6 and that of 5 in 5. For example the product of 5 times 5 will end in 5 and so will 5 times this product and if necessary 5 times this again and to infinity no other concluding term will be found except 5. From 6 too in the same fashion 8 and no other will be the concluding term and so 1 likewise is potentially spherical and recurrent for as is reasonable it has the same property as the spheres and circles. For each one of them circling and turning around ends where it begins. And so these numbers afore aid are the only ones of the products of equal factors to return to the same starting point from which they began in the course of all their increases. If they increase in the manner of planes in two dimensions they are called circular like 1 25 and 36 derived from 1 times 1 5 times 5 and 6 times 6 but if they have three dimensions or are multiplied still further than this they are called spherical solid numbers for example 1 125 216 or again 1 625 1 296

CHAPTER XVIII

[1] Regarding the solid numbers this is for the present sufficient. The physical philosophers however and those that take their start with mathematics call the same and the other 'the principles of the universe and it has been shown that "the same inheres in unity and the odd numbers to which unity gives specific form and to an even greater degree in the squares made by the continued addition of odd numbers because in their sides they share in equality while "the other" inheres in 2 and the whole even series which is given specific form by 2 and particularly in the heteromeic numbers which are made by the continued addition of the even numbers because of the share of the original inequality³ and otherness which they have in the difference between their sides. Therefore it is most necessary further to demonstrate how in these two as in origins and seeds there are potentially existent all the peculiar properties of number of its forms and subdivisions of all its relations of polygonals and the like.

[2] First however we must make the distinction whereby the oblong (promecic) number differs from the heteromeic. The heteromeic is as was stated above 'the product of a number multiplied by another larger than the first by 1 for example 6 which is 2 times 3 or 12 which is 3 times 4. But the oblong is similarly the product of two differing numbers differing however not by 1 but by some larger number as 2 times 4 3 times 6 4 times 8 and similar numbers which in a way exceed in length and overstep the difference of 1.

[3] Therefore since squares are produced from the multiplication of numbers by their own length and have their length the same as their breadth properly speaking they would be called idiomeic or tautomeic for example 4 times 4 3 times 3 4 times 4 and the rest. And if this is true they will admit in every way of sameness and equality and for this reason are limited and come to an end for the equal and the same are so in one definite way. But since the heteromeic numbers are produced by the multiplication of a number by not its own but another number's length they are therefore called heteromeic and admit of infinity and boundlessness.

[4] In this way then all numbers and the

¹Cf II 6 3

²Cf Aristotle, *On the Soul* 406^b 13

³Cf II 17 2

⁴Cf II 17 1

objects in the universe which have been created with reference to them are divided and classified and are seen to be opposite one to another and well do the ancients at the very beginning of their account of Nature make the first subdivision in their cosmogony on this principle. Thus Plato¹ mentions the distinction between the natures of the same and "the other" and again that between the essence which is indivisible and always the same and the one which is divided and Philolaus says that existent things must all be either limitless or limited or limited and limitless at the same time, by which it is generally agreed that he means that the universe is made up out of limited and limitless things at the same time obviously after the image of number for all number is composed of unity and the dyad even and odd and these in truth display equality and inequality sameness and otherness the bounded and the boundless the defined and the undefined

CHAPTER VII

[1] That we may be clearly persuaded of what is being said namely that things are made up of warring and opposite elements² and have in all likelihood taken on harmony—and harmony always arises from opposites for harmony is the unification of the diverse and the reconciliation of the contrary minded—let us set forth in two parallel lines no longer as just previously the even numbers from 2 by themselves and the odd numbers from 1 but the numbers that are produced from these by adding them successively together the squares from the odd numbers and the heteromeic from the even. For if we give careful attention to their setting forth we shall admire their mutual friendship and their cooperation to produce and perfect the remaining forms to the end that we may with probability conceive that also in the nature of the universe from some such source as this a similar thing was brought about by universal providence

[2] Let the two series then be as follows. That of the squares from unity 1 4 9 16 25 36 49 64 81 100 121 144 169 196 225 and that of the heteromeic numbers beginning with 2 and proceeding thus 2 6 12 20 30 42 50 72 90 110 132 156 182 210 240

[3] In the first place then the first square is the fundamental multiple of the first hetero-

meic number the second compared to the second is its sesquialter the third, sesquitercian of the third the fourth sesquiquartan of the fourth then sesquiquintan sesquisextan and so on similarly *ad infinitum*. Their differences too will increase according to the successive numbers from 1 the difference of the first terms is 1 of the second 2 of the third 3, and so on. Next if first the second term of the squares be compared with the first heteromeic number, the third with the second the fourth with the third and the rest similarly they will keep unchanged the same ratios as before but their differences will begin to progress no longer from 1 but from 2 remaining the same as before and according to the advance observed in the former comparison the first to the first will be the first or root form multiple the second to the second the second sesquialter from the root-form the third to the third the third sesquitercian from the root-form and the succeeding terms will go on in similar fashion

[4] Furthermore the squares among themselves will have only the odd numbers as differences the heteromeic even numbers. And if we put the first heteromeic number as a mean term between the first two squares the second between the next two the third between the two following and the fourth between the two next succeeding therein will be seen still more regularly the numerical relations in groups of three terms. For as 4 is to 2 so is 2 to 1 and as 9 is sesquialter to 6 so is 6 to 4 and as 16 to 12 so is 12 to 9 and so on with both numbers and ratios regularly advancing. As the greater is to the mean so will the mean be to the lesser and not in the same ratio but always a different one by an increase. In all the groupings too the product of the extremes is equal to the square of the mean and the extremes plus twice the mean by exchange will always give a square. What is neatest of all from the addition of both there comes about the production of the triangles in due order showing that the nature of the series is more ancient³ than the origin of all things thus 1 plus 2 2 plus 4 4 plus 6 6 plus 9 9 plus 12 12 plus 16 16 plus 20 and by this process the triangles which give rise to the polygons come forth in order

CHAPTER XX

[1] Still further every square plus its own side becomes heteromeic or by Zeus if its side is subtracted from it. Thus the other⁴ is con-

¹Cf Plato *Timaeus* 30

²Plato *Timaeus* 30

³Cf II 17 3 IR 1 and II 7 4 cf 12 8

ceived of as being both greater and smaller than 'the same' since it is produced both by addition and by subtraction in the same way that the two kinds of inequality¹ also the greater and the less have their origin from the application of addition or subtraction to equality [2] This also is sufficient evidence that the two forms partake of sameness and otherness of otherness in an indefinite fashion but of sameness definitely 1 and 2 generically but the odd of sameness after the manner of a subordinate species because it belongs to the same class as 1, and the even of otherness because it is homogeneous with 2

[3] There is also a still clearer reason why the square since it is the product of the addition of odd numbers is akin to sameness and the heteromeic numbers to otherness because it is made up by adding even numbers for as though they were friends of one another these two forms share in their two rows the same differences when they do not have the same ratios and conversely the same ratios when they do not have the same differences For the difference between 4 and 2 in the double ratio is found between 6 and 4 as a superparticular and again the difference between 9 and 6 as a sesquialter is found between 12 and 9 as a sesquitercian and so on What is the same in quality is different in quantity and just the opposite what is the same in quantity is different in quality [4] Again it is clear that in all their relations the same difference between two terms will necessarily be called fractions with names that differ by 1 and be the half of one and the third of the other or the third of one and the quarter of the other or the fourth of one and the fifth of the other and so on

[5] But what will most of all confirm the fact that the odd and never the even is pre-eminently the cause of sameness is to be demonstrated in every series beginning with 1 following some ratio for example the double ratio 1 2 4 ■ 16 32 64 128 256 or the triple 1 3 9 27 81 243 729 2187 and as far as you like You will find that of necessity all the terms in the odd places in the series are squares and no others by any device whatsoever and that no square is to be found in an even place

But all the products of a number multiplied twice into itself that is the cubes which are extended in three dimensions and seen to share in sameness to an even greater extent are the product of the odd numbers not the even 1

8 27 64 125 and 216 and those that go on analogously, in a simple unvaried progression as well For when the successive odd numbers are set forth indefinitely beginning with 1 observe this The first one makes the potential cube the next two added together the second the next three the third the four next following the fourth the succeeding five the fifth the next six the sixth and so on

CHAPTER XXI

[1] After this it would be the proper time to incorporate the nature of proportions a thing most essential for speculation about the nature of the universe and for the propositions of music astronomy, and geometry and not least for the study of the works of the ancients and thus to bring the *Introduction to Arithmetic* to the end that is at once suitable and fitting

[2] A proportion then is in the proper sense the combination of two or more ratios but by the more general definition the combination of two or more relations even if they are not brought under the same ratio but rather a difference or something else

[3] Now a ratio² is the relation of two terms to one another and the combination of such is a proportion so that three is the smallest number of terms of which the latter is composed although it can be a series of more subject to the same ratio or the same difference For example 1 2 is one ratio where there are two terms but 2 4 is another similar ratio hence 1 2 4 is a proportion for it is a combination of ratios or of three terms which are observed to be in the same ratio to one another [4] The same thing may be observed also in greater numbers and longer series of terms for let a fourth term 8 be joined to the former after 4 again in a similar relation the double and then 16 after 8 and so on

[5] Now if the same term one and unchanging is compared to those on either side of it to the greater as consequent and to the lesser as antecedent such a proportion is called continued for example 1 2 4 is a continued proportion as regards quality³ for 4 is equal to 2 1 and conversely 1 2 equals 4 In quantity 1 2 3 for example is a continued proportion for as 3 exceeds 2 so 2 exceeds 1 and conversely as 1 is less than 2 by so much 4 is less than 3

[6] If however one term answers to the lesser term and becomes its antecedent and a

¹Cf I 17 6

²Cf Euclid *Elements* I viii

³Cf II 22 ■ 23 4 below

greater term and another not the same takes the place of consequent and lesser term with reference to the greater such a mean and such a proportion is called *no longer continued* but *disjunct* for example as regards quality, 1 2 4 8 for 2 1 equals 8 4 and conversely 1 2 equals 4 8 and again 1 4 equals 2 8 or 4 1 equals 8 2 and in quantity, 1 2 3 4 for as 1 is exceeded by 2 by so much 3 is exceeded by 4 or as 4 exceeds 3 so 2 exceeds 1 and by interchange as 3 exceeds 1 so 4 exceeds 2 or as 1 is exceeded by 3 by so much 2 is exceeded by 4

CHAPTER XXII

[1] The first three proportions then which are acknowledged by all the ancients Pythagoras Plato and Aristotle are the arithmetic geometric and harmonic and there are three others subcontrary to them which do not have names of their own but are called in more general terms the fourth fifth and sixth forms of mean after which the moderns discover four others as well making up the number ten which according to the Pythagorean view is the most perfect possible It was in accordance with this number indeed that not long ago the ten relations were observed to take their proper number the so-called ten categories the divisions and forms of the extremities of our hands and feet and countless other things which we shall notice in the proper place

[2] Now however we must treat from the beginning first that form of proportion which by quantity reconciles and binds together the comparison of the terms which is a quantitative equality as regards the difference of the several terms to one another This would be the arithmetic proportion for it was previously reported that quantity is its peculiar belonging [3] What then is the reason that we shall treat of this first and not another? Is it not clear that Nature shows it forth before the rest? For in the natural series of simple numbers beginning with 1 with no term passed over or omitted the definition of this proportion alone is preserved moreover in our previous statement we demonstrated that the *Arithmetical Introduction* itself is antecedent to all the others because it abolishes them together with itself but is not abolished together with them and because it is implied by them but does not imply them Thus it is natural that the mean which shares the name of arithmetic will not

unreasonably take precedence of the means which are named for the other sciences the geometric and harmonic for it is plain that all the more will it take precedence over the subcontraries over which the first three hold the leadership [4] As the first and original therefore since it is most deserving of the honor let the arithmetic proportion have its discussion at our hands before the others

CHAPTER XXIII

[1] It is an arithmetic proportion then, whenever three or more terms are set forth in succession or are so conceived and the same quantitative difference is found to exist between the successive numbers but not the same ratio among the terms, one to another For example 1 2 3 4, 5 6 7 8 9 10 11, 12 13 for in this natural series of numbers examined consecutively and without any omissions every term whatsoever is discovered to be placed between two and to preserve the arithmetic proportion to them For its differences as compared with those ranged on either side of it are equal the same ratio however is not preserved among them

[2] And we understand that in such a series there comes about both a continued and a disjunct proportion for if the same middle term answers to those on either side as both antecedent and consequent it would be a continued proportion but if there is another mean along with it a disjunct proportion comes about

[3] Now if we separate out of this series any three consecutive terms whatsoever after the form of the continued proportion or four or more terms after the disjunct form and consider them the difference of them all would be 1 but their ratios would be different throughout If however again we select three or more terms not adjacent but separated separated nevertheless by a constant interval if one term was omitted in setting down each term the difference in every case will be 2 and once more with three terms it will be a continued proportion with more disjunct If two terms are omitted the difference will always be 3 in all of them continued or disjunct if three 4 if four 5 and so on

[4] Such a proportion therefore partakes in equal quantity in its differences but of unequal quality for this reason it is arithmetic If on the contrary it partook of similar quality but not quantity it would be geometric instead of arithmetic

[5] A thing is peculiar to this proportion

that does not belong to any other namely, the mean is either half of or equal to the sum of the extremes whether the proportion be viewed as continuous or disjunct or by alternation for either the mean term with itself or the mean terms with one another are equal to the sum of the extremes

[6] It has still another peculiarity, what ratio each term has to itself, thus the differences have to the differences that is they are equal

Again the thing which is most exact and which has escaped the notice of the majority the product of the extremes when compared to the square of the mean is found to be smaller than it by the product of the differences whether they be 1 2, 3 4 or any number whatever

In the fourth place a thing which all previous writers also have noted the ratios between the smaller terms are larger as compared to those between the greater terms It will be shown that in the harmonic proportion on the contrary the ratios between the greater terms are greater than those between the smaller for this reason the harmonic proportion is subcontrary to the arithmetic and the geometric is midway between them as it were between extremes for this proportion has the ratios between the greater terms and those between the smaller equal and we have seen that the equal is in the middle ground between the greater and the less So much then about the arithmetic proportion

CHAPTER XXIV

[1] The next proportion¹ after this one the geometric is the only one in the strict sense of the word to be called a proportion because its terms are seen to be in the same ratio It exists whenever of three or more terms as the greatest is to the next greatest so the latter is to the one following and if there are more terms as this again is to the one following it but they do not however differ from one another by the same quantity but rather by the same quality of ratio the opposite of what was seen to be the case with the arithmetic proportion

[2] For an example set forth the numbers beginning with 1 that advance by the double ratio 1 2 4 8 16 32 64 and so on or by the triple ratio 1 3 9 27 81 243 and so on or by the quadruple or in some similar way In each one of these series three adjacent terms or four or any number whatever that may be taken

will give the geometric proportion to one another as the first is to the next smaller so is that to the next smaller and again that to the next smaller and so on as far as you care to go and also by alternation For instance, 2 4 8 the ratio which 8 bears to 4 that 4 bears to 2, and conversely they do not, however have the same quantitative difference Again 2, 4 8, 16 for not only does 16 have the same ratio to 8 as before though not the same difference but also by alternation it preserves a similar relation—as 16 is to 4 so 8 is to 2 and conversely as 2 is to 8 so 4 is to 16 and disjunctly as 2 is to 4 so 8 is to 16 and conversely and in disjunct form as 16 is to 8 so 4 is to 2 for it has the double ratio

[3] The geometric proportion has a peculiar property shared by none of the rest that the differences of the terms are in the same ratio to each other as the terms to those adjacent to them the greater to the less and vice versa Still another property is that the greater terms have as a difference with respect to the lesser the lesser terms themselves and similarly difference differs from difference by the smaller difference itself if the terms are set forth in the double ratio in the triple ratio both terms and differences will have as a difference twice the next smaller in the quadruple ratio thrice in the quintuple four times and so on

[4] Geometric proportions come about not only among the multiples but also among all the superparticular, superpartient and mixed forms and the peculiar property of this proportion in all cases is preserved that in the continued proportions the product of the extremes is equal to the square of the mean but in disjunct proportions or those with a greater number of terms even if they are not continued but with an even number of terms that the product of the extremes equals that of the means

[5] As an illustration of the fact that in all the relations all kinds of multiple superparticulars superpartients and mixed ratios the peculiar property of this proportion is preserved let that suffice and be sufficient for us wherein we fashioned beginning with equality by the three rules all the kinds of inequality out of one another when they were in both direct and reverse order for each act of fashioning and each series set forth is a geometric proportion with all the aforesaid properties as well as a fourth namely that they keep the

¹Cf Euclid Elements VII Def 21

²Cf I 23 7 ff

sponse for diagnostic purposes (1175). Dehydroisoandrosterone excretion may aid in the differential diagnosis of adrenal hyperfunction (370). Similarly, increased urinary pregnanediol and pregnanetriol, in cases of adrenal hyperplasia, has been reduced by cortisone therapy (100) and urinary estrogens may also be reduced (371-771). The therapeutic use of cortisone to depress adrenal secretion in nonmalignant cases has been encouraging.

Adrenal tumors in the mouse have been discussed in a review by Woolley (1237), and the adrenal-penital relationship reviewed by Zuckerman (1262).

B. OVARIAN DISTURBANCES

The use of cortisone in various ovarian disturbances and its use in cases of sterility have been described (598). ACTH caused a two- to eight fold increase in urinary output of gonadotropins in eight of nine patients treated (1042). From the studies on adrenalectomized patients it appears that the adrenals normally produce estrogens (203). In rats, cortisone caused a slight augmentation of the increase in ovarian weight after administration of chorionic gonadotrophin (330). Cortisone alone did not exert any sex hormone or gonadotrophic activity (787). Szego found that the acute stimulation of the uterus by estrogens could be antagonized by cortisone or hydrocortisone but not by DCA. Such action was independent of pituitary function (1128). Others observed a similar action using ovariectomized rats (1132). In castrated male guinea pigs cortisone in small doses exerted a stimulating estrogen like action on nipple growth. Larger doses however inhibited nipple growth (1202). The correlation between ovarian and adrenal cyclic function in guinea pigs has been suggested as related to the estrogen stimulation of the anterior pituitary to release ACTH (1259). In the dog cortisone did not exert androgenic effects nor did it antagonize the action of testosterone propionate on the prostate (1132). A recent detailed review containing extensive experimental data on the influence of the adrenals on the reproductive system has been published by Moore (787).

C. MAINTENANCE OF LACTATION

The controversy over the effectiveness of DCA or cortisone and 11 dehydrocortisone in maintaining lactation in adrenalectomized rats has been reinvestigated by Cowie. Whereas no agent alone was completely effective it was found that in the rat cortisone pellets (2 of 11 mg.) and DCA (1 of 50 mg.) allowed complete restoration of lactation (184). The relationship between the adrenal cortex and the mammary gland has been reviewed by Folley (333).

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sponse for diagnostic purposes (1175). Dehydroisoandrosterone excretion may aid in the differential diagnosis of adrenal hyperfunction (370). Similarly, increased urinary pregnanediol and pregnanetriol in cases of adrenal hyperplasia has been reduced by cortisone therapy (100) and urinary estrogens may also be reduced (371, 771). The therapeutic use of cortisone to depress adrenal secretion in nonmalignant cases has been encouraging.

Adrenal tumors in the mouse have been discussed in a review by Woolley (1237), and the adrenal genital relationship reviewed by Zuckerman (1262).

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CHAPTER XV

Clinical Endocrinology

BY H. E. PASCHKIS AND A. E. RAKOFF

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I Introduction

The task of reviewing, within the framework of this book, recent progress in clinical endocrinology is not easy. In view of the extensive literature we had to impose upon ourselves certain limitations. It would hardly appear likely that the student or the practicing physician would turn for detailed practical guidance, to a book dealing quite predominantly with the chemistry and physiology of the hormones. Rather the biochemist and physiologist would seek here information on the clinical applications of the scientific achievements in his field. In fact, the line between physiological and clinical investigations is being reduced almost to the vanishing point. We are therefore presenting advances in principles and approaches rather than details of procedures or of therapeutic results.

II Anterior Pituitary Gland

A GENERAL REMARKS

In the not so distant past the anterior pituitary was supposed to secrete some twenty different hormones. For every and any effect of crude pituitary extracts a separate hormone entity was postulated and a name coined. Chemical isolation procedures have yielded six hormones and a better knowledge of metabolic interrelations has led to an understanding of multiple effects of individual single hormones.

The anterior pituitary then produces six hormones: growth hormone (also called somatotrophic hormone), thyrotrophic hormone (TSH), adrenocorticotrophic hormone (ACTH) and the three gonadotrophic hormones: follicle stimulating hormone (FSH), luteinizing or interstitial cell stimulating hormone (LH, IC₅H) and luteotrophic hormone (LTH), identical with lactogenic hormone.

At present the trend appears to reverse itself. Observations have been recorded suggesting the existence of two adrenocorticotrophic hormones (one responsible for adrenocortical growth, another for secretory function) (8, 125) as well as of two thyrotrophic hormones (73). Regarding growth hormone, a separation of the diabetogenic factor from the protein anabolic growth hormone has been reported (169). These problems are under current investigation and cannot as yet be evaluated from the standpoint of the physiologist or clinician. Even if the fractionations and chemical separations mentioned above should be borne out by further work, the question will have to be answered whether the pituitary gland at any time secretes these factors separately or whether it secretes a complex hormone with different actions residing in different parts of the molecule, parts which could be separated *in vitro*.

Regarding ACTH, various preparations have been made and have been

come available for clinical use. The action of these preparations is qualitatively identical. A highly purified preparation designated ACTX (Corticotropin A) is much less subject to destruction than is ACTH (the crude preparation) when administered intramuscularly. Actide corticotropin (Corticotropin B) is a preparation obtained by peptic hydrolysis of ACTH. It evidently represents the active peptides of the larger ACTH molecule. It is readily inactivated or destroyed in the extravascular tissues and at present appears to offer little advantage for clinical use (239).

As far as the six hormone entities are concerned it appears that their secretion is to a large extent independent and that the gland does not function as a unit. There is no panhyperpituitarism, hyperfunction of the entire anterior pituitary, nor is there a concerted hypofunction, panhypopituitarism, except that due to destruction of the glandular tissue. As a matter of fact, interrelations of secretion of individual pituitary hormones appear to be the result of retrograde action upon the pituitary from secretions of the secondary glands. Examples of this type of interrelation may be the dependence of growth hormone secretion upon thyroid hormone secretion and the release of LH through changes in estrogen levels.

The independence of secretion of individual pituitary hormones has become more intelligible through the discovery of individual cell types responsible for elaboration of the individual hormones. Traditionally the hormone secreting cells of the anterior pituitary are classified on the basis of the staining properties of their granules and of the morphologic characteristics of the Golgi apparatus as acidophiles (eosinophiles) and basophiles. The elaboration of FSH, TSH, and ACTH is tentatively ascribed to the basophiles and that of the other three hormones to the eosinophiles. It has, however, become evident that basophilic staining is the common property of specialized cells, differentiated by histochemical reactions. A basophilic FSH producing cell is different from the basophilic TSH producing cell (78, 79, 165, 166). No similar recent differentiations have been reported within the eosinophilic cells, but an older observation of carminophilic cells in the cat pituitary believed responsible for LH production points in the same direction (60).

B ACROMEGALY AND GIGANTISM

The clinical picture of gigantism and acromegaly was outlined in *The Hormones*, Volume II. More information has accumulated regarding the natural history of acromegaly. The excessive secretion of growth hormone by the eosinophilic cells occurs periodically. Periods of excessive secretion are followed by periods of relative quiescence. Such cycles may be repeated several times in the course of years. It is of more than academic interest to recognize the active phase of the disease earlier than in retro

spect by increased growth phenomena such as enlargement of feet or separation of teeth. Bio assay of growth hormone in the blood has been reported in one case (115) but assay methods which would lend themselves readily to clinical work have yet to be developed. Increased levels of serum inorganic phosphate have been found in the active phase of acromegaly, the levels return to normal under treatment with sex steroids as well as with radiation therapy (115).

Diabetes has long been known to be a frequent complication of acromegaly. In many respects it appears to be the human equivalent to the growth hormone-induced diabetes of dogs. Even the reversibility of early stages of growth hormone-induced diabetes in dogs has been paralleled in diabetes of acromegaly. In an acromegalic, observed and studied repeatedly for several years, a diabetic glucose tolerance made its appearance and reverted to normal following intensive roentgen irradiation of the pituitary (132). It has been pointed out that diabetes in acromegaly is more frequent in acromegalics in whose families non acromegalic members have diabetes. This would tentatively suggest that an inherited 'vulnerability' of the islets ('doglike' islets) is a prerequisite for the growth hormone to exert its diabetogenic effect. The bearer of 'nonvulnerable' islets would respond much like those species (e.g., the rat) in whom diabetes cannot readily be induced by growth hormone.

It has long been known that goiter is a frequent occurrence in acromegalics. Also in many cases a high BMR is recorded (39). In view of the well known stimulation of the thyroid by anterior pituitary TSH, the obvious assumption was that in this type of hyperpituitarism an increased secretion of TSH was associated with the increased secretion of growth hormone and that the former caused goiter and increased function of the thyroid. The situation is undoubtedly more complicated and the whole story has yet to be told. Cushing and his collaborators pointed out early that the histologic appearance of goiters of acromegalics differed markedly from that in thyrotoxicosis; also thyroidectomy caused regression of the elevated BMR and of other manifestations only in exceptional cases of acromegaly (39). More recently it has been found that the I 131 uptake of the thyroid gland of acromegalics was normal regardless as to whether there was a goiter or whether the BMR was elevated or normal (136). These observations have suggested, as did the older clinical observations of Cushing and associates that the high BMR of many acromegalics may be due to factors other than increased thyroid function (136). TSH levels in the serum of acromegalics have been found elevated regardless of the presence or absence of goiter and of the levels of the BMR (42). Neither the significance of this finding in the pathophysiology of acromegaly nor

the cause of excessive secretion of TSH (believed to be secreted by the basophilic anterior pituitary cells) is at present understood.

Treatment In the opinion of most clinicians indication for surgical treatment is not the demonstrable evidence for presence of an adenoma but rather a rapidly progressive encroachment of the optic nerves. Other cases should be subjected to roentgen ray treatment; its effectiveness has been greatly improved with the use of higher doses of radiation than were previously employed (112). If there is no evidence of increased hormonal activity nor evidence of expansive growth of an adenoma treatment should be withheld. Treatment with androgens or with androgens and estrogens has been tried in an attempt to suppress growth hormone secretion through the suppressive action of the steroid hormones upon the anterior pituitary. With this form of therapy return of the elevated serum inorganic phosphorus levels to normal as well as improvement of the diminished carbohydrate tolerance has been observed but it is as yet too early to judge the practical usefulness of steroid hormone therapy in acromegaly (115).

C. HYPOPHYSECTOMY

Destruction of the anterior pituitary resulting from tumor (chromophobe adenoma, parasellar tumors *e.g.* craniopharyngioma) or post partum necrosis causes panhypopituitarism. Depending on the amount of pituitary tissue destroyed the severity of the disorder may vary from complete loss of pituitary function to milder forms. It is interesting that in the incomplete forms not all pituitary functions are diminished or abolished in the same proportions (151). It is our impression that the gonadotropin secretion is most sensitive and is usually lost first. The various combinations of functional deficiencies seen in tumor cases in which the nontumorous part of the gland is exposed to compression from the tumor cannot be explained readily by topographic arrangement of the cells responsible for the respective hormone secretion but perhaps rather by differences of vulnerability of these cells to the pressure exerted on all.

The essential findings are described in Volume II. It should be added that sodium loss evidently due to adrenocortical salt hormone deficiency is found in severe cases (151); this is of interest because it has been stated that in rodents salt hormone function of adrenal cortex is not governed by the anterior pituitary. The lack of response of the adrenal cortex to ACTH mentioned in Volume II pertains only to single intramuscular injections. With repeated injections and especially with intravenous drip infusion the atrophic adrenal can be stimulated and evidence of function be obtained (rise of 17 ketosteroid and 11 oxysteroid excretion eosinopenia). We have been able to stimulate the adrenal cortex 18 years after

post partum neurosis (152). The therapeutic regime as outlined in Volume II now includes cortisone and hydrocortisone.

1 *Single Hormone Hypopituitarism*

In contradistinction to panhypopituitarism due to destruction of the anterior pituitary the secretion of individual single hormones may be deficient, in the presence of normal secretion of the others.

Hypogonadotropism occurs both in males and females and causes clinical pictures indistinguishable from that due to primary gonadal failure. The distinction is made chiefly by assay of urinary levels of gonadotropin excretion which is high in primary gonadal failure and absent in hypogonadotropism. This will be further discussed in the chapters on Testes (VII) and Ovaries (V).

Hypothyrotropism is indistinguishable clinically from hypothyroidism due to primary thyroid failure. A distinction based on thyrotropin assay, analogous to the gonadotropin studies mentioned above, can be attempted only in laboratories equipped for TSH assays. In view of the difficulties of this assay procedure the two types of hypothyroidism are usually differentiated by the response to injections of TSH (102, 159). An increase of I 131 uptake, protein bound iodine (PBI), or BMR can be expected only if the thyroid gland is intact, and can therefore be accepted as evidence of hypothyrotropism.

There is at present no information as to whether these single hormone deficiencies of the anterior pituitary are the result of an anatomical derangement or whether the lack of secretion of the respective hormones is due only to a functional disturbance.

The growth hormone deficiency causing pituitary dwarfism of the Loran-Levy type does not belong strictly in the category of single hormone deficiencies because in this disorder growth hormone deficiency is usually associated with gonadotropin deficiency and consequent hypogonadism. In exceptional cases however gonadal structure and function are normal (88, 180) and these cases legitimately can be considered under the heading of single hormone deficiencies. As is discussed below under the heading of pituitary dwarfism there is evidence for morphological changes in the eosinophilic cells of the anterior pituitary.

2 *Pituitary Dwarfism*

Dwarfism due to lack of growth hormone secretion is seen in children in whom the anterior pituitary is destroyed or encroached upon by tumors. In most instances craniopharyngiomas. Almost always the growth deficiency in these cases is associated with deficiencies of the tropic hormones, in other words the growth deficiency is part of panhypopituitarism. In

cases of pituitary dwarfism without clinically or radiologically demonstrable pituitary lesion (Lorain Levy type) the diagnosis is difficult. In most cases hypogonadism is present (but of course recognizable only after the age of puberty) but normal sexual development and even fertility have been recorded in exceptional cases (55-180). Delayed closure of the epiphyses appears to be typical. Increased insulin sensitivity might be expected in view of the anti-insulin action of growth hormone, however few observations are available and it would appear far from certain whether a normal insulin sensitivity would rule out growth hormone deficiency in the presence of normal adrenocortical function. The pituitary dwarfism of the Lorain Levy type may be compared with the dwarfism of the dwarf Silver mouse. In the latter dwarfism and hypogonadism occur as a recessive hereditary trait (188-189). The pituitary shows absence of eosinophilic cells. A marked diminution of the number of eosinophilic cells and degenerative changes in the present have been reported in a case of pituitary dwarfism in a woman (55).

Whereas most growth hormone preparations have been found to be entirely ineffective in man inducing neither Δ retention nor growth in length (in spite of good potency in the rat (145)) positive results have been obtained recently by several investigators (114-154). The effective growth hormone preparations are not yet commercially available.

II Thyroid Gland

A DIAGNOSTIC PROCEDURES

A considerable amount of work during the last several years has been concerned with diagnostic methods based on the role of the thyroid gland in iodine metabolism. The interest in these aspects has overshadowed that in the older procedures especially in the determination of the BMR. Conditions other than those of abnormal thyroid function are known to effect the oxygen consumption. In clinical diagnostic use one of the most common and most disturbing factors is that of emotional unrest causing high oxygen consumption probably by release of epinephrine. Techniques have been introduced for performing the test in barbiturate induced sleep or anesthesia under such conditions the high oxygen consumption resulting from nervous factors is reduced to normal values whereas the high BMR due to hyperthyroidism is not appreciably affected (13-172).

The use of I 131 tracer studies and PBI determinations has added materially to the knowledge of the pathophysiology of thyroid disease and these procedures have proved invaluable as diagnostic tools. Obviously a study of these parameters of thyroid function gives no information regarding the target response of the body tissues. This is highlighted by the unex-

plained syndrome of "hypometabolism" in which the basal oxygen consumption may be as low as in severe hypothyroidism, but in which the manufacture and release of thyroid hormone is entirely normal, as are all other body functions dependent on thyroid hormone.

1 Tracer Studies

The physiological basis for the diagnostic use of I 131 tracers has been discussed in Volume II. In many clinics the uptake is determined 24 hours after an oral tracer dose by measuring the radiation over the thyroid region. It has generally been confirmed that the uptake is high (more than 40%) in hyperthyroidism and low (less than 10%) in hypothyroidism. The tracer dose employed depends largely upon the sensitivity of the measuring instruments and of course on the particular parameters to be studied. With the use of scintillation counters the dose can be as low as 1 to 10 microcuries (3). Urinary excretion studies can supplement the measurements over the thyroid being merely related to the latter. Most of the iodine not taken up by the gland is rapidly excreted in the urine. It should be remembered that conditions other than thyrotoxicosis cause increased uptake of I 131: the high uptake of iodine-deficiency goiter long postulated on the basis of animal experiments, has now been demonstrated in an endemic goiter population not exposed to iodine prophylaxis such as the use of iodized table salt (195). The high uptake in some goitrous cretins is discussed on page 837. Drug induced situations of internal iodine starvation also may cause a temporary rise of uptake considerably above normal. This is the case in individuals in whom the iodine trapping mechanism was blocked by thiocyanate with development of goiter. Following withdrawal of thiocyanate the uptake by the iodine starved goitrous gland is very high. A similar situation occurs after withdrawal of thiouracils in experimental animals but has not been definitely observed in man (41).

Conversely, low uptakes occur in conditions other than hypothyroidism. Antithyroid drugs (thiourea derivatives, imidazoles) as well as thiocyanate block the uptake, though in different ways. Thiocyanate truly blocks iodide uptake under ordinary conditions; only the presence of very large amounts of iodide would force the latter into the gland through the block. The antithyroid drugs do not materially interfere with the iodide concentrating mechanism but inhibit incorporation of molecular iodine into the amino acid. Since 24 hours after the tracer dose mainly the protein bound moiety remains, the result at this time interval is one of decreased uptake. Low uptake values are also obtained during and for several weeks after medication with thyroid hormone evidently through suppression of TSH secretion from the anterior pituitary. Administration of large amounts of iodine either as iodide or as organically bound compound

(as used in roentgenographic studies of cholecystograms uterosalpingograms urograms bronchograms encephalograms) also causes low tracer uptakes. Two factors may be involved. The iodide may cause a real block perhaps by inactivating TSH. Furthermore if the iodide pool (circulating and extracellular) is considerably increased the tracer is diluted much more than in an average pool the uptake of the tracer may thereby be smaller owing to dilution. In renal disease the urinary excretion of iodide is diminished both because of impaired glomerular filtration and increased tubular reabsorption. Tracer studies based on urinary excretion of I 131 would give erroneous results. The increased I 127 pool also causes a greater dilution of the administered tracer and the accumulation of I 131 in the thyroid is delayed. However, values obtained by direct measurement over the thyroid at 26 hours are within normal range (157).

The physiological and pharmacological factors influencing I 131 uptake are of interest not only because of the light they throw on the pathophysiology of thyroid disease but also because they are pitfalls in diagnostic interpretation of the results of such studies. Another interesting finding is the almost complete obliteration of I 131 uptake in acute thyroiditis without other evidence of hypothyroidism.

Several procedures, using I 131 tracers have been introduced which add materially to the information obtained by determining the uptake after 24 hours. By repeated measuring of activity over the neck following a tracer dose the concentration of I 131 in the thyroid is found to be a linear function of the square root of time. The slope of this curve is called the accumulation gradient (9-74).

The mathematical relationship is simple but rather obscure in terms of biological significance. The linear relationship just mentioned is found in euthyroid and hypothyroid individuals but not in hyperthyroidism. The method is valuable inasmuch as it establishes the accumulation gradient within 1 to 2 hours after administration of the tracer and permits study of factors influencing the iodine accumulation of the thyroid (168). For example the activity of antithyroid drugs has been screened by this method (200).

Determination of thyroidal and renal plasma I 131 clearance rates has been recommended as a routine diagnostic test. Whereas determinations of the 24 hour uptake fail to correlate with the clinical condition in 7% to 10% of cases the claim is made that the clearance rates show no overlapping of values in normal and hyperthyroid individuals (17). It will be interesting to see whether such a sharp dividing line between euthyroid and hyperthyroid individuals will be confirmed on a larger material which will include mild cases of hyperthyroidism. It may appear questionable to some whether there is biologically a sharp line of demarcation between a thyroid

gland with 'top normal' function and that functioning slightly in excess of normal

Another parameter of thyroid function which can be studied by the use of I 131 tracers is that of release of thyroid hormone from the gland. By direct determination of the radiation over the thyroid for several days a measure for the secretion of thyroid hormone can be obtained (9).

This is also referred to as 'physiological decay,' and the time during which the gland loses 50% of its activity (corrected for physical decay) is called the biological half life. For more precise determinations it is necessary to administer antithyroid drugs (thiourea derivatives or imidazole derivatives) after the tracer dose in order to block uptake of that iodide moiety originating from the secreted thyroid hormone (9). Because of the necessary length of observation period this method is not generally used in clinical practice. Measurements of protein bound iodine I 131 following a tracer dose reflect the secretory activity of the thyroid (186). Under certain circumstances the values of PBI₁₃₁ do not correlate with the clinical physiological status of the individual. In some cases of postoperative myxedema the PBI₁₃₁ values have been found high in the presence not only of clinical manifestations of hypothyroidism but also of low PBI₁₂₇ values. This has been thought to indicate a very small but very active thyroid remnant with rapid turnover rate but small total output (19). By determining total plasma I 131 and protein bound plasma I 131 a conversion ratio has been calculated by the formula conversion ratio =

$$\frac{\text{PBI}_{131}}{\text{total plasma I 131}} \times 100$$

This ratio is believed to be related to the rate at which the gland utilizes iodine for hormone synthesis and release. There is some doubt as to the validity of such a simple formula on physiological grounds; correlation with the clinical condition has been found only in hyperthyroidism whereas the conversion ratio of hypothyroid and euthyroid individuals did not differ materially (183).

A further application of the tracer technique made possible through the introduction of the highly sensitive scintillation counter is the scanning procedure (68, 69). One or several scintillation counters are moved by a motor driven device across the thyroid region; the impulses are transmitted to a stylus which records them on graph paper. Such 'scintigrams' record the outlines of the thyroid gland and have been used for determining thyroid weight as a basis for calculating therapeutic doses (69). Inactive and hyperactive nodules can be demonstrated inasmuch as the scintigram will show regions of diminished or absent density or regions of increased density (68). The procedure has furthermore been used as an aid in differential diagnosis of masses of the neck and by scanning body regions other than the neck for detection of thyroid cancer metastases (10, 195).

2 Protein Bound Iodine (PBI) (207)

Methods for determination of PBI have been improved a discussion of these methods is beyond the scope of this review. In interpreting PBI values as indicative of abnormal thyroid function, sources of error other than those of a technical nature must be kept in mind. The most important error results from the presence in the body fluids of large amounts of nonhormonal iodine. This may be the result of ingestion of large amounts of iodide organic iodine compounds used as contrast media in various diagnostic procedures (cholecystography, pyelography, bronchography, encephalomyelography, uterine alpingography) cause even greater and more prolonged spuriously high PBI values. It appears that in the presence of large amounts of iodide in the serum some iodine will come down with the protein precipitate causing high yields in this fraction. High PBI levels are regularly found in pregnancy (129). This finding poses an intriguing question. Evidently the normal pregnant woman exhibits no signs of hyperthyroidism. Is this because the body needs more thyroid hormone in pregnancy? Spuriously low PBI levels have been obtained during the use of mercurial diuretics, during the technical procedures of digestion insoluble mercuric iodates or iodides are formed (139).

B THYROTOXICOSIS

No progress has been made regarding the etiology of thyrotoxicosis.

Advances in diagnostic methods are discussed on page 827. Diagnosis is not difficult in fairly typical cases certain types of patients pose a more difficult diagnostic problem in which the use of the newer procedures outlined above are of particular value. One is the thyrocardiac individual presenting cardiac failure frequently without a history typical of thyrotoxicosis nor are the clinical manifestations unequivocal. Frequently a failure to respond to digitalis first arouses suspicion. Signs and symptoms such as auricular fibrillation, tachycardia and nervousness can be the result of the cardiac condition and therefore help little in establishing the diagnosis of thyrotoxicosis. In the presence of cardiac failure and passive congestion a true basal metabolic rate cannot be determined. PBI and tracer studies have been of great diagnostic value in these cases. Control of the thyrotoxicosis will improve the cardiac condition significantly. Whether thyrotoxicosis can cause failure and consequent passive congestion of an otherwise entirely normal heart is debatable. The fact that the majority of thyrocardiac patients are in the older age group would lend support to the assumption that thyrotoxicosis causes failure only of a damaged heart. The argument loses practical significance if one keeps in mind that the pre-existing cardiac damage (most frequently arteriosclerotic in origin) may be subclinical and may revert to a symptomless stage after the thyrotoxic component is controlled.

Another group of thyrotoxic patients in whom the diagnosis can be established only with the aid of the tests described is that of certain severely agitated psychoneurotics. It is generally believed that psychic factors can precipitate or aggravate thyrotoxicosis, equally thyrotoxicosis is apt to aggravate or bring into the open psychoneurotic tendencies. Regardless of what is cause or effect in an individual case control of thyrotoxicosis if present is of greatest importance. Sleeping BMR, PBI, and tracer studies will help to establish the presence or absence of thyrotoxicosis.

Therapy. No basically new approach in the treatment of thyrotoxicosis has evolved. However a great amount of experience has accumulated with the use of the methods outlined in Volume II.

Numerous antithyroid drugs have been tested. Those which have proved most useful and are generally used, are two thiouracil derivatives propylthiouracil and methylthiouracil and an imidazole derivative 1-methyl-2-mercaptoimidazole (trade name Tapazole). They can be used in preparation for surgery or for definitive medical treatment. If given preoperatively the increased vascularity and friability of the hyperplastic gland may cause technical difficulties at operation. Iodide (e.g. in the form of liquor iodi comp. Lugol's solution) causes involution of the hyperplastic gland with decreased vascularity and friability; iodide is therefore administered together with the antithyroid drug for a few days prior to operation. The mechanism of action of iodide under the conditions is but poorly understood. It may enter the gland but is not oxidized and rapidly leaves the gland again. It may possibly inactivate endogenous TSH. If the antithyroid drugs are used for definitive medical treatment they should be administered for long periods of time. It is advisable to continue maintenance doses for 8 to 12 months after a euthyroid condition has been attained. Then treatment is discontinued. Fifty to sixty per cent of the patients remain in remission; the others relapse. It has been possible in some instances to keep up maintenance treatment for several years; if for any reason the other forms of therapy—surgery or radiation—were inadvisable.

Toxic reactions are observed in a small number of cases both with propyl and methyl thiouracil and with Tapazole. The total incidence is less than 2% in a few instances agranulocytosis has occurred.

Experience with radioactive iodine has also been greatly extended. There still is considerable difficulty in determining the dose which will reduce thyroid activity to normal but not cause hypothyroidism (see also p. 830). In many cases there occurs an exacerbation of the thyrotoxicosis during the first two to three weeks following the administration of I 131 owing to the liberation of stored hormone incident to the destruction of thyroid tissue. This exacerbation is rarely serious but thyroid crisis has

been observed. We have seen the most gratifying results of I 131 therapy in the thyrocardiac patient but the temporary exacerbation of the thyrotoxicosis may cause the patient to go through a period of more severe cardiac failure.

The possibility, that the radiation from therapeutic doses of I 131 may be carcinogenic is considered remote by most investigators. However thyroid cancer has been observed in rats treated with I 131 (6). Furthermore the possibility of late development of radiation induced fibrosis has to be considered. Many clinics therefore prefer not to treat thyrotoxicosis with I 131 in individuals younger than 35 to 40 years.

1 Ophthalmopathy

Interest in the ophthalmopathic facet of Graves disease continues. There appears to be no question that the severe ophthalmopathy is caused by a pituitary factor as outlined in Volume II. In the older experiments in which exophthalmos had been induced in experimental animals with pituitary extracts this action had been ascribed to TSH or to a pituitary factor closely allied to TSH. Dobyns and Steelman have succeeded in separating the exophthalmos producing principle from the thyroid stimulating hormone (49). This still leaves open the question whether the pituitary actually secretes the two factors separately, or whether the separation is a chemical one only. The most severe forms of ophthalmopathy are usually observed following thyroidectomy for thyrotoxicosis; occasionally they occur during treatment with antithyroid drugs. As long as the thyrotoxicosis is uncontrolled the ophthalmic picture is usually less severe; in exceptional cases the dissociation of thyrotoxicosis and ophthalmopathy is complete, the latter occurring in euthyroid individuals or in spontaneous myxedema, in either case without any indication of a preceding hyperthyroid episode.

The early changes in the retrobulbar tissue both in exophthalmos produced experimentally by injections of TSH and in human cases are those of edema; the edema fluid is rich in mucopolysaccharides (127). The muscle changes are swelling, edema and lymphocytic infiltration (pseudo hypertrophy). Whereas these muscular changes occur quite predominantly in the extrinsic muscles of the eye they are demonstrable to a minor extent also in skeletal muscles.

Occasionally severe ophthalmopathy occurs simultaneously with pretibial localized myxedema (38, 224). The cutaneous localized infiltration shows the same chemical characteristics as the retrobulbar tissue (227). Injection of hyaluronidase into such lesions has been reported to cause resolution, evidently owing to the specific depolymerizing action of this enzyme on the mucopolysaccharides (77, 142) in one case of our own

observation we were not able to show any changes following local hyaluronidase injection (132). Similarly, hyaluronidase has been injected into the retrobulbar tissue in cases of severe ophthalmopathy, but results have been unsatisfactory (77-107). Because TSH was believed to be the active factor from the anterior pituitary, several investigators have studied the TSH levels in blood or the urinary excretion of TSH in ophthalmopathic cases (5-42, 161). In most instances, high blood levels or high urinary excretion was found (5) but this appears not to be invariably the case (42, 161). If the *in vitro* separation of an exophthalmos inducing factor from the thyroid stimulating moiety is confirmed, the TSH studies just described would suggest that TSH secretion usually, but not invariably, parallels that of exophthalmos inducing factor.

The question whether the ophthalmopathy associated with Graves disease is always and in all cases due to a pituitary factor (so called thyrotropic ophthalmopathy) can as yet not be definitely answered. Most investigators favor a unitary etiology for both the severe ophthalmopathy and the ocular manifestations of classical Graves disease. However, the occurrence of two etiologically different types of ophthalmopathy has been suggested, thyrotropic and thyrotoxic in nature (143). In the latter the ocular changes are supposed to be the direct effect of excessive thyroid hormone secretion. If this type actually exists, the eye manifestation would be expected to subside after the thyrotoxicosis is brought under control. In many instances, improvement of the eyes following control of the thyrotoxicosis is more apparent than real: the lid retraction (a definitely thyrotoxic manifestation) subsides and with weight gain the cheeks fill out, giving the eyes a more normal appearance, where as the exophthalmos may not recede or actually increase even if slightly. However, occasional observations do suggest a real regression of exophthalmos and of paresis of extrinsic ocular muscles.

It is most important to prevent the development of the most severe or malignant forms. Since they occur mostly postoperatively, thyroidectomy should be advised against in thyrotoxic patients in whom severe ophthalmopathy, including swelling of the lids, pain, extensive muscle paresis and edema of the conjunctiva is present. In rare instances, severe deterioration of the ophthalmopathy has been observed during thiouracil therapy. In such cases the thyrotoxicosis should be brought under control very gradually by relatively small doses of antithyroid drugs (propyl thiouracil, Tapizok) or by small repeated doses of I 131 or by external roentgen radiation.

The treatment of the ophthalmopathy is directed to the anterior pituitary. If at the time of treatment the patient is euthyroid or hypothyroid (regardless of whether this was preceded by thyrotoxicosis or not), desic-

cated thyroid given to tolerance is frequently beneficial. Results appear better, the earlier treatment is started. Thyroid hormone probably acts by suppressing secretion of TSH (or an exophthalmos factor) from the anterior pituitary. Estrogens in large doses have been used successfully in severe cases; they also probably act by suppressing the anterior pituitary. Roentgen ray irradiation to the pituitary was up to recently, used mostly as an adjunct treatment of questionable value combined with either thyroid or estrogen medication; recently, however, cases have been reported in which roentgen radiation to the pituitary alone yielded good results (15).

2 *Thyrotoxic Myopathy (141, 167, 240)*

It has long been known that some muscle wasting is a common occurrence in thyrotoxicosis. The increased (endogenous) creatinuria as well as the increased creatinuria following an exogenous creatine load (decreased creatine tolerance) is probably the result of the muscle dysfunction induced by the excess circulating thyroid hormone. In rare instances of thyrotoxicosis muscle wasting is very much more pronounced and may be the presenting feature of the disease; these cases are referred to as thyrotoxic myopathy. Doubt has been voiced as to whether this syndrome represents only a quantitative exaggeration of the more moderate muscle involvement commonly seen in many severe thyrotoxic patients; it has been suggested that a qualitatively different but as yet unknown process is involved. Good recovery is observed when the thyrotoxicosis is brought under control. Diagnosis may be difficult inasmuch as the myopathy can dominate the picture to the point of masking the thyrotoxicosis; careful history and examination and appropriate tests will aid in establishing the diagnosis. When the presence of thyrotoxicosis and of myopathy has been established the cases have to be differentiated from myasthenia gravis. The latter disease is occasionally associated with thyrotoxicosis. Whether there is any causal relationship is as yet unknown. Commonly the response to prostigmine is used to differentiate between myasthenia gravis as associated with thyrotoxicosis and thyrotoxic myopathy. The former does and the latter does not respond to the drug. However the specificity of the response to prostigmine has been questioned and cases responding to a test dose of prostigmine have been reported as thyrotoxic myopathies. This must remain a debate in terminology until other tests characteristic for myasthenia gravis are also employed in the study of these cases (response to curare, electromyogram, etc.). The review of clinical problems in neuromuscular physiology by Denny Brown is helpful in understanding these complex problems (46). In patients simultaneously afflicted with myasthenia gravis and with thyrotoxicosis the myopathy has been reported to show improvement, deterioration, or no change after

control of the thyrotoxicosis. It is trite to state that more studies of this problem are needed.

Cases are on record of simultaneous occurrence of thyrotoxicosis and periodic familial paralysis (141, 228). As far as can be judged from the reported cases the periodic paralysis appears to be improved but not abolished following control of the thyrotoxicosis.

3 Thyroid Crisis

The exact nature of thyroid crisis or thyroid storm has yet to be completely unraveled. The factors most commonly precipitating crisis are (1) any operations performed upon untreated or incompletely controlled thyrotoxicosis cases (surgical storm) and (2) severe infections as well as major upsets (medical storm). Whether these factors act as 'stressors' through the adrenal cortex (see below) is at present a matter for speculation. There seems, however, little doubt that thyroid crisis represents thyrotoxicosis at peak severity. Why, in this condition, the heat regulating processes of the body break down, with the skin becoming hot and dry rather than dripping with perspiration, is not understood. It has long been suggested that an acute adrenocortical insufficiency occurs during thyroid storm and may in part be responsible for the fatal outcome. Treatment with cortisone has been successful and in several cases the development of post-operative 'surgical' storm was prevented by early treatment with ACTH or with cortisone, when an emergency operation became necessary in uncontrolled thyrotoxicosis (209).

C. CRETINISM

Cretinism is clinically subdivided into endemic and sporadic cretinism. Prima facie this is a geographical classification of cretinism occurring in goiter areas (iodine deficiency areas) or in nongoitrous regions. The endemic cretin is identified with the bearer of a degenerated thyroid, usually but not always goitrous, incapable of manufacturing thyroid hormone. The sporadic cretin on the other hand is identified with a congenital defect—a thyroid aplasia.

Tracer studies with I 131 on cretins believed to have congenital thyroid aplasia have in some instances yielded some small uptake in the region of the thyroid indicating the presence of some thyroid tissue. This confirms old histological reports in such cases in which small amounts of thyroid tissue rather than complete absence was reported. It will be interesting to see whether studies including the response to ISH of a larger material will detect early cases of TSH hypopituitarism in sporadic cretins.

The occurrence of goitrous cretins outside areas of goiter endemics has been described. Studies reported in some of these cases raise intriguing

problems regarding the pathophysiology. The uptake of I 131 is rapid and of a high order (81 122 137 199) such as is usually seen in the thyrotoxic gland and has been found in endemic iodine deficiency goiter (198). The utilization of the iodine by the thyroid gland appears to differ in different cases. In Stanbury's cases the rapid purging of the gland by thiocyanate indicates that the iodide was not oxidized, and therefore not incorporated into organic compounds (199). The deficiency in these cases was similar to that induced by thiourea and imidazole compounds. In contradistinction in McGirr's cases thiocyanate failed to purge the gland suggesting oxidation of iodide with subsequent organic binding. This was further supported by the presence of protein precipitable I 131 in the blood. It was assumed that this was an abnormal compound which was not identified (137). In Hamilton's cases analysis of thyroid tissue following administration of I 131 yielded radioactive fractions corresponding to thyroxine and diiodotyrosine. These studies were done before chromatographic analysis of these compounds had been introduced. In these cases it was assumed that the thyroid gland was capable of synthesizing thyroid hormone but not of releasing it (81).

Hurvith has claimed that some goitrous cretins subsequently, and without treatment become euthyroid or even hyperthyroid (94 96). These cases have not been reported in great detail and from the data recorded the diagnosis of cretinism may be questioned.

It is now generally recognized that treatment of cretins even when started early and continued uninterruptedly frequently fails to bring about normal mental development. Mental development improves during therapy, but in many instances does not reach normality. The nature of the changes underlying the lack of mental development is not known. Recent observations indicate the presence of electroencephalographic abnormalities in untreated cretins (43 217). It has been suggested that the EEG pattern becomes normal under thyroid medication only in those cases in whom mental development is fully restored and not in those who remain mentally deficient in spite of treatment (217). This has not been borne out regularly in our as yet limited experience (47).

On the basis of the concept of a reciprocal relationship of the pituitary tropic hormones with those of the secondary glands one would expect a high TSH secretion from the anterior pituitary in severe hypothyroidism including congenital thyroaplasia (an analogy could be drawn between the latter and ovarian agenesis associated with high FSH excretion). However this is not always the case. In a study of cretins Paschalis and D'Angelo found absence of TSH in the blood of an untreated cretin 11 years of age. TSH becoming detectable only after thyroid medication. This suggests that the anterior pituitary which has not been exposed to

thyroid hormone may be incapable of manufacturing and/or secreting TSH. In untreated cretinous babies the expected high TSH levels have been found. Studies regarding secretion of hormones other than TSH by such pituitaries will be of interest.

Familial incidence of sporadic cretinism has been reported (22) (familial incidence is of course common in endemic cretinism). On the other hand, the occurrence of cretinism in one of two identical twins has been reported unfortunately very briefly and without detailed data (175).

D. COITER

Iodine deficiency is firmly established as the most common cause of goiter (see Chapter V by Salter, in Volume II of *The Hormones*). Interesting historical aspects are touched upon by Salter (177). Certain aspects of the pathophysiology of human iodine-deficiency goiter had not been studied with modern methods because iodine prophylaxis in the form of iodized table salt has been in general use for several decades in the goiter regions and goiter belts of Europe and North America. Recently such studies were carried out in an endemic goiter area in the Argentine Andes where the natural conditions of the iodine-deficiency goiters had heretofore never been interfered with by administration of extra iodide. The great avidity of these goiters for iodine was demonstrated by the very high uptake of tracer doses of I 131 (198). Further results of this study will be awaited with interest.

Colloid goiters (clinically and pathologically indistinguishable from those occurring in iodine deficiency regions) are observed sporadically in regions such as the Atlantic seaboard where iodine supply in water and food is adequate. Endemic occurrence of goiter in locations not deficient in iodine has long been recorded (e.g. Derbyshire goiter in England) (139). Following the observations of goitrogenic action of cabbage brussels seeds and other vegetable matter in rabbits and rat, Astwood isolated an active goitrogen, 1,5,2-vinyl thiocyanazolidone from rutabaga and studied the goitrogenic action of a great number of vegetables (6,7). The chemical composition as well as the goitrogenic action is similar to that of thiourea and its derivatives. The subject has been reviewed by Greer (75) and by Fertman *et al.* (54). Only very few observations in human beings are recorded in whom the presence of a goiter could with great probability be traced to the ingestion of large amounts of goitrogenic vegetables (55, 138). Meins has coined the term *struma cibaria* for this type of goiter (139). Careful nutritional histories should be obtained in all goiter cases especially outside iodine deficiency areas. However it appears very doubtful to us that the majority of colloid goiters occurring in regions with adequate iodine supply can be explained on this basis.

The etiology of most cases is still obscure; an endogenous deficiency in iodine utilization may be thought of hypothetically (76).

In the medical treatment of colloid goiter medication with desiccated thyroid has been revived (76). Thyroid medication evidently acts by suppressing endogenous TSH secretion. The latter was previously increased causing enlargement of the thyroid because the normal size gland has for some reason been unable to produce adequate amounts of thyroid hormone.

Γ THYROIDITIS

Whereas the etiology of acute nonsuppurative thyroiditis remains obscure in most instances, an interesting physiopathologic aspect has been detected. The uptake of tracer doses of I 131 is greatly diminished or absent. No other manifestations of hypothyroidism are present. PBI values are normal or even slightly elevated. The reason for the depressed I 131 uptake is not evident; administration of TSH will increase the uptake though to a lesser degree than in euthyroid individuals in thyrotoxicosis factitia or in hypopituitarism. Curiously, administration of TSH brings about an improvement, at least a temporary one, of the condition. Various hypotheses have been offered in explanation of the low I 131 uptake and the response to TSH, but none is entirely satisfactory (176).

Milder cases last only a few days, but in more severe cases fever, pain, and swelling may continue for several weeks. In the absence of any knowledge regarding the etiology (bacteriological cultures have been almost uniformly sterile), treatment is symptomatic. Antibiotics have been without effect in almost all instances. There have been reports of beneficial action of external radiation and of thiouracil therapy; more recently cortisone or ACTH has been used successfully; their beneficial action is probably due to the antiinflammatory action of the adrenal steroids (230).

1 Subacute Thyroiditis (36a)

Attention has been drawn to this form of thyroiditis, which is not infrequently confused with Riedel's struma. A history of an acute phase is frequently obtained, but sometimes the onset is insidious and fever and severe general constitutional manifestations are absent. The gland is enlarged and very hard; it may or may not be tender. Even in the absence of fever the sedimentation rate is elevated and the I 131 uptake quite as low as in acute thyroiditis. Extensive fibrosis of the surrounding structures characteristic of Riedel's struma is absent. The histological picture (surgical or needle biopsy) is characterized by pseudotubercles and

giant cells. In contradistinction to Riedel's struma subacute thyroiditis is self limited and responds rather well to roentgen ray therapy, operations are unnecessary. The etiology is as obscure as that of other forms of thyroiditis.

IV The Adrenal Gland

A ADRENOCORTICAL HYPOFUNCTION

Chronic hypoadrenocorticism in human beings is due to (1) Destruction of the adrenal cortex by chronic disease (Addison's disease), (2) adrenalectomy performed for hypertension or advanced cancer of prostate or breast, (3) hypopituitarism resulting in lack of stimulation of the adrenal cortex by ACTH, (4) 'salt losing adrenogenital syndrome' (see p 831). Acute adrenocortical insufficiency probably plays a role in Waterhouse-Friderichsen syndrome (acute fulminant bacteremia mostly meningococemia, with extensive hemorrhage into the adrenal cortices), and perhaps as a phenomenon of adrenocortical exhaustion in severe infection without hemorrhage into the adrenals.

B ADDISON'S DISEASE

'Cortical atrophy' so called and tuberculosis are the causes of Addison's disease in the vast majority of cases. The term atrophy is a misnomer unfortunately ingrained in the clinical literature. The anatomical changes in the adrenal cortex in these cases are the result of cytotoxic damage, necrosis and fibrosis. In a small number of cases histoplasmosis involving the adrenal cortex causes Addison's disease; cases of amyloidosis (84-225) leucemic infiltration and extensive tumor metastases are rare occurrences (29). It is believed that severe clinical manifestations occur only after 90% of the adrenal cortex is destroyed.

1 Diagnostic Procedures

The absence of response to ACTH has been widely employed, with ACTH given either in a single intramuscular injection or by intravenous drip infusion. The end points studied are the decrease in circulating eosinophils and the rise in urinary 17 ketosteroids and 11 oxysteroids after the more intensive stimulation by continuous infusion. The eosinophil response to epinephrine however cannot be used as a test for adrenal responsiveness since it is very doubtful whether in man epinephrine in doses employed actually fires the pituitary-adrenal system (213). The most significant tests are those based on the disturbance of electrolyte metabolism. The response to NaCl deprivation combined with potassium administration (Cutler-Power-Wilder) still appears to be the most reliable and simple procedure if it is carried out with due safeguards for the possi-

ble precipitation of a crisis (40) The deranged Na and K metabolism can be demonstrated by testing the composition of sweat (35-126) or saliva (57-226) These studies have yielded interesting results regarding hormonal control of electrolyte balance but they would at present appear less suitable than urine studies for diagnostic purposes Tests designed to demonstrate the abnormal electrolyte metabolism are the most significant not only because most of the clinical manifestations of Addison's disease are due to the abnormal electrolyte metabolism but also because only the "salt hormone deficiency" is a constant finding in all cases of Addison's disease Disturbances of carbohydrate metabolism are present in the majority of cases in varying severity Severe hypoglycemia after prolonged fast and severe hypoglycemia following the hyperglycemia induced by intravenous glucose administration are indicative of the sugar hormone deficiency In many instances this deficiency is not severe enough to be demonstrable by these procedures Increased sensitivity to insulin (hypoglycemia irresponsiveness) is present in the majority of cases Because most Addison patients lapse into hypoglycemic coma when tested with intravenous administration of insulin by the standard procedures modifications of the test have been devised Thorn suggests giving insulin in increasing doses with the meals (214) Engel has devised an insulin glucose test in which the glucose administration prevents severe hypoglycemic manifestations (52) Cases of dissociated Addison's disease are of considerable interest in the presence of the classical picture with pigmentation hypotension weakness and demonstrable defect in Na and K metabolism the response to insulin is normal and low normal urinary excretion of glucocorticoids is present (152) Similarly the very low or absent 17 ketosteroid excretion is not always found in some instances values in the lower range of normal excretion have been reported (131) These dissociated cases cannot be adequately interpreted at this time Even if in man the different hormones are secreted by different zones of the cortex as has been suggested in the rat and mouse it is difficult to visualize a zonal destruction in the cortex No autopsy reports have come to our attention which would throw light on this problem if indeed the pathologist will hold the answer Also follow up studies of such patients will be desirable in order to see whether eventually the dissociated disturbance will in later stages of the disease embrace all hormones resulting in what may be called panhypocorticalism

2 Addison's Disease and Pregnancy

Before adrenocortical hormones became available for treatment of Addison's disease the occurrence of pregnancy in Addison patients was extremely rare and almost invariably fatal (25) With present therapy

several women have been carried through pregnancy successfully and have given birth to viable children. The most critical periods are the first trimester, during which profuse vomiting may precipitate crisis and the delivery with the incident stress and blood loss. Adrenalectomized pregnant rats have been known not only to tolerate pregnancy but to be maintained easier and better. This was ascribed to the deoxycorticosterone like action of progesterone. However in human beings there is evidence that the placenta produces corticoids (and probably ACTH). The *c* compounds are not produced in amounts sufficient for complete maintenance but may contribute to make maintenance after the third month easier (2, 99, 150).

3 Treatment of Addison's Disease

The use of deoxycorticosterone acetate and salt was discussed in Volume II of *The Hormones*. The compound is now available in 'buccal tablets' resorption is through the mucosa of the oral cavity into the major circulation. Dosage is of necessity less accurate than with administration by injection or pellet implantation; the degree of accuracy or inaccuracy depends entirely upon the intelligent and meticulous co-operation of the patient. Recently deoxycorticosterone trimethylacetate was introduced. This is a 'long acting' preparation permitting of maintenance by one intramuscular injection about every four weeks (215). The required dose of deoxycorticosterone and of salt must be determined empirically for every case; overdosage must be avoided because it causes hypertension, edema, and cardiac failure. Standardization is carried out with the use of body weight, blood pressure, hematocrit and serum electrolyte levels as criteria; one of the most valuable and sensitive effects is that on heart size as determined by roentgenograms. Cortisone is used as an adjunct to the treatment with deoxycorticosterone. Even when treatment with the latter succeeds in normalizing the patient by all objective criteria additional therapy with small doses of cortisone (12.5 to 20 mg. per day) frequently adds to the sense of well being. Whether this is due to the action on carbohydrate metabolism or rather to the cerebral euphoria, the action of cortisone is not immediately apparent. Perhaps the fact may be significant in this respect that the abnormal electroencephalographic pattern of Addison's disease is corrected by cortisone therapy but not by the use of deoxycorticosterone. Perhaps the most important place of cortisone is in prevention of crisis. Patients can be instructed to take extra cortisone if they have a sore throat or any other febrile episode even before they report to their physician. Corticosterone (Compound B of Kendall) has been investigated as a therapeutic agent in Addison's disease but this steroid has not been available for large scale clinical testing (34).

The fact that glycyrrhizic acid, the active ingredient of Liquorice is capable of correcting the electrolyte disturbance of Addison's disease is at this time at least less of practical than of theoretical interest (24-71)

Addisonian crisis is treated by intravenous infusion of glucose saline plasma antibiotics and aqueous adrenocortical extract and by intramuscular injections of cortisone and deoxycorticosterone acetate

CUSHING'S SYNDROME

1 Pathogenesis

Cushing's syndrome has been recognized as a form of hyperadrenocorticism as was described in Volume II page 692. The adrenals are bilaterally diffusely hyperplastic in some cases whereas in others an adrenocortical cancer is present. In a small number of cases no morphological changes are found in the adrenals if no ovarian tumor is present an excessive function of adrenal of normal size and structure may be assumed. This however is an assumption since it is conceivable that normal amounts of hormone might be secreted by the glands and the picture of hyperfunction be simulated by abnormal metabolism of the hormone. The tumors of the ovary mentioned above are probably adrenal rest tumors such tumors are rare and most of those reported have caused adrenogenital syndrome rather than Cushing's the occurrence of the latter however has been established.

Cases of hyperplasia may be the link between the newer concept of Cushing's syndrome as hyperadrenocorticism and Cushing's original thesis that these cases were due to basophilic hyperpituitarism. Whether or not a basophilic adenoma is present excessive secretion of ACTH by the pituitary basophils would explain hyperplasia and hyperfunction of the adrenal cortex. Older reports of increased ACTH levels in the blood of such patients are inconclusive because of the inadequacy of the methods employed. With the use of Sayer's method increased levels of ACTH have now been demonstrated in some instances (23-70) but not in others (208a, 209a). Correlation with anatomical findings in adrenals and pituitary will be desirable. If the findings of increased ACTH levels in the blood of Cushing cases with adrenocortical hyperplasia are confirmed in a larger number of cases a further problem arises. We have reason to assume that hypersecretion of cortical hormones will suppress ACTH secretion from the anterior pituitary making endogenously caused adrenocortical hypersecretion self limited. This would of course not apply to adrenocortical carcinomas which may be autonomous and continue to function at high levels in spite of the retrograde suppression of ACTH secretion the contralateral adrenal in such cases is however atrophic because of this mechanism. In cases of hyperplasia the question therefore

arises why the excessive ACTH should persist, and not be suppressed by the liberated adrenal steroids. There is at present no answer, one could suggest the hypothesis that the control mechanism in such cases is 'set' at a higher level, either in the pituitary or in the hypothalamus.

The latter has been implicated in the pathogenesis of Cushing's syndrome (63), and whereas the evidence is far from conclusive the experimental data on hypothalamic control of ACTH secretion (26, 56, 93, 134) should stimulate further study of clinical material.

In a discussion of the pathogenesis mention should be made of the Cushing-like pictures which have been observed (1) after therapeutic administration of large doses of cortisone (197) and (2) in certain forms of hepatic cirrhosis (21). From a purely therapeutic angle, the occurrence of Cushing-like features in patients treated with cortisone is an undesirable 'side effect'. However these cases are of considerable interest inasmuch as they would seem at first sight to nullify the concepts of the pathophysiology of Cushing's syndrome as first proposed by Albright (1) and Kepler (110). According to this concept Cushing's syndrome would represent 'panhypercorticalism' some of the manifestations would be the result of excessive secretion of deoxycorticosterone like salt hormone (hypertension, cardiac enlargement, edema) with others to be ascribed to the excess sugar hormones' (diabetes, obesity, deossification, purple striae) and yet others (acne, hirsutism, plethora) to increased androgen secretion. After all these manifestations were induced by excess cortisone administration it appeared doubtful to some investigators whether the concept of Cushing's syndrome as 'multihormone excess' or 'panhyperticalism' could be upheld. In view of the fact that "single hormone" hypersecretion as well as hyposecretion appears to occur and in view of the fact that the adrenal gland can perform remarkable interchanges in the steroid structure the induction of Cushing's syndrome by cortisone administration does not appear conclusively to invalidate the previous concepts.

The observation of Cushing-like pictures in certain forms of hepatic cirrhosis is of interest. This has been explained by assuming that the diseased liver is deficient in inactivating or destroying endogenous cortical hormones. The urinary excretion of formaldydrogenic steroids was found increased in these cases—a finding compatible with the explanation (21). Hepatic inactivation of cortisone has been demonstrated *in vitro* and *in vivo*.

To return to the pituitary basophils the only constant finding in all cases of Cushing's syndrome has been that of hyalinization of many basophils these changes are usually referred to as 'Crooke's changes'. The physiological significance of this finding was not apparent the hyaliniza-

tion was interpreted as possibly indicative of excessive ACTH secretion, or as the late Dr Kepler suggested as an effect of the excessive adrenocortical secretion on the pituitary (109). The latter explanation may now be accepted in view of the fact that identical changes have been found in the pituitaries of subjects treated with ACTH or with cortisone for various conditions (66, 121).

2 Hormone Secretion

Following the development of methods for determining adrenocortical 11 17 hydroxy compounds in blood, high levels were found in cases of Cushing's syndrome (156). Urinary excretion studies have been carried out with various methods. 17 Ketosteroids have been found to be within normal range, or moderately elevated (131). This would correlate well with the moderate degree of virilizing manifestations seen in Cushing's syndrome in contradistinction to the marked degree observed in the various forms of adrenogenital syndrome. Corticoid excretion determined by bio assay, is elevated and this finding correlates well with isolation studies which have yielded large amounts of 17 hydroxycorticosterone (hydrocortisone). Reports of studies using either reducing methods or formaldehyde generation are less consistent. excretion levels have been found high in some and within normal range in other cases. most of these studies were carried out without β glucuronidase hydrolysis. On the other hand the enzyme appears to liberate substances not related to corticosteroids which react both in the reducing and formaldehydogenic procedures. Even with technical refinements of the formaldehydogenic method (90 237) there are still considerable sources of error in this procedure. With the use of methods based on the Porter Silber reaction (30 64 173 179 223) more consistent results are obtained even though (or perhaps because) this reaction determines a more limited range of compounds (17 hydroxycorticosteroids). Determination of the 17 hydroxycorticosteroids and especially of the enzyme liberated Porter Silber positive compounds (17 hydroxycorticosteroids) has yielded high values in cases of Cushing's syndrome (173 178). additional information can be obtained by separate determination of the free and the glucuronidase liberated 17 hydroxysteroids (33).

3 Metabolic Changes

The retention of sodium and loss of potassium in Cushing's syndrome is a well established fact (see Volume II). Eventually this may lead to a severe hypokalemic hypochloremic alkalosis. To what extent the electrolyte changes are due to the action of C₁₁ oxygenated compounds alone or to a concomitant increased secretion of salt hormone is as yet unsettled.

Studies for the presence of aldosterone¹ will shed light on this problem. An interesting abnormality of the electrolyte metabolism pertains to the response to deoxycorticosterone acetate. Whereas in the normal individual as well as in the Addisonian injection of this compound together with intravenous administration of sodium chloride is followed by sodium and chloride retention, sodium and chloride loss occurs under the same conditions in cases of Cushing's syndrome (191, 192). There is at present no adequate explanation for the paradoxical action.

The disturbance of carbohydrate metabolism is clearly the result of excess secretion of C_{21} oxygenated steroids ('sugar hormones'). As would be expected the resulting diabetes is relatively insulin resistant. Differences of severity of impairment of carbohydrate metabolism ranging from severe diabetes to 'latent' diabetes (impaired glucose tolerance), may in part at least be due to differences of β cell 'reserve capacity'. The diabetogenic action of the C_{21} oxygenated corticosteroids is due to increased gluconeogenesis and to decreased glucose utilization. An interesting aspect of disturbed carbohydrate metabolism in Cushing's syndrome has received some attention recently. It has been shown previously that in normal individuals the fasting blood values for glucose, lactate and pyruvate were in the ratio of 100:10:1; this ratio is maintained following ingestion of glucose. In diabetics the rise of blood sugar during a glucose tolerance test is not associated with a proportional rise of lactate and pyruvate; these metabolites either fail to rise or this rise lags appreciably. In contradistinction in some cases of Cushing's syndrome the fasting levels of lactate and pyruvate were disproportionately high, and following glucose load the diabetic glucose curve was associated with high levels of lactate and pyruvate (89-111). This has been interpreted to indicate that in this form of 'steroid diabetes' the islets respond to the glucose load with good insulin secretion in spite of the diabetic shape of the glucose tolerance curve. High blood pyruvate levels following glucose load have also been reported after administration of cortisone and ACTH (111). If the high blood levels indicate increased formation of pyruvate this could conceivably also have some bearing on the fat formation via acetate and help to explain the obesity observed both in Cushing's syndrome and following cortisone and ACTH therapy.

¹ A number of important papers regarding the physiologic and therapeutic action of aldosterone have appeared after this manuscript was submitted. (B. Gaunt et al. *Endocrinol.* 55: 236 (1954); F. T. G. Prunty et al. *Lancet* III: 620 (1954); G. L. Farrell et al. *Proc. Soc. Exptl. Biol. Med.* 87: 141 (1954); J. A. Luetscher Jr. et al. *J. Clin. Invest.* 33: 276 (1954); J. A. Luetscher Jr. et al. *J. Clin. Endocrinol. and Metabolism* 14: 1086 (1954).)

4 Diagnosis

There has been no significant new contribution with regard to diagnosis. The frequency of occurrence of signs and symptoms as well as of abnormal laboratory data has been compiled by Plotz *et al* (163). Determination of urinary 17 ketosteroids is of limited diagnostic value because values are within normal range in many cases. Determination of 17 hydroxysteroids promises to be of great significance.

After the diagnosis of Cushing's syndrome is made it is obviously of greatest importance to determine presence or absence of a cortical tumor. An interesting biological test is based upon the response to exogenous cortisone. In the normal individual as well as in cases of adrenocortical hyperplasia administration of cortisone suppresses 17 ketosteroid excretion probably by suppressing ACTH secretion from the anterior pituitary with consequent suppression of secretion of the adrenal cortex. In cases of adrenocortical cancer cortisone administration is not followed by a decrease of 17 ketosteroid excretion evidently because the adrenocortical cancer is autonomous and continues to function in the absence of ACTH stimuli (97). In a case of Cushing's syndrome which one of us (K. E. P.) discussed in a CPC conference injection of a test dose of insulin was followed by profound and prolonged hypoglycemia usually seen in Addison's disease and in panhypopituitarism. This was interpreted as indicating the presence of an autonomous adrenocortical cancer associated with the typical atrophy of the other adrenal.

The biological tests (response to cortisone response to insulin) will have to be confirmed in a larger number of cases before the reliability and practical usefulness can be evaluated.

At present direct evidence for the presence of a tumor must still be sought in every case. To the older methods of roentgenography following urography and periadrenal air insufflation roentgenography following introduction of air into the retrorectal space (retroperitoneal pneumography) has been added (203). This procedure appears to be safe and has yielded good pictures in several cases. As before exploratory operation has to be resorted to in doubtful cases.

5 Course of Disease

The natural history of Cushing's syndrome has been critically reevaluated in a study by Plotz *et al* based on 33 observed cases and analysis of 189 cases reported in the literature (163). The rare occurrence of spontaneous remissions has been reemphasized by Knowlton (117) this may be important in trying to evaluate results of therapy.

6 Treatment

The treatment of choice in cases of tumor is clearly surgical removal of the tumor.

As mentioned in Volume II treatment in cases due to adrenocortical hyperplasia had not been very satisfactory. Roentgen irradiation of the pituitary is now being carried out with much larger doses (3000 to 5000 tissue r) than were employed previously. Good results with remission of long duration have been obtained in two series of cases (51, 103). However even these large radiation doses fail to bring about any improvement in some cases (33, 103). Partial hypophysectomy has been carried out in a few cases (128). B. Arner *et al.* report a good result following exposure and electrocoagulation of the pituitary (4). Beneficial results of subtotal and of total adrenalectomy have been reviewed (196). 2,2-bis (para-chlorophenyl) 1,1-dichloroethane (D D D) has been found to cause atrophy of the zona fasciculata of the adrenals of dogs rendering them insulin sensitive (146). It is also active in rats increasing their insulin sensitivity and alleviating alloxan diabetes but without causing adrenocortical atrophy (27, 28). Similar observations have been made in rabbits (101). However, in two cases of Cushing's syndrome this drug failed to exert any effect on corticoid excretion and on carbohydrate metabolism (33, 182).

D THE ADRENOGENITAL SYNDROME

As indicated in Volume II the clinical manifestations are due to increased secretion of androgenic 17 ketosteroids from the adrenal cortex. The anatomical change in the latter is either diffuse bilateral hyperplasia or tumors of one cortex. The picture will differ depending on the age and sex of the individual as summarized in Table I.

Excretion of 17 ketosteroids is increased above the levels normal for the respective age group. Higher values are found in tumor cases but there is some overlap. However excretion of more than 150 mg/24 hours indicates the presence of adrenocortical cancer. Tumor cases excrete proportionately more β 17 ketosteroids (mainly dehydroepiandrosterone) than do hyperplasia cases. In the latter α and β 17 ketosteroids are increased in approximately normal ratio the β fraction comprising up to 25% of the total. Separation into α and β fraction can be carried out by Digitonin precipitation or with the use of color reaction based on the Pettenhofen reaction. However cases of adrenocortical cancer with low β fractions have been recorded (62, 132) and in our experience also the differentiation of tumor and hyperplasia by determination of the α - β ratio is not reliable. Increased 17 ketosteroid values have recently also been found in blood in adrenogenital syndrome (61). Recently high urinary excretion values of pregnanetriol in adrenogenital syndrome have been

TABLE I
SIGNIFICANT ANDROGENIC SYNDROME

| Age of onset | Anatomy of adrenal | Clinical manifestations | | Remarks |
|--------------|------------------------------|---|--|---|
| | | Female | Male | |
| 1 Fetal | Diffuse hyperplasia | Female pseudohermaphrodite Persistent urogenital sinus Large clitoris Later in addition features like #2 | Macrogenitosoma; precocious Manifestations not conspicuous at birth | Minimal incidence in some instances Type 1 Effects of excess androgens only Type 2 Salt losing Effects of excess androgens plus salt hormone deficiency Type 3 Effects of excess androgens plus sugar hormone deficiency Tumor more frequent than in hyperplasia Similar syndrome in females from virilizing ovarian tumors in males from Leydig cell tumors and teratomas |
| 2 Childhood | Tumor or diffuse hyperplasia | Heterosexual precocious puberty Large clitoris Body and facial hair Deep voice Advanced bone age Rapid growth in length Amenorrhea Absent mammary tissue | Isosexual precocious puberty Large penis Small testes Body and facial hair Deep voice Advanced bone age Rapid growth in length Azospermia | Same as #2 |
| 3 Adult | Tumor or diffuse hyperplasia | Large clitoris Deep voice Body and facial hair Amenorrhea Atrophy of breasts | | |

found (20, 132-140), perhaps the high pregnanediol excretion reported in earlier studies (63, 221) actually was due to the presence of the triol in the precipitate. The possibility that the pregnanetriol is a metabolite of 17 hydroxyprogesterone, probably a precursor of hydrocortisone, has been discussed (20).

The pathogenesis of the diffuse hyperplasia poses a challenging problem. There is no evidence to suggest a primary hypothalamic or pituitary disorder causing excessive secretion of ACTH. However, high blood levels of ACTH have been reported (208). It is not immediately apparent how continuous stimulation by excessive ACTH causes excess androgen secretion only, and not panhypercorticism as seen in Cushing's syndrome. A very interesting hypothesis has been proposed by F. Bartter *et al.* (14). According to this schema the primary abnormality of the adrenal cortex in these cases is the decreased ability to produce C₁₁ oxygenated steroids ("sugar hormones") or, with other words, a low target responsiveness of 11 oxysteroid production to ACTH. It has been subsequently suggested that the 'block' might be at the conversion of 17 hydroxyprogesterone to hydrocortisone and that perhaps the high excretion of pregnanetriol (probably a metabolite of 17 hydroxyprogesterone) is due to this block (20). The low levels of circulating 11 oxysteroids take the brakes off the anterior pituitary's ACTH secretion, the larger amounts of ACTH now in circulation are adequate to maintain a fair 11 oxysteroid production but the 17 ketosteroid production now becomes excessive because the sensitivity of the processes responsible for their production is normal; this hypothesis assumes that the idional androgens are much weaker inhibitors of ACTH secretion than the C₁₁ oxysteroids. A number of observations in patients with adrenogenital syndrome would be explicable on the basis of Bartter's schema: ACTH administration fails to induce eosinopenia, nitrogen loss and increased urinary levels of reducing steroids in cases of adrenogenital syndrome. Also ACTH administration was not followed by Na retention or K loss (14-124). Low blood levels of C₁₁ 17 oxygenated steroids have been reported in several cases (105); this is in apparent discrepancy with the normal urinary values of reducing or formaldehydogenic steroids found in other cases (14). The discrepancy may be more apparent than real since methods determining reducing and formaldehydogenic steroids are not entirely specific. On the other hand it is curious and not immediately understandable that individuals with low secretion of C₁₁ C₁₇ oxysteroids should have no manifestations referable to this deficiency. Furthermore studies indicate that these adrenals are incapable of increasing their 11 oxysteroid production under additional ACTH stimulation (14-124) yet the individuals appear to tolerate infections and other stresses quite well. It would appear that

the steroid deficiency differs in degree in different cases being relatively mild and subclinical in most instances in others severe deficiency with hypoglycemia has been observed (231-235). Hypertension has been recorded in a number of cases (236).

In some cases of adrenogenital syndrome a disturbance of electrolyte metabolism similar to that present in Addison's disease is found (12-36). In this type designated as the salt losing type of adrenogenital syndrome inability to retain sodium lead to Addisonian crisis. From a clinical standpoint the most difficult cases are those occurring in very young infants in boys there may be little in the appearance of the patient to suggest virilizing adrenocortical hyperplasia because masculinization may not appear before the end of the first year of life. An assay of urinary 17 keto-steroid excretion will permit of early diagnosis.

In some cases an aplasia of the zona glomerulosa in addition to the hyperplasia of the inner zone of the cortex was found (18). In view of the fact that at least in rodents the zona glomerulosa is believed to elaborate salt hormone it was suggested that in these cases salt loss was due to a deficiency of salt hormones (234). Another explanation was suggested based on the observation that these patients responded to ACTH injection with sodium loss instead of with the sodium retention observed in normal individual. It is possible that these adrenals elaborate an abnormal hormone which causes Na loss. As yet such a compound has not been demonstrated. The effect of this hypothetical factor has to be overcome with large doses of deoxycorticosterone acetate and salt until cortisone successfully suppresses its secretion (101-124).

The familial incidence of congenital adrenocortical hyperplasia causing pseudohermaphroditism in girls and macrogenitosomia precox in boys has been known for some time. Bentinck and co-workers have analyzed all reported instances they point out that in no instance has the abnormality been found in more than one generation. Generally they believe that the evidence available militates against the assumption of a genetic factor (16).

The differentiation of cases due to adrenocortical tumor from those resulting from diffuse hyperplasia is of course of greatest practical importance. As mentioned above neither the urinary excretion level of total 17 keto-steroids nor the ratio of urinary alpha and beta 17 keto-steroid give reliable information. Whether pregnanetriol excretion in hyperplasia will prove a reliable differentiating feature from the tumor cases will have to await study of a larger material. Except in the congenital form which is invariably due to hyperplasia a tumor must always be suspected. The visualization of the adrenal glands on roentgenogram following air injections has been greatly improved by the use of retro-

found (20, 132, 140), perhaps the high pregnanediol excretion reported in earlier studies (63, 221) actually was due to the presence of the triol in the precipitate. The possibility that the pregnanetriol is a metabolite of 17 hydroxyprogesterone probably a precursor of hydrocortisone has been discussed (20).

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stance adrenergic in nature which passes along the portal vessels from the hypothalamus to the adenohypophysis is (130)

Not only organic lesions but also various stress reactions both physical and emotional may influence this mechanism markedly and in at least several different ways to produce several syndromes of pituitary ovarian dysfunction

A GENETIC ABNORMALITIES

1 Ovarian Agenesis (Turner Syndrome (218) Turner Varney Syndrome)

In this condition there is a congenital absence of the germ cells and failure of the follicular epithelium to develop, so that the ovary consists of a thin band of fibrous tissue lined by germinal epithelium lying in the mesosalpinx parallel to the Fallopian tube. It is not known whether this condition is due to failure of production of the germinal cells or to failure of the germinal cells to migrate from their site of origin in the hind gut into the gonad.

The question has been raised (232) as to whether all of these patients are in reality females as their external genital development denotes. Since in the male the Mullerian system appears to regress under the influence of androgen from the developing testes is it not possible that many of the essentially agonaladal individuals are genetic males? On the other hand if this were so some other explanation would be required for the occasional male who presents a similar syndrome (p 865). Experimental observations made in young rabbit embryos by Jost (106) and in mice by Wells (229) support this hypothesis.

Many other genetic and endocrine features in this syndrome require clarification. The shortness of stature is hardly to be expected in view of the congenital hypogonadism. There is a remarkable tendency for most of these people to range in height from 54 to 56 inches. The skeletal proportions are usually normal rather than eunuchoid. The bone age is generally within the normal range occasionally being moderately delayed. Pubic hair is present in some but absent in other cases. Webbing of the neck is often noted. Among the other congenital defects sometimes seen are a receding chin, hypertension, coarctation of the aorta, osteoporosis with resulting orthopedic abnormalities and polydactylism. Some patients may show no congenital defects except for the hypogonadism and shortness of stature.

Gonadotropin (FSH) excretion is uniformly elevated and indeed serves as an important point in differential diagnosis from pituitary infantilism.

* A recent report by Wilkins *et al.* *J Clin Endocrinol and Metabolism* 14: 170 (1954) indicates that the chromosomal sex is male in most cases.

peritoneal pneumography (203) Air is injected into the retrorectal space and spreads upward retroperitoneally outlining all retroperitoneal structures. This procedure appears to be technically simpler, and above all safer than the perirenal air insufflation. If for some reason adequate visualization is not obtained, surgical exploration is necessary.

Treatment in tumor cases consists of course in surgical removal of the tumor. Treatment of hyperplasia cases was very unsatisfactory up to the introduction of cortisone therapy (232, 233, 236). Treatment with cortisone suppresses the function of the adrenal cortex, evidently by suppressing ACTH secretion from the anterior pituitary. The inhibition of androgen production causes arrest of the virilization in females as well as of the rapid growth and bone maturation with premature closure of epiphyses. Development is normalized and a normal female pubertal development in girls with menarche and breast development takes place. Evidently exogenously administered cortisone adequately replaces the suppressed secretion of the other cortical hormones.

1. *Treatment of the Salt Losing Type (36, 232)*

Treatment with cortisone alone even with administration of extra salt, is usually inadequate for maintaining electrolyte metabolism. Deoxycorticosterone acetate (DCA) is used in combination with cortisone the latter chiefly for suppressing androgenic 17 ketosteroid excretion the former for correction of the inability to retain sodium. It has been found that large doses of deoxycorticosterone and NaCl are necessary. This has been tentatively considered supportive evidence for the theory that the salt loss is due not to a deficiency of salt hormone but to the production of a Na losing adrenal compound the action of which has to be overcome by DCA. The requirement for the latter is appreciably reduced after cortisone therapy has brought about suppression of adrenocortical activity.

V. Ovary

Normal cyclic function of the ovaries with regard to both ovulation and hormone secretion involves not only a complex but also a very labile mechanism involving particularly (1) neurohumoral factors regulating the cyclic release of three gonadotropic substances (FSH, LH and LTH) (2) a precise sequential response of the germ cells to gonadotropic stimulation and (3) the cyclic secretion of steroid hormones by the granulosa and theca cells in proper amounts and at the proper time to favor a self continuing push pull relationship between the gonad and the hypophysis. The hypothalamus is undoubtedly an important relay station in the neurohumoral control of the pituitary-ovarian axis and present evidence suggests that gonadotropin release is mediated by a neurohumoral sub-

closure of the epiphyses will be hastened the secondary sex characters will develop as will the vagina and uterus and the vasomotor symptoms will be controlled but spontaneous ovarian function rarely follows the withdrawal of substitutional therapy

C PITUITARY HYPOGONADISM

Pituitary hypogonadism in women may be associated with pituitary dwarfism (p 826) or may be a single gonadotropic deficiency. The skeletal development may therefore range from marked dwarfism to eunuchoidism depending apparently upon the degree of growth hormone deficiency. In some instances with moderate retardation of growth the clinical picture may resemble that of ovarian agenesis but can be distinguished from the latter by absence of FSH in the urine. Other congenital defects are usually absent the bone age is more markedly retarded and the urinary 17 ketosteroids are lower (often less than 10 mg per 24 hours) (232). As indicated above the differential diagnosis from a primary ovarian deficiency is of importance because of the possibility of a response to gonadotropic therapy

D NONNEOPLASTIC CYSTS OF THE OVARY

1 *Follicle Cysts of the Ovary*

The etiologic factors in follicular cystic disease of the ovary appear to be many and varied including genetic abnormalities inflammatory lesions of the ovary degenerative processes and endocrine factors. Usually the physiologic disturbance results in a menstrual dysfunction and impairment of ovulation. Estrogen secretion varies the follicular fluid in some of the large follicle cysts of the ovary contains small to moderate amounts of estrogenic activity whereas others are hormonally inactive. Urinary excretion of estrogens also vary from diminished titers in most cases to excessive levels in occasional instances. The endometrial pattern is likewise diverse ranging from an early proliferative type to one of cystic hyperplasia. A secretory pattern with apparently normal ovulation may occur in the patient with one or a few follicle cysts of the hormonally inactive type in which the ovaries are otherwise normal.

Little is known of the endocrine factors which may cause large follicle cysts of the ovary. Occasionally these may be induced by too vigorous gonadotropic therapy (44) suggesting that a disturbance of the pituitary ovarian endocrine mechanism may be the etiologic factor in some cases. In this respect it is of interest that the use of estrogens or androgens given in full dosage for several weeks may cause rapid regression of simple follicle cysts.

in questionable case. Estrogens in the urine are very low. Despite the pattern of high gonadotropins and low estrogens vasomotor symptoms do not occur in these patients as they so often do in acquired types of prepubertal hypogonadism. The 17 ketosteroid excretion varies considerably; in some cases it is normal and in others it is quite low.

As a rule, the diagnosis of this condition is not established with certainty until the postpubertal years when the hypogonadism becomes evident. However, in instances in which it is suspected on the basis of dwarfism together with other congenital stigmata, the diagnosis can be made with certainty if there is a high urinary gonadotropic excretion, a feature distinguishing this condition from pituitary and other types of dwarfism. Wilkins (232) has noted that urinary FSH was absent or low during early childhood but rose as puberty was approached. The administration of estrogens produces a gratifying development of the breasts, secondary sex characters and growth of the vagina suitable for sexual purposes. This should be continued in cyclic fashion to maintain development but in moderate dosage since otherwise marked endometrial hyperplasia with troublesome bleeding can occur. We have not noted a stimulation of skeletal growth with estrogens even when treatment has been started in late childhood.

B. INFANTILE OVARIES WITH EUNUCHOIDISM

In this syndrome, the ovaries are of the elongated ovoid (infantile) type containing apparently normal numbers of primordial follicles. The follicles fail to grow and mature despite adequate FSH secretion as evidenced by high urinary titers. The irresponsiveness of the ovary to gonadotropic stimulation is believed to be due to a genetic defect although the possibility of a severe childhood infection affecting the ovaries must also be considered. These patients show the characteristic stigmata of prepubertal hypogonadism; they are relatively tall with eunuchoid proportions, the secondary sex characteristics and reproductive organs are immature, they have primary amenorrhea and hot flashes. The latter symptoms, as well as the high urinary FSH titers, distinguish them from the prepubertal eunuchoid hypogonad of pituitary origin. Furthermore, these patients rarely improve spontaneously, whereas the pituitary hypogonad may sometimes go into a delayed puberty in her late teens or early twenties and may eventually attain normal reproductive function. The latter may be encouraged or hastened by the cyclic administration of gonadotropic hormones with follicle stimulating activity, whereas the primary ovarian eunuchoid can be treated only symptomatically with estrogens or with estrogens and progesterone in cyclic fashion. The

closure of the epiphyses will be hastened the secondary sex characters will develop as will the vagina and uterus and the vasomotor symptoms will be controlled but spontaneous ovarian function rarely follows the withdrawal of substitutional therapy.

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2 The Stein Leventhal Syndrome

A particular type of microfollicular cystosis of the ovary which is of interest because of its endocrine stigmata is that seen in the Stein Leventhal syndrome (201, 202, 123). This condition is characterized by a menstrual dysfunction (usually amenorrhea but sometimes irregular or excessive bleeding), infertility and hirsutism of varying degree, together with bilateral cystic enlargement of the ovaries. The condition generally begins in the late teens or early twenties, usually after prior normal menstrual function. Although the hirsutism may become quite marked in some cases, the clitoris is rarely appreciably enlarged or the voice deepened. The 17 ketosteroid excretion is generally within the normal range, occasionally being slightly elevated (up to 20 mg per 24 hours). A moderately high β fraction (20% to 40%) is often noted. Estrogens are present but usually in moderately low titers and the vaginal smear shows a persistent early follicular reaction. T-H is often not demonstrable in the urine but high titers occur sporadically, especially early in the condition.

The appearance of the ovaries on exploration is quite characteristic. They are enlarged, flattened and pale and have a thickened gray capsule; they resemble oysters. The cut surface reveals many small cystic areas of varying size. On histologic examination the significant abnormalities are the marked fibrous thickening of the ovarian capsule, the presence of numerous follicle cysts, with a paucity of normal growing follicles and hyperplasia of the theca surrounding the cystic follicles. The latter finding is not consistently present.

A small type of Stein Leventhal ovaries is sometimes noted, in which the ovaries are only slightly enlarged and ovoid but have a markedly thickened capsule, microfollicular cysts and theca hyperplasia. These patients often have excessive and irregular bleeding rather than amenorrhea.

The etiology of this condition is not known. There is little to support an inflammatory basis for the excessive fibrosis of the capsule which could secondarily produce follicular cystosis. A genetic dysfunction likewise seems improbable in view of the previous apparently normal menstrual history and absence of other stigmata in most cases. It seems more likely that the condition is due to an endocrine dysfunction. One wonders whether the hyperplastic theca may not be secreting increased amounts of an unusual steroid, perhaps not excreted as a 17 ketosteroid which is responsible for the mild masculinization and inhibition of FSH. In this regard, it is to be noted that hirsutism has been observed in association with ovarian hyperthecosis (162, 167, 204, 211).

This condition may be differentiated (72) from an adrenogenital syndrome on the basis of lack of true masculinization and normal 17 ketoster

oids and from a Cushing's syndrome by lack of the typical habitus, purple striae and metabolic disturbances. A masculinizing tumor of the ovary may also be differentiated by lack of true masculinization and the fact that the ovarian enlargement is bilateral and cystic.

Generous wedge resection of the ovaries frequently results in prompt and striking improvement of this syndrome, with return to normal ovulatory cycles in a high percentage of cases as indicated by basal temperature charts, endometrial biopsies and improved fertility (202). This improvement in endocrine function following removal of ovarian tissue is difficult to explain.

Hormonal therapy is not often of help in this condition. In its early stages occasional cases benefit from cyclic progesterone therapy or treatment with cortisone (p. 860).

E. OVARIAN DEFICIENCIES OF PSYCHOGENIC ORIGIN

It is well known that nervous or emotional disturbances may be followed by impairment of ovarian function resulting in anovulatory infertility, secondary amenorrhea, dysfunctional uterine bleeding, cystic mastitis or other manifestations of disturbed ovarian function. The clinical findings, together with hormonal and other endocrine studies of such patients indicate that there are several patterns or syndromes of ovarian hypofunction which may occur on a psychogenic basis (163, 171, 206).

1. **Hypogonadotropism.** Inhibition of pituitary gonadotropic secretion with secondary ovarian deficiency appears to be most common in such disturbances. In its mildest form the menses may occur at fairly regular intervals but there is interference with normal ovulation causing relative infertility. This may progress to dysfunctional bleeding or to secondary amenorrhea depending upon the degree of inhibition of the pituitary ovarian mechanism.

2. **Inhibition of gonadotropins plus other pituitary hormones.** These patients generally have secondary amenorrhea with absence of gonadotropin excretion in the urine plus clinical and metabolic evidences of mild hypothyroidism. In severe cases the clinical picture is that of anorexia nervosa with loss of weight, tiredness, hypotension and moderately decreased 17 ketosteroids. In occasional instances the clinical manifestations may be so marked as to require differentiation from Simmonds disease (57).

3. **Normal FSH with ovarian deficiency.** This is the pattern seen in patients with so called 'hypothalamic amenorrhea' as described by Klinefelter, Albright and Griswold (116). It has been presumed by these authors that the ovarian hypofunction results from lack of LH secretion to synergize with FSH to stimulate the ovary. The clinical

manifestations are those of secondary amenorrhea with anovulatory infertility. In our experience, this group is not as commonly seen as those described under the first classification although the clinical picture is often the same.

4 Primary ovarian deficiency with high FSH. This syndrome is characterized clinically by secondary amenorrhea and infertility and usually accompanied by hot flashes and other menopausal symptoms. It generally occurs after an acute psychic trauma and is often of long duration and highly resistant to treatment.

From the endocrine patterns which have been described above, it would appear that psychogenic factors may influence ovarian function by way of the pituitary, the hypothalamus, or the ovary directly. Unfortunately, very little is known of the intermediary mechanisms by which psychogenic factors influence these structures.

The therapeutic approach to psychogenic ovarian hypofunction is often difficult. In milder cases readjustment of environmental factors and superficial psychotherapy may be all that is required. For this reason, many such cases eventually undergo spontaneous correction. On the other hand, in many other instances the syndrome may be prolonged and may fail to respond even to intensive psychotherapy. Such patients are also known to be very resistant to endocrine therapy of various types. The use of gonadotropic hormones in patients with low FSH only rarely results in stimulation of ovarian function. Preparations containing follicle stimulating activity should be tried in cyclic fashion in the milder cases. In the more severe cases substitutional therapy with estrogen followed by estrogen and progesterone may bring on cyclic bleeding. These patients seem to require much higher doses of estrogen and progesterone to obtain an endometrial response and bleeding than patients with other types of amenorrhea (67, 171). The repeated induction of cyclic bleeding is occasionally followed by resumption of spontaneous ovarian function. This approach with substitutional therapy is of course the only plan of endocrine therapy indicated in patients with a primary ovarian deficiency.

Low dosage irradiation to the pituitary or ovaries is not usually helpful in these cases of psychogenic ovarian hypofunction (170) and is reserved for cases of infertility in which all other methods of treatment have failed. Better results are obtained in patients with a gonadotropic deficiency than in those with primary ovarian failure. The effects of the irradiation therapy appear to be mediated by way of the ovary rather than the pituitary (170).

F METABOLIC DISTURBANCES AFFECTING OVARIAN FUNCTION

In some patients excessive gain of weight may result in ovarian hypofunction with secondary amenorrhea or oligomenorrhea. This is usually due to inhibition of pituitary FSH secretion although the mechanism by which this occurs is not known. It is remarkable to observe in some patients the manner in which menstrual function may return and then disappear again depending upon their gain or loss of weight by dieting.

In this respect it is also to be noted that some young women may develop a very severe or resistant type of secondary amenorrhea following rapid loss of weight below their usual normal by severe dietary restriction (90). In these patients there is also an FSH deficiency. Unfortunately in some cases ovarian function does not resume spontaneously following a return to normal diet and normal weight. It is believed that diets deficient in protein and certain members of the B complex are particularly likely to cause ovarian dysfunction.

Disturbances in liver function may indirectly affect the function of the ovaries because of failure of normal inactivation of the ovarian hormones. Thus in hepatitis and in cirrhosis of the liver disturbances of ovarian function causing menorrhagia, polymenorrhea or cystic mastitis are sometimes encountered (98).

G EFFECT OF ADRENAL HYPERFUNCTION ON THE OVARIES

It is of course well known that marked degrees of adrenal cortical hyperfunction will inhibit ovarian function by depressing the gonadotropins. In the adrenogenital syndrome this may be attributed primarily to the androgens which are produced in excess. In some cases there is also increased estrogen secretion by the adrenal (90) the effects of which are masked by the greater increase in the androgens. However in some cases of Cushing's syndrome in which the 17 ketosteroids are not elevated and the total estrogen excretion is diminished the inhibition of gonadotropic function may be attributed to other steroids of the adrenal cortex.

The histologic picture of the ovaries in patients with an adrenogenital syndrome due to an adrenal cortical tumor has been of particular interest to us. In two such cases in which ovarian biopsies were obtained there was a striking resemblance of the histologic findings in the ovary to those seen in the Stein-Leventhal syndrome. Although the ovaries were not enlarged there was marked thickening of the capsule, microfollicular cystosis and hyperplasia of the theca cells.

Another type of ovarian dysfunction which appears to be related to a mild degree of adrenal cortical hyperfunction or dysfunction is seen in patients with mild hirsutism, anovulatory infertility, oligomenorrhea or

secondary amenorrhea and 17 ketosteroid excretion which is only slightly elevated above the normal, ranging from 15 to 22 mg per 24 hours. Such patients will sometimes respond favorably to the administration of cortisone in a dosage of 20 to 30 mg daily by mouth. Jones *et al* (105) have drawn attention to the use of cortisone in stimulating greater follicular activity in patients of this type. The mechanism by which cortisone produces this favorable action would appear to be similar to that described by Wilkins in patients with congenital adrenal hyperplasia, namely the depression of adrenal androgens by cortisone with subsequent release of FSH.

H. OVARIAN HYPERFUNCTION

The most common and the most perfect example of ovarian hyperfunction is that which is seen in early pregnancy, during which the corpus luteum is stimulated to produce maximal amounts of estrogen and progesterone by the rising titer of chorionic gonadotropin secreted by the cytotrophoblast. Even in states of anterior pituitary hyperfunction the hypophysis is not capable of secreting sufficient luteinizing and luteotropic hormone to maintain and stimulate the corpus luteum to a degree comparable to that seen in early gestation. There are however several syndromes in which this may occur to a lesser extent.

I. CORPUS LUTEUM CYST

In some instances in which there has been excessive hemorrhage into a corpus luteum this structure may become cystic and persist with functional activity beyond the usual two week period so regularly noted in the human cycle. Menstruation may be delayed because of a persistently high estrogen and progesterone level. There may be signs and symptoms of early pregnancy or the clinical picture may suggest an ectopic pregnancy. Also, in some cases, the pregnancy test may be weakly positive, suggesting increased secretion of LH and LTH.

Rare patients may often or regularly have a slightly prolonged corpus luteum phase, lasting for 16 to 18 days as indicated by basal temperature charts. This occurs occasionally in patients who are particularly anxious for pregnancy but who do not develop a true pseudocyesis syndrome.

I. PSEUDOCYESIS

The pseudocyesis syndrome is a striking example of the influence of psychogenic factors upon the pituitary-ovarian mechanism to produce a state of ovarian hyperfunction. This condition occurs primarily in women who have a great desire to become pregnant but who have been unable to do so or sometimes in women who have a great fear that they have become pregnant illegitimately or against their wishes. Endocrine studies (58-59)

indicate that they have persistent corpus luteum function characterized by hyperhormonal amenorrhea associated with a secretory endometrium, persistently good levels of estrogen and pregnandiol plus such other evidences of ovarian hyperfunction as increased glandular tissue of the breasts, pigmentation of the nipples, softening of the cervix and slight enlargement of the uterus. FSH in the urine is generally absent but assays for prolactin have been positive. These findings explain the milky secretion which may often be expressed from the nipples of these patients. It would seem that psychogenic factors have in some way interfered with the pituitary hypothalamic mechanism so as to favor continued secretion of LH and LTH and persistence of the corpus luteum. The endocrine disturbance together with the other psychogenic manifestations accounts for the full blown syndrome of pseudocyesis. Psychiatric studies on these patients indicate that the psychosis is a type of hysteria. Psychotherapy is only of limited value in the treatment of this condition which also has a great tendency to be recurrent. Testosterone is sometimes helpful apparently by inducing regression of the corpus luteum. This is generally followed by a menstrual flow suggesting to the patient that she is no longer pregnant.

VI Placenta

The human placenta is a versatile and prodigious producer of both protein and steroid hormones. There is well-documented evidence that the placenta secretes large amounts of chorionic gonadotropin, estrogens and progestogens. There is also good evidence suggesting that the placenta secretes glucocorticoids (220-100-150). Studies of the ketosteroids in the urine of pregnant women (48) show both qualitative and quantitative differences from those of nonpregnant women suggesting that the placenta produces other steroid precursors or that pregnancy induces alterations in steroid metabolism or both. Evidence for the secretion of a protein hormone containing ACTH activity has also been presented (101).

There is increasing evidence (205-238) that the cytotrophoblast (Langhans layer) is the site of origin of chorionic gonadotropin and possibly of other protein hormones whereas histochemical studies suggest that the syncytial cells are the site of formation of the steroid hormones.

Comparatively little is known concerning the functions of the placenta in gestation. Chorionic gonadotropin appears in the blood and urine in rapidly rising titers in the early days of pregnancy and apparently maintains the corpus luteum of pregnancy. During the last half of pregnancy the chorionic gonadotropin levels are much lower. The function of this hormone during the placental phase is not known; it has been suggested that it stimulates the production of estrogens and progesterone by the syncytial cells (190). The early appearance of chorionic gonadotropin in

blood and urine and its high concentration make it possible to use its assay as the basis for the more reliable pregnancy tests including the Aschheim Zondek, Friedman, South African frog, American male frog, and rat-ovary hyperemia tests.

Estrogens and progesterone are secreted in increasing amounts as pregnancy progresses. It has been generally believed that during the first trimester the bulk of these steroids are of corpus luteum origin and that the corpus luteum is essential for the maintenance of pregnancy during this period. More recent studies (118) would indicate that in the human being the corpus luteum is essential for a much briefer period of time and that removal of the corpus luteum during the first weeks of pregnancy does not necessarily result in termination of the pregnancy or diminution in pregnanediol excretion, thus suggesting that the trophoblast may adequately take over this function very early.

Estrogen and progesterone appear to be essential for growth and quiescence of the uterine muscle during gestation (174). They also induce many other changes favorable for mother and fetus such as the further growth and development of the breasts, softening of the cervix, thickening of the vaginal epithelium, and increased vaginal acidity. With regard to some of these changes the action of the two hormones appears to be synergistic whereas in others they are antagonistic.

The intermediary metabolism of estrogen and progesterone during pregnancy is discussed in Volume I of *The Hormones*.

A. PREGNANCY COMPLICATIONS

1. Abortion

Several patterns of hormonal imbalance have been observed (104) in patients with sporadic or habitual abortion. The latter condition has been particularly useful for such studies since the hormonal excretion patterns have been studied from early gestation to the time of abortion. In one type the titer of chorionic gonadotropin is low from the very early days of pregnancy and progressively falls. Estrogen and pregnanediol excretion is also diminished. In these pregnancies the fetus is often resorbed and there is reason to suspect a genetic defect. In another type pregnanediol and estrogen levels are diminished in spite of normal titers of chorionic gonadotropin, suggesting poor response by the corpus luteum or syncytiotrophoblast. In this group substitutional therapy with large doses of progesterone and estrogens appears to be particularly indicated. In a third type abortion may occur repeatedly in spite of normal urinary levels of all three hormones, suggesting a uterine factor or possibly a disturbance of other unknown hormonal factors. Diminished thyroid function must be

considered it has been suggested (158) that failure of the serum protein bound iodine to rise during pregnancy may be of significance in this regard

2 *Hydatidiform Mole and Chorionepithelioma*

In many instances these conditions are associated with unusually high titers of chorionic gonadotropin which may be helpful in their diagnosis. Since hydatidiform mole is most often suspected during early pregnancy when the titer is normally quite high a high serum chorionic gonadotropic titer will be indicative of hydatidiform mole only if it is several times higher than the peak values for normal pregnancy or if a moderately high level is sustained significantly beyond the period when the titer normally drops (12th to 16th weeks). A rising titer of chorionic gonadotropin following expulsion of a mole or after an abortion or normal delivery is presumptive evidence of chorionepithelioma provided the possibility of another pregnancy has been ruled out. Patients who have expelled a mole or have had surgery for a chorionepithelioma should have repeated follow up assays over a long period of time in order to detect recurrence or metastases. It should however, be emphasized that occasionally chorionepitheliomas produce little or no chorionic gonadotropin particularly in their early stages (212). The secretion of estrogens and progesterone by moles and chorion epitheliomas varies much more than that of chorionic gonadotropin the assay of the former is therefore less useful from a diagnostic standpoint. Some of these tumors may secrete very large amounts of chorionic gonadotropic hormone with but little if any estrogens or progesterone whereas others may secrete considerable amounts of all three hormones (148).

3 *Toxemias of Pregnancy*

There are many reasons to suspect a placental factor possibly of hormonal nature of etiologic significance in the toxemias of pregnancy, but satisfactory evidence for such an endocrine factor has not yet been obtained. Results of hormonal excretion studies in toxemias of pregnancy are quite controversial. The pattern reported by the Smiths (190) in many instances namely increased urinary chorionic gonadotropin and diminished urinary pregnanediol and estrogen as well as changes in the partition of the urinary estrogens has in part been confirmed by some workers. On the other hand these alterations are often absent in patients with severe pregnancy toxemia (155, 160, 193).

An increase in the urinary excretion of corticoids in toxemia of pregnancy has been reported (47, 148, 216) this finding has been tentatively correlated with the sodium and water retention observed in toxemia. The increase above that occurring in normal pregnancy is probably of questionable significance especially in view of the methods employed in these studies.

Even if such an increase will be confirmed with the use of more specific methods such as those based on the Porter-Silber reaction, the question will still remain what is cause, and what effect? The occurrence of a severe toxemia could cause a non specific rise of adrenocortical secretion. It has been suggested that increased production of a mineral corticoid might play a role in this condition (147). This entire problem appears to be quite unsettled and further advances will depend in part on better assay methods for these steroids, particularly for the mineralocorticoids. Similarly the question of the use of cortisone or ACTH in the treatment of late pregnancy toxemias is controversial (147, 211).

VII Testes

The hypophyseal testicular interrelationships have been discussed in Volume II. Several problems in this sphere remain unsolved. Although FSH is believed to be primarily responsible for stimulation of spermatogenesis, it is uncertain whether this occurs without the synergism of the other gonadotropins or of androgens. It is equally uncertain whether luteotropin (LTH prohetin) plays any role in the male, perhaps as a synergist of LH (ICSH) in stimulating Leydig cell function. The question of a second testicular hormone awaits final answer. The existence of such a hormone has been postulated on the basis of analogy with the female Interstitial tissue (and corpus luteum) in the female and interstitial tissue (Leydig cells) in the male producing progesterone and testosterone respectively is regulated by LH (ICSH). Ovarian follicles and estrogen secretion in the female and spermatogenic tubules in the male are governed by FSH. Is there a tubular hormone in the male corresponding to the follicular estrogen?

In addition to such general considerations certain clinical observations suggest the existence of a tubular hormone. In Klinefelter's syndrome (Volume II p. 679) hyalinization of the seminiferous tubes and intact Leydig cell function are associated with excessive urinary FSH excretion. This suggests that under normal circumstances the tubules secrete a hormone the absence of which releases pituitary FSH secretion. It has been proposed that this hormone might be an estrogen. Estrogens have been extracted from the testis of human beings as well as certain animal species and are believed by some to be secreted by the Sertoli cells. This assumption is based chiefly on the findings in Sertoli cell tumors of the testis which induce feminization in dogs and in man. From Sertoli cell tumors of dogs estrogens have been extracted. However Nelson has questioned the origin of these testicular tumors from Sertoli cells and believes them to be Leydig cell tumors (144). If androgens and estrogens are according to

Nel on produced in the Leydig cell the postulated tubular hormone still goes begging

A HYPOGONADISM (91)

Hypogonadism is discussed in Volume II of *The Hormones*

In view of the fact that in many instances nothing is known regarding the etiology of the various syndromes hypogonadism may be classified, on a symptomatological basis as follows Failure (1) of both tubular (spermatogenic) and Leydig cell function (2) of tubular function with intact Leydig cell function and (3) of Leydig cell function with intact tubular function All three types can at least theoretically be caused (a) by a primary testicular deficiency and (b) by a testicular deficiency secondary to pituitary gonadotropic failure

1 Failure of Tubular and Leydig Cell Function

This type is represented by the surgical castrate (eunuch) and by the eunuchoid in whom testicular failure is due to an anlage defect (functional castrate 86), or due to impaired blood supply incident to operations for cryptorchidism or hernia or due to bilateral orchitis

These cases may be indistinguishable in their physical appearance and in their functional deficiencies from cases of pituitary hypogonadotropicism Differentiation of primary testicular eunuchoidism and eunuchoidism secondary to hypogonadotropicism is readily accomplished by assay of urinary gonadotropins (FSH assay) which yields very high values in the former whereas gonadotropin excretion is absent in the latter

The castrates and the primary testicular eunuchoid are treated with testosterone In hypogonadotropic eunuchoids treatment with gonadotropic hormones (chorionic gonadotropin and FSH containing pituitary preparations) has been successful

Whether an anlage defect of the testis ever is associated with a syndrome analogous to that of ovarian agenesis (Turner's syndrome) is as yet uncertain As is discussed elsewhere (p 83) the absence of estrogen effects and of germ cells and the presence of high urinary gonadotropin (FSH) levels in ovarian agenesis are associated with features which are different from those of the prepubertal castrate shortness of stature frequent occurrence of other congenital malformations including webbing of the neck usually no delay of closure of the epiphyses None of the few cases reported as Turner's syndrome in the male (194) appear to us convincingly analogous to the female syndrome The suggestion has been made (p 83) that some patients with ovarian agenesis may in fact be genetic males and thus represent instances of testicular aplasia

Even if such an increase will be confirmed with the use of more specific methods, such as those based on the Porter-Silber reaction, the question will still remain what is cause, and what effect? The occurrence of a severe toxemia could well cause a non specific rise of adrenocortical secretion. It has been suggested that increased production of a mineral corticoid might play a role in this condition (147). This entire problem appears to be quite unsettled and further advances will depend in part on better assay methods for these steroids particularly for the 'mineralocorticoids'. Similarly the question of the use of cortisone or ACTH in the treatment of late pregnancy toxemias is controversial (147, 211).

VII Testes

The hypophyseal testicular interrelationships have been discussed in Volume II. Several problems in this sphere remain unsolved. Although FSH is believed to be primarily responsible for stimulation of spermatogenesis it is uncertain whether this occurs without the synergism of the other gonadotropins or of androgens. It is equally uncertain whether luteotropin (LTH prolactin) plays any role in the male perhaps as a synergist of LH (ICSH) in stimulating Leydig cell function. The question of a 'second' testicular hormone awaits final answer. The existence of such a hormone has been postulated on the basis of analogy with the female. Interstitial tissue (and corpus luteum) in the female and interstitial tissue (Leydig cells) in the male producing progesterone and testosterone respectively, is regulated by LH (ICSH). Ovarian follicles and estrogen secretion in the female and spermatogenic tubules in the male are governed by FSH. Is there a tubular hormone in the male corresponding to the follicular estrogen?

In addition to such general considerations certain clinical observations suggest the existence of a tubular hormone. In Klinefelter's syndrome (Volume II p. 679) hyalinization of the seminiferous tubes and intact Leydig cell function are associated with excessive urinary FSH excretion. This suggests that under normal circumstances the tubules secrete a hormone the absence of which releases pituitary FSH secretion. It has been proposed that this hormone might be an estrogen. Estrogens have been extracted from the testis of human beings as well as certain animal species and are believed by some to be secreted by the Sertoli cells. This assumption is based chiefly on the findings in Sertoli cell tumors of the testis which induce feminization in dogs and in man. From Sertoli cell tumors of dogs estrogens have been extracted. However Nelson has questioned the origin of these testicular tumors from Sertoli cells and believes them to be Leydig cell tumors (144). If androgens and estrogens are according to

(119-135-151) The evidence for the latter varied somewhat in individual cases: eunuchoid body proportions, sparseness or absence of pubic and axillary hair, poor beard growth, high pitched voice, undersized penis as well as diminished urinary 17 ketosteroid values have been reported. Testicular biopsy specimens showed diminished number and pyknotic appearance of Leydig cells and production of mature sperm in the tubules. In some instances an ejaculate could not be obtained, in one case the fructose content of the ejaculate was very low (119), the fructose content of the semen is considered to be a measure of androgen secretion (120). In other cases the ejaculate was normal and fertility was claimed for one of these cases. McCullagh has applied in his cases an assay method believed to show LH (ICSH) in urinary extract. In three of his five cases I H was not demonstrable, suggesting a pituitary deficiency limited to this one hormone. Some response was obtained by treatment with chorionic gonadotropin; this would add further support to the assumption of a pituitary deficiency (135).

B TESTOSTERONE AS A GONADOTROPIN

Several older studies are on record indicating that under narrowly circumscribed experimental conditions testosterone can exert a gonadotropic function. In the male hypophysectomized rat treatment with testosterone initiated immediately after hypophysectomy will maintain spermatogenesis and prevent testicular atrophy, but treatment with testosterone will not repair the previously atrophied testicle nor restore spermatogenesis once the posthypophysectomy changes have taken place (145-187).

There are for obvious reasons no observations in man which would duplicate the experimental conditions in the rat. However in several cases of hypopituitarism sperm production occurred under treatment with testosterone (95-113). In contradistinction to the hypophysectomized animal the possibility cannot be ruled out in such a case that testosterone therapy induced the pituitary to secrete gonadotropins. Kinsell rejects this possibility because no rise of urinary gonadotropin titer was observed during androgen therapy (113).

C TESTICULAR TUMORS

Tumors of the testis will be discussed briefly only with regard to endocrine aspects. Leydig cell tumors secrete large amounts of androgens and 17 ketosteroids and androsterone sulfate has been isolated from the urine. In prepubertal boys these tumors cause precocious pseudopuberty (p. 869). It has been pointed out that probably because of the protein anabolic action of the androgen secreted by the tumor tumor cachexia may be absent even in advanced cases with widespread metastasis (133). It is

2 *Impairment of Tubular Function with Intact Leydig Cell Function*

Typically the individuals are normally developed males, with normal secondary sex characteristics, normal libido and potency, and normal LH and testosterone excretion. The size of the testes varies from normal to markedly diminished. The sperm content of the ejaculate varies from oligospermia to azospermia. Testicular biopsy specimens show normal Leydig cells; the germinal epithelium may be well developed, with spermatogenesis progressing to the spermatide stage ('spermatogenesis'), but with absent sperm maturation ('spermogenesis'). In other cases the tubules are completely denuded with only Sertoli cells left. Peritubular fibrosis is present in varying degrees in most but not in all cases of the latter type. Urinary gonadotropin excretion is normal in some and very high in other cases, but a definite correlation of the FSH excretion pattern with the histological picture of the testis is as yet not possible. There is no evidence that any of these cases is due to a pituitary gonadotropic deficiency. The etiology is unknown.

It has been claimed that seminal failure can be recognized already in the histologic picture of the pubertal testis by marked retardation of tubular development (31).

The Klinefelter syndrome discussed in Volume II, characterized by hyalinization of the tubules, normal Leydig cells, gynecomastia, and high FSH excretion, would appear to be a special entity within the group of seminal failure without Leydig cell failure (45-116). Cases have been reported which show features of Klinefelter syndrome associated with varying degrees of Leydig cell failure (85-87).

Treatment. Cases with azospermia and extensive fibrosis and/or hyalinization of the tubules as seen in the biopsy specimen are obviously beyond any possibility of repair. However, results are also unsatisfactory in most instances of milder forms with oligospermia and maturation arrest. Various gonadotropic preparations have been used on a purely empirical basis. One would expect results if any from preparations containing adequate amounts of FSH such as pituitary gonadotropins or Synapoidin. Chorionic gonadotropin has also been used and occasional good results have been claimed with any of the preparations. Treatment with large doses of testosterone has also been suggested; complete suppression of sperm production is reported to be followed by a rise of viable sperm cells in the ejaculate to values permitting fecundation ('rebound effect') (82). Confirmation has to be awaited.

3 *Leydig Cell Deficiency with Normal Tubular Function*

Recently three reports have appeared concerning patients who had spermatogenesis in the presence of evidence for impaired Leydig cell function

of the sex steroids. These amounts are too small to induce somatic changes as evidenced by the absence of secondary sex characteristics in normal children and by the absence of detectable somatic changes in prepubertal castrates. However the fact that the prepubertal castrate (80) as well as the child with ovarian agenesis (185) may secrete appreciable amounts of FSH indicates that the child's gonad is not wholly inactive. The pubertal spurt of sex steroid secretion from the gonad is caused by a spurt of gonadotropin secretion from the anterior pituitary. This can be shown by urinary gonadotropin assay and is evident from the fact that administration of exogenous gonadotropin causes sex hormone secretion and development of secondary sex characteristics years before the physiological onset of puberty.

The anterior pituitary probably is triggered into gonadotropin release at the time of puberty by an hypothalamic mechanism presumably by the secretion of a neurohumeral agent which reaches the anterior pituitary by way of the portal venous system of the pituitary stalk. Just as the gonad is capable of responding to gonadotropic stimulation long before the physiological onset of puberty, the anterior pituitary of animals (181) and human beings (11) contains appreciable amounts of gonadotropic hormones without releasing them. The release of gonadotropic hormones from the anterior pituitary through the hypothalamic mechanism has been demonstrated for the LH release in the rabbit (130); the fact that certain organic lesions of the hypothalamus in children cause precocious puberty suggests that a similar mechanism operates in the initiation of the pubertal chain of events in the human being.

Precocious puberty is found in cases of hypothalamic lesions (tumors, encephalitis, hydrocephalus); perhaps the precocious puberty associated with polyostotic fibrous dysplasia (Albright's disease) is due to cerebral changes resulting from deformities of the base of the skull. In the majority of cases of precocious puberty no organic lesion is present; these cases are referred to as constitutional precocious puberty. It may be assumed that for some reason the intrinsic triggering mechanism in the hypothalamus is set off abnormally early in life. Because in precocious puberty both in that due to organic lesions and in the constitutional type the entire physiological pubertal mechanism is duplicated, the precocious development includes that of the gonad involving both its endocrine secretions and maturation of ova and sperm. Pregnancy has been reported in a girl $5\frac{1}{2}$ years old.

Tumors or hyperplasia of the adrenal cortex, tumors of the testis or ovary, and teratomas containing testicular tissue cause precocious development of secondary sex characteristics, either isosexual or heterosexual, but not maturation of the gonad. This type of precocity is called precocious pseudopuberty.

interesting that the estrogen excretion may be increased simultaneously in such cases (222). This finding would add weight to the theory that the Leydig cell is the source of both androgens and estrogens (p. 864). However, too few cases have been adequately and completely studied to permit any generalizations or final conclusions, unfortunately reports on this rare tumor continue to appear in the literature with little information other than a histologic study of the tumor.

Tumors of the testis believed to originate from Sertoli cells, have been found in man (210) and more commonly in the dog (92). These tumors secrete estrogens and cause gynecomastia. Nelson questions their origin from Sertoli cells and believes them to be a variety of the Leydig cell tumor (144) (see also p. 864).

Certain malignant tumors of the testis contain chorion like tissue (choriocarcinoma) they are characterized by excretion of large amounts of gonadotropins which are biologically identical with chorionic gonadotropins of pregnancy. Pregnancy tests (Asheim Zondek Friedman, rat ovary hyperemia test) carried out with urine of such cases are positive. Increased urinary estrogen levels and the occurrence of gynecomastia in some cases suggest that the choriocarcinoma duplicates not only the gonadotropin production but also the estrogen production of normal chorionic tissue. There is suggestive evidence that in some cases of choriocarcinoma of the testis there is increased urinary excretion not only of chorionic gonadotropin but also of FSH. Because of the technical difficulties of assaying the two gonadotropins separately in a mixture of both the data should at present not be considered final. The presence of large amounts of urinary FSH in such cases has been tentatively explained as the result of destruction of testicular tissue (castration by tumor) but in view of the fact that the same hormone pattern occurs in unilateral tumors with intact second testis this explanation is not wholly satisfactory. Similar observations of increased urinary FSH levels have been reported in other testicular tumors. Regarding the significance of this finding the same uncertainty has to be recorded as that mentioned above in the case of choriocarcinomas. No essentially new facts have been reported recently a good review of the subject appeared in 1946 (219).

VIII Puberty (149)

In our population the onset of puberty normally takes place at the age of about 11 to 12 years in girls and 12 to 13 years in boys. From this mean age of onset there occurs considerable deviation with onset delayed up to 18 to 20 years or advanced to as early as 1 to 2 years of age (precocious puberty). The onset of pubertal changes is brought about by the beginning of the secretions of estrogens and androgens respectively. The prepubertal gonad appears to be not completely inactive but to secrete small amounts

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